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Research Article

Functors induced by comma categories

SUDDHASATTWA DAS*

ABSTRACT. Category theory provides a collective description of many arrangements in mathematics, such as topological spaces, Banach spaces and game theory. Within this collective description, the perspective from any individual member of the collection is provided by its associated left or right slice. The assignment of slices to objects extends to a functor from the base category, into the category of categories. Slice categories are a special case of the more general notion of comma categories. Comma categories are created when two categories $\mathcal{A}$ and $\mathcal{B}$ transform into a common third category $\mathcal{C}$, via functors F, G. Such arrangements denoted as $[F \ ; \ G]$ abound in mathematics, and provide a categorical interpretation of many constructions in mathematics. Objects in this category are morphisms between objects of $\mathcal{A}$ and $\mathcal{B}$, via the functors F, G. We show that these objects also have a natural interpretation as functors between slice categories of $\mathcal{A}$ and $\mathcal{B}$. Thus even though $\mathcal{A}$ and $\mathcal{B}$ may have completely disparate structures, some morphisms in $\mathcal{C}$ lead to functors between their respective slices. We present this relation in the form of a functor from $\mathcal{C}$ into the category of left slices. The proof of our main result requires a deeper look into associated categories, in which the objects themselves are various commuting diagrams.

Keywords: Comma categories, functors, slices, orbits.

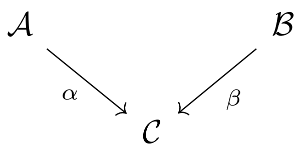
2020 Mathematics Subject Classification: 18A40, 18A35, 37A99.

1. INTRODUCTION

Category theory has emerged as an useful alternative to the descriptive language of set-theory. Instead of specifying mathematical objects from the ground-up, i.e. by their constituents, it provides a collective description of their relative arrangements. Categorical approaches have yielded surprising simplifications of deep results in all fields of mathematics, such as topology [8, 33, 40], probability theory [19, 20, 34], dynamical systems theory [7, 11, 44], and game theory [21, 22]. Readers can obtain a basic understanding of categories and functors from sources such as [31, 39].

Our discussion is based on the following general arrangement of categories and functors:

(1.1)



We shall see several examples of how such a general arrangement is prevalent all over mathematics. Our focus will be on a category built upon such arrangements, called the comma category $[\alpha \ ; \ \beta]$. The objects in this category are

$$ob([\alpha \ ; \ \beta]) := \{(a, b, \phi) \ : \ a \in ob(\mathcal{A}), \ b \in ob(\mathcal{B}), \ \phi \in \mathrm{Hom}_{\mathcal{C}}(\alpha a; \beta b)\},$$

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*Corresponding author: Suddhasattwa Das; iamsuddhasattwa@gmail.com
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and the morphisms comprise of pairs $\{(f, g) \,:\, f \in \mathrm{Hom}(D), g \in \mathrm{Hom}(E)\}$ such that the following commutation holds:

$$(a, \phi, b) \xrightarrow{(f,g)} (a', \phi', b') \quad \Leftrightarrow \quad \begin{array}{c} a \\ \downarrow f \\ a' \end{array}, \quad \begin{array}{c} b \\ \downarrow g \\ b' \end{array}, \text{ s.t. } \begin{array}{ccc} \alpha a & \xrightarrow{\alpha f} & \alpha a' \\ \downarrow \phi & & \downarrow \phi' \\ \beta b & \xrightarrow{\beta g} & \beta b' \end{array}$$

Thus $[\alpha \,;\, \beta]$ may be interpreted as the category of bindings between α, β, via their common codomain $\mathcal{C}$. Comma categories contain as sub-structures, the original categories $\mathcal{A}, \mathcal{B}$, via the forgetful functors

$$\mathcal{A} \xleftarrow{\pi_1} [\alpha \,;\, \beta] \xrightarrow{\pi_2} \mathcal{B}$$

whose action on morphisms in $[\alpha \,;\, \beta]$ can be described as

$$\begin{array}{c} a \\ \downarrow f \\ a' \end{array} \quad \xleftarrow{\quad \pi_1 \quad} \quad \begin{array}{ccc} \alpha a & \xrightarrow{\alpha f} & \alpha a' \\ \downarrow \phi & & \downarrow \phi' \\ \beta b & \xrightarrow{\beta g} & \beta b' \end{array} \quad \xrightarrow{\quad \pi_2 \quad} \quad \begin{array}{c} b \\ \downarrow g \\ b' \end{array}$$

Comma categories prevail all over category theory and mathematics in general, such as graph theory [23], in the theory of lenses and fibrations [30], iterative algebras [2], stochastic processes [5], Paré et al.'s work on double categories [26, 27], connectedness [37], and mathematical logic [38, 43]. If a category can be presented as a comma category, then one obtains additional results to prove the existence of (co)-limits [7, 24]. We now examine two simpler examples for motivation:

Example 1.1 (Measured dynamical systems)**.** *Suppose $\mathcal{T}$ is a semigroup, then it is representable as a 1-object category. Let* [Topo] *be the category of topological spaces and continuous maps. Then the class of topological dynamics is the functor category* $\mathbb{F}(\mathcal{T} \,;\, [\mathrm{Topo}])$*. Now let [Euc] be the subgroup of* [Topo] *comprised only of Euclidean spaces. Then the following arrangement is of the pattern given in* (1.1)*:*

$$\mathbb{F}(\mathcal{T} \,;\, [\mathrm{Topo}]) \xrightarrow{\ \mathrm{dom}\ } [\mathrm{Topo}] \xleftarrow{\ \subset\ } [Euc]$$

The domain functor dom *above assigns to every dynamical system its domain. A typical object in the resulting comma category is a dynamical systems* (Ω, Φ^t) *along with a measurement* $\phi : \Omega \to \mathbb{R}^d$*:*

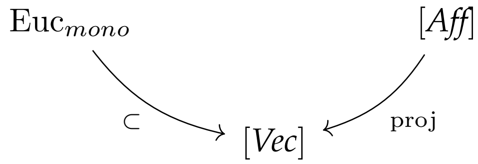
$$\Omega \xrightarrow{\Phi^t} \Omega \xrightarrow{\phi} \mathbb{R}^d.$$

A typical morphism in this comma category is a change of variables $h : \Omega \to \Omega'$ *that leads to the following joint commutation:*

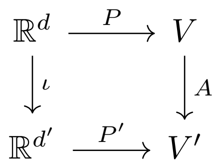
$$\begin{array}{ccccc} \Omega & \xrightarrow{\Phi^t} & \Omega & \xrightarrow{\phi} & \mathbb{R}^d \\ h \downarrow & & h \downarrow & & \downarrow \iota A \\ \Omega' & \xrightarrow{\Phi'^t} & \Omega' & \xrightarrow{\phi'} & \mathbb{R}^{d'} \end{array}, \quad \forall t \in \mathcal{T}.$$

This comma category encapsulates the collection of all measured, topological dynamical systems. This category has been instrumental in a categorical study of data and reconstruction theory [11].

Example 1.2 (Finite subspaces)**.** *Let [Aff] be the category of vector spaces and affine maps, [Vec] be the category of vectors spaces and linear maps,* Euc *be the category of finite dimensional vector spaces and linear maps, and* Euc_{mono} *be the subcategory of* Euc *comprising only of injective maps. Then the following arrangement*



leads to a comma category in which the objects are affine linear embeddings $P : \mathbb{R}^d \to V$ *of finite dimensional vector spaces in possible infinite dimensional vector spaces. The morphisms between these objects are now allowed to be affine. Shown below is a morphism from an object* P *to an object* P'*:*

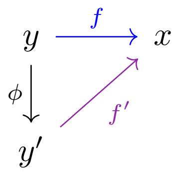


in which ι *must be injective, and* A *is an affine map. The study of these objects and morphisms lead to a generalized notion of "null" and "everywhere"* [9, 29].

A particular instance of comma categories are slice categories. Henceforth, we shall use the symbol $\star$ to denote the category with a single object with no non-trivial morphism. Take any category $\mathcal{X}$, and an object x in it. This object may be interpreted by a unique functor from $\star$ to $\mathcal{X}$, which we shall also denote by $\star \xrightarrow{x} \mathcal{X}$. Now, set

$$\mathcal{B} = \star,\ \mathcal{A} = \mathcal{C} = \mathcal{X},\ \beta = x,\ \alpha = \text{Id}_{\mathcal{X}},$$

in (1.1). The resulting comma category $[\text{Id}_{\mathcal{X}}\ ;\ x]$ is known as the left slice of x in $\mathcal{X}$, and will be denoted more briefly as $[\mathcal{X}\ ;\ x]$. A typical morphism in this category is shown below



The objects are the morphisms shown in blue, and a morphism ϕ from f to f' is a morphism $\phi : y \to y'$ such that the above commutation holds. One can similarly define the right slice of an object within its category. An important example of a slice category is the right slice of the pointed space in the category [Topo] of topological spaces. This corresponds to the category of pointed topological spaces. If $\mathcal{X}$ is a preorder category, the left or right slice of an object x is the *down-set* or *up-set* of the object. If $\mathcal{X}$ is the collection of subsets of a superset $\mathcal{U}$ ordered by inclusion, then the left slice of any subset x of $\mathcal{U}$ is the power set of x, also ordered by inclusion.

Example 1.3. *Let* $\mathcal{U}$ *be any topological space. Then its left slice* $[[\text{Topo}]\ ;\ \mathcal{U}]$ *in* [Topo] *is the category formed by the collection of all continuous maps into* $\mathcal{U}$*.*

Example 1.4. *Let* $\mathcal{U}$ *be a set and* $2^{\mathcal{U}}$ *be the power set of* $\mathcal{U}$*. Thus* $2^{\mathcal{U}}$ *is a preorder and a category. Then the left slice of any set* $S \subset \mathcal{U}$ *in* $2^{\mathcal{U}}$ *is the power set of* S*.*

Example 1.5. *More generally, let* $\mathcal{C}$ *be any category and* $\mathcal{C}_{mono}$ *be the subcategory comprising only of monomorphisms. Then the left slice of any object* c *of* $\mathcal{C}$ *in* $\mathcal{C}_{mono}$ *is the* subobject category *of* c*.*

The forgetful functors inbuilt into comma categories may also be arranged into diagrams similar to (1.1). Consider the following more abstract example:

Example 1.6. *Given the arrangement as in* (1.1) *one has the following diagram:*

$$[\alpha\ ;\ \beta] \xrightarrow{\pi_2} \mathcal{B} \xleftarrow{b} \star$$

where b is an object of $\mathcal{B}$ and $\star$ is the 1-point category. This arrangement creates the comma category $\left[\pi_2^{[\alpha\ ;\ \beta]}\ ;\ b\right]$*, whose objects are*

$$ob\left(\left[\pi_2^{[\alpha\ ;\ \beta]}\ ;\ b\right]\right) := \left\{ \begin{array}{c} \alpha a' \\ \downarrow f \\ \beta b' \end{array} , \begin{array}{c} b' \\ \downarrow g \\ b \end{array} \right\}. \tag{1.2}$$

Every such object creates an L-shaped diagram:

$$\begin{array}{ccc} \alpha a' & & \\ f \downarrow & & \\ \beta b' & \xrightarrow{\beta g} & \beta b' \end{array}$$

This is a diagram within $\mathcal{C}$ in which one morphism f is drawn from $\mathcal{C}$, while another g is drawn from $\mathcal{B}$.

Example 1.6 reveals how various diagrams following a certain pattern create a category of its own. Example 1.6 is a generalization of the more concrete Example 1.1. The comma category (1.2) will play a significant role in the next section, where we state some technical results. Yet, another important manifestation of comma categories are arrow categories. If we set

$$\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathcal{X},\ \alpha = \beta = \mathrm{Id}_{\mathcal{X}},$$

in (1.1), then the resulting comma category $[\mathrm{Id}_{\mathcal{X}}\ ;\ \mathrm{Id}_{\mathcal{X}}]$ is called the arrow category of $\mathcal{X}$, and is denoted by Arrow $[\mathcal{X}]$. The objects in this category are the arrows or morphisms in $\mathcal{X}$. A morphism between two morphisms $x \xrightarrow{f} x'$ and $y \xrightarrow{g} y'$ is a pair of morphisms $x \xrightarrow{\phi} y$ and $x' \xrightarrow{\phi'} y'$ such that the following commutation holds:

$$\begin{array}{ccc} x & \xrightarrow{f} & x' \\ \phi \downarrow & & \downarrow \phi' \\ y & \xrightarrow{g} & y' \end{array}$$

Thus, Arrow $[\mathcal{X}]$ reveals how the arrows of $\mathcal{X}$ are bound to each other via the commutation relations in $\mathcal{X}$. Some important examples are

Example 1.7. *The arrow category Arrow* $[[\mathrm{Topo}]]$ *in* $[\mathrm{Topo}]$ *corresponds to the category of topological pairs.*

Example 1.8. *The arrow category Arrow* $[[Vec]]$ *in* $[Vec]$ *corresponds to the category of linear maps, with linear change of variables serving as morphisms.*

Example 1.9. *Recall the notations in Examples 1.3 and 1.4. Let $\mathcal{U}$ be a topological space. Take $\mathcal{X}$ to be the power set $2^{\mathcal{U}}$ and $\mathcal{Y}$ to be the subcategory of* [Topo] *formed by all topological subspaces of $\mathcal{U}$. Then one has an obvious inclusion $\iota : \mathcal{X} \to \mathcal{Y}$. Note that the comma category $[\iota \,;\, \iota]$ has the same objects as the arrow category Arrow $[\mathcal{Y}]$: any object $F \in$ Arrow $[\mathcal{Y}]$ is a continuous map between topological spaces $F : X \to Y$. But a morphism from F to another such object $F' : X' \to Y'$ is just inclusion, i.e., $X' \supseteq X$, $Y' \supseteq Y$ and F is a restriction of F'.*

Comma, slice and arrow categories thus represent finer structures present within categories, and also how objects from different categories assemble together to produce more complex categories. The language of comma categories contributes to the universality of category theory as an alternate formulation for descriptive set theory [32]. Comma categories have been used in a variety of ways, from being a descriptive tool to higher constructions in category theory. Classical expositions on category theory [31, 39] rely on slice categories for pointwise description of Kan extensions. The descriptive strength has also been an integral part of a categorical reformulation of dynamical systems theory [11, 12]. One important consideration in our analysis is the fact that the objects of comma and slice categories are morphisms, and thus composable in nature. This composability of comma-objects were utilized in [36] to develop a string-diagrammatic language for ordinary categories. One of the most important applications of comma categories is Paré et al.'s work on double categories [26, 27], which are categories with two orthogonal dimensions of structure. This very concept, along with its associated notion of connectedness [37] for categories, rely on a heavy use of comma categories. In the next section, we present the main result which analyzes how the objects in a comma category could induce functors between left slices. The functor is created when one tries to complete L-shaped diagrams into universal commuting squares. The topic of functors induced by universal properties of comma categories has not been explored much. The reader might find interest in the recent work [28] on functors induced by comma categories involving exact functors and Abelian categories.

2. Main Results

The main difference of a categorical description of a subject from the classical set-theoretic description, is that properties are not internal to an object, but defined entirely in terms of their relations to external objects. Set-theoretic discourses start with several elementary ideas such as points, sets, maps, numbers and addition, and more advanced ideas are built from various combinations of these primitive concepts. On the other hand category theory only has morphisms and objects as the primitive concepts. So the more advanced concepts of mathematics have to be realized as patterns or diagrams in a categorical discourse. Various conclusions and theorems are extracted from the inter-relations between various diagram classes, and from properties defined by universality.

The simplest diagram in a category is a morphism, which expresses a relation between two objects. A more advanced diagram would be a commutation square,

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{i} & D \end{array} \;\Rightarrow\; \begin{array}{ccc} A & & \\ \downarrow{\scriptstyle f} & & \\ B & \xrightarrow{i} & D \end{array} \;,\; \begin{array}{ccc} A & \xrightarrow{g} & C \\ & & \downarrow{\scriptstyle h} \\ & & D \end{array} ,\; \begin{array}{ccc} & & C \\ & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{i} & D \end{array} ,\; \begin{array}{ccc} A & \xrightarrow{g} & C \\ \downarrow{\scriptstyle f} & & \\ B & & \end{array}$$

which not only expresses a set of four relations between four objects, but also a relation binding these four relations. Every such square contains four smaller L-shaped diagrams as shown above. They are the four corners of the commutation square and are objects of certain comma categories. As objects of these categories they are also inter-related functorially. Adjacent corners overlap on a common arrow. These overlaps can be expressed as constraints that bind the functors between the various comma-categories. The paper presents a deep dive into this diagrammatic language of category theory. One can various questions, such as given any corner L-diagram, what is the minimal or maximum square to which it can be completed? If the bottom edge i is fixed, one can look for lower-left corners, lower-right corners that extend that edge into an L-diagram. Let these classes be named $LL(i)$ and $LR(i)$ respectively. Are these classes also categories? If each object in $LL(i)$ and $LR(i)$ can be completed into a universal square, is this correspondence functorial, and how does it correspond to the i we started out with? While these questions are entirely diagrammatic, we will discover important applications to classical branches of mathematics. We focus on an arrangement of the form $\mathcal{X} \xrightarrow{\iota} \mathcal{Y} \xleftarrow{\iota} \mathcal{X}$, which is a special instance of (1.1). More precisely:

Assumption 2.1. *There are complete categories $\mathcal{X}$ and $\mathcal{Y}$, with initial objects $0_{\mathcal{X}}$ and $0_{\mathcal{Y}}$ respectively, and there is a continuous functor $\iota : \mathcal{X} \to \mathcal{Y}$ such that $0_{\mathcal{Y}} = \iota(0_{\mathcal{X}})$, and ι is injective on objects.*

Let $\text{LeftSlice}(\mathcal{X})$ denote the category whose objects are left-slice categories $[\mathcal{X} \,;\, \Omega]$ for various $X \in \mathcal{X}$, and morphisms are the functors between these categories. Thus $\text{LeftSlice}(\mathcal{X})$ is a full subcategory of [Cat], the category of small categories. Let $\iota(\mathcal{X})$ denote the full subcategory of $\mathcal{Y}$ generated by objects of ι. Note that the morphisms in $\iota(\mathcal{X})$ are precisely the objects of $[\iota \,;\, \iota]$. Recall that a morphism f in any category is said to be surjective or equivalently, an *epimorphism*, if for any composable morphisms g, g', if $g'f = gf$, then $g = g'$. Similarly, a morphism is said to be injective or equivalently, a *monomorphism*, if for any composable morphisms g, g', if $fg' = fg$, then $g = g'$. We need the following assumptions:

Assumption 2.2. *For every monomorphism f in $\mathcal{Y}$, there are morphisms g in $\mathcal{Y}$ and h in $\mathcal{Y}$ such that $f = (\iota h)g$.*

Assumption 2.3. *The image under ι of every morphism in $\mathcal{X}$ is injective in $\mathcal{Y}$.*

Assumption 2.4. *The category $\mathcal{Y}$ is balanced, i.e., any morphism in $\mathcal{Y}$ which is both surjective and injective is an isomorphism.*

Assumption 2.4 is satisfied in categories such as topoi [4, 17, 25]. Two prime examples of topoi are [Set] and [Topo]. Another important category in which Assumption 2.4 is satisfied is [Group], the category of groups and homomorphisms. Our main result establishes a functor

$$\iota(\mathcal{X}) \xrightarrow{\text{Dyn}} \text{LeftSlice}(\mathcal{X}) \tag{2.3}$$

that achieves certain universal diagram completions, as discussed before. Recall that a typical morphism in $\iota(\mathcal{X})$ is a $\mathcal{Y}$-morphism $\iota\Omega \xrightarrow{F} \iota\Omega'$. The functor in (2.3) should convert this into a functor between the left-slice categories $[\mathcal{X} \,;\, \Omega]$ and $[\mathcal{X} \,;\, \Omega']$ associated to the endpoints of F.

Theorem 2.1. *Let Assumptions 2.1, 2.2, 2.3 and 2.4 hold. Then, there is a functor as in (2.3) which maps every object $\Omega \in \mathcal{X}$ to the left slice $[\mathcal{X} \,;\, \Omega]$. It maps an morphism $\iota\Omega \xrightarrow{F} \iota\Omega'$ into a functor*

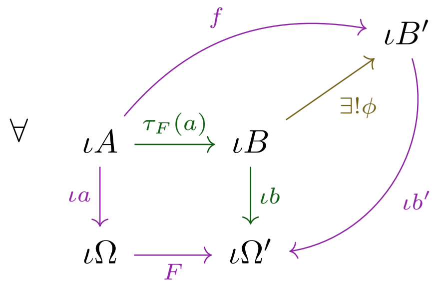
$$[\mathcal{X} \,;\, \Omega] \xrightarrow{\tau_F} [\mathcal{X} \,;\, \Omega']$$

such that for every slice-object $A \xrightarrow{a} \Omega \in [\mathcal{X} \,;\, \Omega]$, $\tau_F(a)$ creates a commutation square

$$\begin{array}{ccc} \iota A & & \\ {\scriptstyle \iota a}\downarrow & & \\ \iota\Omega & \xrightarrow[F]{} & \iota\Omega' \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \iota A & \xrightarrow{\tau_F(a)} & \iota B \\ {\scriptstyle \iota a}\downarrow & & \downarrow{\scriptstyle \iota b} \\ \iota\Omega & \xrightarrow[F]{} & \iota\Omega' \end{array} \tag{2.4}$$

for some $B \xrightarrow{b} \Omega \in [\mathcal{X} \,;\, \Omega']$. Moreover, this square is universal in the sense that for any slice-object $b' : B' \to \Omega'$ and any morphism $f : \iota A \to \iota B'$, if the blue commutation square shown below holds

(2.5)



then there is a unique morphism ϕ which factors the outer commutation loop into the inner commutation loop.

The statement of Theorem 2.1 is pictorial. It says that beginning with any L-shaped diagram as shown on the left of (2.4), one has the commuting diagram as shown on the right of (2.4). Moreover, this diagram is universal in the sense that all other possible commutation squares over the L-diagram can be recovered from it, as shown in (2.5). The completion into a square is essentially created by the top horizontal arrow $\tau_F(a)$, which is being claimed by Theorem 2.1 to be the image of a functor. The left slice objects in LeftSlice $(\mathcal{X})$ reside within the structure of $\mathcal{X}$. The claim thus implies that morphisms from a different category $\mathcal{Y}$ naturally creates functors between the various left-slices of $\mathcal{X}$.

There are numerous examples of the arrangement of Theorem 2.1 in mathematics. One of the most important among them is the following:

Example 2.10. *Recall the notations in Example 1.9. Any object $F \in [\iota \,;\, \iota]$ is a continuous map between topological spaces $F : X \to Y$. Then τ_F can be interpreted to be the induced map from the power set $[\iota \,;\, X]$ of X, into the power set $[\iota \,;\, Y]$ of Y. The correspondence between F and τ_F is itself functorial. Overall, we have a functor $\mathrm{Dyn} : [\iota \,;\, \iota] \to$ LeftSlice $\left(2^{\mathcal{U}}\right)$. Note that LeftSlice $\left(2^{\mathcal{U}}\right)$ is the category in which each object is a power set of some $S \subset \mathcal{U}$, and morphisms are inclusion preserving maps between these power sets.*

Example 2.10 can be generalized based on the following observation:

Lemma 2.1. *Given any complete category $\mathcal{C}$, the subcategory $\mathcal{C}_{mono}$ formed by monomorphisms is complete, and has the same initial object as $\mathcal{C}$.*

Lemma 2.1 and Theorem 2.1 have the following important consequence:

Corollary 2.1. *Suppose $\mathcal{C}$ is a complete, balanced category. Then:*

(i) The inclusion $\iota : \mathcal{C}_{mono} \to \mathcal{C}$ satisfies Assumptions 2.1, 2.2, 2.3 and 2.4.
(ii) There is a functor Dyn from $\mathcal{C}$ into the full subcategory of $[\mathrm{Cat}]$ spanned by the sub-object categories of $\mathcal{C}$.

Some examples of categories $\mathcal{C}$ which satisfy the conditions of Corollary 2.1 are [Set], [Topo] and [Group]. As a result we have the following instances of Corollary 2.1:

Example 2.11. *Continuing the discussion in Example 1.8, any linear map $A : U \to V$ between vector spaces induces a mapping between the subspaces of U and V respectively. This correspondence is a functor from [Vec] to the category of collections of vector subspaces.*

As a slight variation to Example 2.11, we have:

Example 2.12. *Take $\mathcal{X} = [Vec]_{mono}$ and $\mathcal{Y} = [Aff]$. The initial objects $0_\mathcal{X}$ and $0_\mathcal{Y}$ are both the zero-dimensional vector space. Note that there is an inclusion functor $\iota : \mathcal{X} \to \mathcal{Y}$ which maps $0_\mathcal{X}$ into $0_\mathcal{Y}$. Thus according to Theorem 2.1 any affine map is functorially related to a map between the collection of vector subspaces of the corresponding spaces.*

If $\mathcal{X}$ is a preorder, a left slice $[\mathcal{X} \,;\, a]$ can be interpreted as the set $\{a' \in \mathcal{X} \,:\, a' \leq a\}$. This is a sub-preorder of $\mathcal{X}$, and is called the *down-set* of a.

Corollary 2.2. *Let $\mathcal{X}$ be a complete preorder, and $\iota : \mathcal{X} \to [Group]$ a functor that maps $0_\mathcal{X}$ into the trivial group with one object. Let a, b be objects in $\mathcal{X}$ and $F : \iota(a) \to \iota(b)$ be a group homomorphism. Then F induces an order preserving map τ_F between the down-sets $[\mathcal{X} \,;\, a]$ and $[\mathcal{X} \,;\, b]$. Moreover this correspondence is functorial.*

An example of $\mathcal{X}, \iota$ from Corollary 2.2 is when $\mathcal{X}$ is a cellular complex, comprised of a collection of inclusions between face maps. Recall that singular homology is a functor

$$\text{Homology} : [\text{Topo}] \to [\text{Group}]\,,$$

mapping each topological space into its sequence of homology groups.

Example 2.13. *Let $\mathcal{X}$ be a simplicial complex of dimension n, comprised of simplices of various dimensions from 0 to n. Each simplex of dimension less than n is included as a face map of a simplex of a higher dimension. Thus $\mathcal{X}$ is a preorder with a finite number of objects and morphisms contained within $[\text{Topo}]$. Then for any two faces a, b of $\mathcal{X}$, the down-sets $[\mathcal{X} \,;\, a]$ and $[\mathcal{X} \,;\, b]$ are the sub-simplexes of these faces. Any group homomorphism*

$$F : Homology(a) \to Homology(b)$$

induces a unique simplicial map between these sub-complexes whose induced map between the homology groups is precisely F.

The last statement in the example above is supported by the commutation in (2.5).

Example 1.1 presented a functorial interpretation of dynamical systems. Category theoretic reformulations of dynamical systems have become of increasing interest due to the simplicity of presentation of many of the deeper results in dynamical systems theory [7, 11, 35, 44]. This is the advantage provided by the constructive/synthetic language of category theory, as opposed to the descriptive nature of set-theoretic language. The new challenge that emerges is that many basic definitions which are trivial in a set-theoretic presentation, becomes harder to present in a category theoretic setting. A prime example is the notion of an orbit. One can associate orbits to both topological, smooth, or measurable dynamical systems. However, the notion of orbit itself is as a minimal object in [Set] satisfying certain properties. In a category theoretic presentation, objects lose all their inner details and are presented as part of a larger collection. The focus shifts from the content of objects, to their relational and compositional structure. Thus orbits cannot be simply defined to be a union of successive images. A categorical definition of orbits is a major gap in the category theoretic reformulation of dynamics, and the functor Dyn discovered in Theorem 2.1 fills this gap.

Example 2.14. *Recall the notations from Examples 1.3, 1.4 and 2.10. Let $Topo(\mathcal{U})$ denote the subcategory of $[\text{Topo}]$ generated by all topological subspaces of $\mathcal{U}$. Recall from Example 1.1 that a functor $\Phi : \mathcal{T} \to Topo(\mathcal{U})$ is a topological dynamical system in the universe $\mathcal{U}$. Its time semigroup $\mathcal{T}$ is typically $\mathbb{N}_0$, $\mathbb{Z}$ or $\mathbb{R}$. Then one has the following composable sequence of functors*

$$\mathcal{T} \xrightarrow{\quad \Phi \quad} Topo(\mathcal{U}) \xrightarrow{\quad \text{Dyn} \quad} LeftSlice\left(2^{\mathcal{U}}\right)$$

The composition of these functors is a set-theoretic dynamical system with time semigroup $\mathcal{T}$. Thus the interpretation of a topological dynamical system as a set-theoretic dynamical system is also functorial.

Dynamical systems theory is about the study of orbits, and their asymptotic properties. There has been a recent interest in developing a categorical language for dynamical systems, and a major gap has been a functorial description of orbits. Example 2.14 along with Theorem 2.1 fills this gap. Given a dynamical system $\Phi^t : \Omega \to \Omega$ on a space Ω, the orbit of a subset $S \subseteq \Omega$ is the union $\cup_{t\in\mathcal{T}} \Phi^t(S)$. Alternatively it can be defined to be the smallest subset of Ω that contains all the images $\Phi^t(S)$. This definition is very suitable for a category theoretic presentation. Note that the composite functor $\tilde{\Phi}$: $\mathrm{Dyn} \circ \Phi$ from Example 2.14 leads to a functor $2^\Omega \times \mathcal{T} \to 2^\Omega$. Then the orbit of Φ is the functor shown in the diagram below:

$$\begin{array}{ccc} 2^\Omega \times \mathcal{T} & \xrightarrow{\tilde{\Phi}} & 2^\Omega \\ {\scriptstyle \mathrm{proj}_1}\downarrow & \Uparrow \searrow & \\ 2^\Omega & \xrightarrow{\text{Orbit}} & 2^\Omega \end{array} \tag{2.6}$$

The bottom horizontal arrow is created by a construction called a right-Kan extension. Kan extensions are a purely diagrammatic/categorical notion, and is elaborated in Section 7. Example 2.14 and Diagram (2.6) is a succinct but precise way of stating the following facts:

Corollary 2.3. *Every topological dynamical system in the universe $\mathcal{U}$ is a functor $\Phi : \mathcal{T} \to Topo(\mathcal{U})$. It leads to the following notions :*

(i) This functor combines with Dyn *from Theorem 2.1 to get a dynamical system $\mathcal{T} \to LeftSlice\left(2^{\mathcal{U}}\right)$.*
(ii) This leads to a functor $\tilde{\Phi} : 2^\Omega \times \mathcal{T} \to 2^\Omega$.
(iii) The existence of the orbit functor follows from the existence of a right Kan extension, as shown in (2.6).
(iv) The minimality of the orbit follows from the universal property of a right Kan extension.

These examples and Corollaries 2.1–2.3 highlight the prevalence of the arrangement described in Theorem 2.1. This ends the presentation of some examples of manifestations of slice categories and applications of Theorem 2.1. Slice categories have had recent applications [15] in optimization and approximation theory [3, 10, 16]. The possibility of applications of Theorem 2.1 to this emerging field is an interesting prospect.

Outline. Theorem 2.1 has several layers to it. Firstly, it associates a functor between left slice categories to every morphism of the form $F : \iota\Omega \to \iota\Omega'$. Secondly (2.3) states that this correspondence itself is functorial, which means that composition of morphisms become composition of functors. Thirdly, the correspondence is defined by the unique and minimal commutation square (2.4) that it creates. We shall unravel the categorical principles that contribute to each claim, over the course of the next sections. The main ingredient of Theorem 2.1 is Assumption 2.1. Assumption 2.1 is slightly generalized into Assumption 3.5 next in following Section 3. This generalization allows us to formulate two Theorems 3.2 and 3.3 which cover part of the claims of Theorem 2.1. The compositionality is proved next in Section 4 via Theorem 4.5. Theorem 2.1 is proved in Section 4, as a consequence of Theorems 3.2, 3.3 and 4.5. See Figure 1 for an outline of the proof of Theorem 2.1, and how the other main results fit into the proof. We take a deep look at comma and arrow categories in Sections 5 and 6. The uniqueness of the commutation in (2.4) is next established via a special categorical construction in Section 7. Finally Theorems 3.2 and 3.3 are proved in Section 8.

3. The Induced Functor Between Slices

Theorem 2.1 was about the comma category $[\iota\ ;\ \iota]$ which is a special case of (1.1). We now make an assumption on (1.1), which turns out to be a generalization of Assumption 2.1.

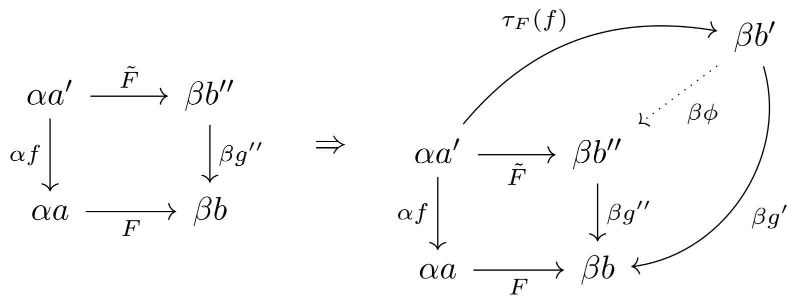
Assumption 3.5. *The category $\mathcal{A}$ and $\mathcal{B}$ from (1.1) are complete, the functor β is continuous. Categories $\mathcal{A}$ and $\mathcal{C}$ have initial elements $0_{\mathcal{A}}$ and $0_{\mathcal{C}}$ respectively, and $\alpha(0_{\mathcal{A}}) = 0_{\mathcal{C}}$.*

Our first result arises from the simple situation when two objects a, b are picked from $\mathcal{A}, \mathcal{B}$ in (1.1), mapped into $\mathcal{C}$, and bound by a morphism F in $\mathcal{C}$. The objects a, b have their own left-slice categories in $\mathcal{A}, \mathcal{B}$, which are independent of each other as well as $\mathcal{C}$. We shall see how the morphism ϕ induces a functor between these two categories.

Theorem 3.2 (Induced functor). *Assume the arrangement of (1.1), and let Assumption 3.5 hold. Fix an object $\alpha a \xrightarrow{F} \beta b$ of the comma category $[\alpha\ ;\ \beta]$. Then there is a functor $\tau_F : [\mathcal{A}\ ;\ a] \to [\alpha\ ;\ \beta]$ such that for any object $a' \xrightarrow{f} a$ in $[\mathcal{A}\ ;\ a]$, there is an object $b' \xrightarrow{g} b$ in $[\mathcal{B}\ ;\ b]$, such that the following commutation holds*

$$\begin{array}{ccc} a' \\ {\scriptstyle f}\downarrow \\ b \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \alpha a' & \xrightarrow{\tau_F(f)} & \beta b' \\ {\scriptstyle \alpha f}\downarrow & & \downarrow{\scriptstyle \beta g} \\ \alpha a & \xrightarrow[F]{} & \beta b \end{array} \tag{3.7}$$

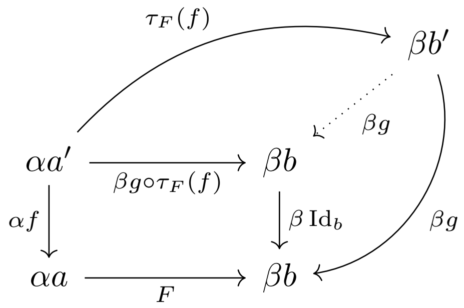
Moreover, $\tau_F(f)$ is minimal in the sense for any other object $b'' \xrightarrow{g} b$, if the commutation shown below on the left holds :



then there is a unique morphism $b' \xrightarrow{\phi} b''$ such that the commutation on the right holds.

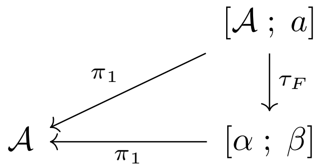
Thus the correspondence τ_F associates to every object f in the left slice $[\mathcal{A}\ ;\ a]$ an object $\tau_F(f)$ in the comma category $[\alpha\ ;\ \beta]$. This object $\tau_F(f)$ itself is an morphism in $\mathcal{C}$ and creates a commutation square involving f and F.

Remark 3.1. *The minimality so described is hardly surprising, since whenever a commutation such as (3.7) holds, the following commutation also holds*



This diagram is a special case of the second claim of Theorem 3.2, with $\tilde{F} = \beta g \circ \tau_F(f)$ and $\phi = g'$.

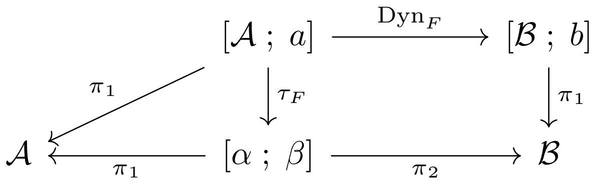
Remark 3.2. *One of the consequences of Theorem 3.2 and the commutation in (3.7) is*



This means that the domain of the morphism $\tau_F(f)$ is the same as the domain of the morphism f.

Remark 3.3. *Any object $a' \xrightarrow{f} a$ in $[\mathcal{A} ; a]$ is sent by π_1 into a', whereas it is sent by τ_F into $\tau_F(f)$, which is then sent by π_1 into a'. This commutation be extended as follows :*

(3.8)



The diagram presents a new functor Dyn_F between the slice categories associated to the terminal points of the comma object F.

Recall the category $\left[\pi_2^{[\alpha ; \beta]} ; b\right]$ (1.2) presented in Example 1.6. The compound objects in $\left[\pi_2^{[\alpha ; \beta]} ; b\right]$ lead to a projection functor

$$\left[\pi_2^{[\alpha ; \beta]} ; b\right] \xrightarrow[\text{Restrict}]{} [\mathcal{B} ; b]$$

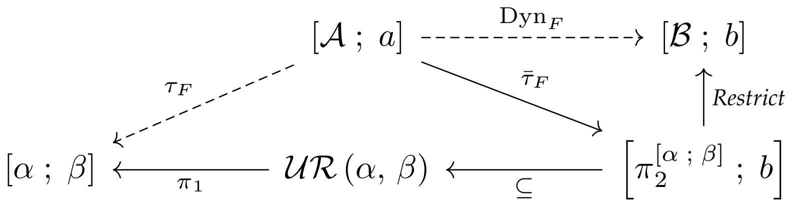
Both Theorem 3.2 and (3.8) are consequences of the following more general result:

Theorem 3.3. *Under the same assumptions as Theorem 3.2 and the category in (1.2) there is a functor*

$$\bar{\tau}_F : [\mathcal{A} ; a] \to \left[\pi_2^{[\alpha ; \beta]} ; b\right]$$

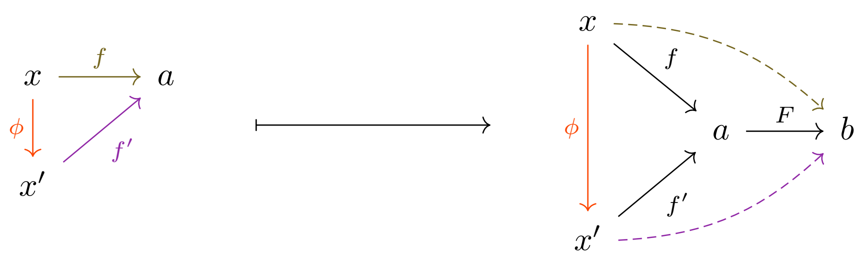
such that the functors τ_F and Dyn_F are created via composition:

(3.9)



Theorem 3.3 jointly implies the statements of Theorem 3.2 and (3.8). Theorem 3.3 is proved in Section 8. See Figure 1 for a summary of the various results and their logical connections.

Remark 3.4. *When $\mathcal{A} = \mathcal{B} = \mathcal{C}$ in (1.1), and $\alpha = \beta = \mathrm{Id}_{\mathcal{A}}$, then $[\alpha ; \beta]$ is just the arrow category Arrow $[\mathcal{A}]$. Any object $a \xrightarrow{F} b$ in this category induces a functor between the slice categories :*



The yellow and blue arrows represent different objects in the respective slice categories, and the red arrows represent morphisms between these objects. The diagram on the right is obtained from the left by simply composing with F. This functorial relation coincides with τ_F.

Remark 3.5. *While τ_F has a simple interpretation when all the functors in (1.1) are identities, determining an induced functor in the more general setting is not trivial. One notable approach relies on the existence of special factorization systems [1, 18]. This approach has been extended to an axiomatic study of topology [6, 13, 14, 42].*

In the next section, we look more closely at the correspondence between F and τ_F.

4. Algebra of Induced Functors

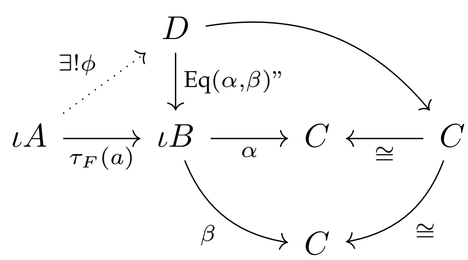
Theorem 3.2 presents how an object in a comma category induces a functor between the left-slices associated to the two endpoints of the object. The functor is realized through morphisms in $\mathcal{C}$ binding an object in a left slice object in $\mathcal{A}$, to a left slice object in $\mathcal{B}$. In this section we shift our attention back to the case when $\mathcal{A} = \mathcal{B}$. In that case all the left slices involved are within the same category. Our first important realization will be that the induced morphisms $\tau_F(f)$ are surjective.

Theorem 4.4. *Assumptions 2.1 and 2.2 hold. Then the induced morphisms $\tau_F(f)$ from Theorem 3.2 are surjective.*

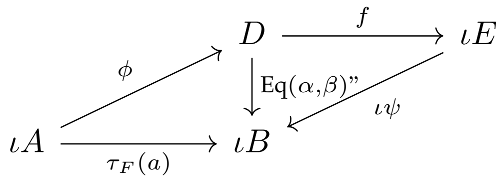
The proof requires the following lemma :

Lemma 4.2. *In any category, an equalizer is injective.*

Proof of Theorem 4.4. To prove surjectivity we need to show that for any pair of morphisms $\alpha, \beta : \iota B \to C$, if $\alpha \circ \tau_F f = \beta \circ \tau_F f$ then $\alpha = \beta$. Since $\mathcal{Y}$ is complete it has equalizers. Consider the following diagram in which the equalizer of α, β has been shown.



Since the equalizer is by definition, the universal morphism γ such that $\beta\gamma = \alpha\gamma$, the morphism $\tau_F(f)$ must factor through the equalizer via the morphism ϕ as shown. Now by Lemma 4.2, the morphism Eq(α, β) is injective. By Assumption 2.2, Eq(α, β) factorizes as shown below.



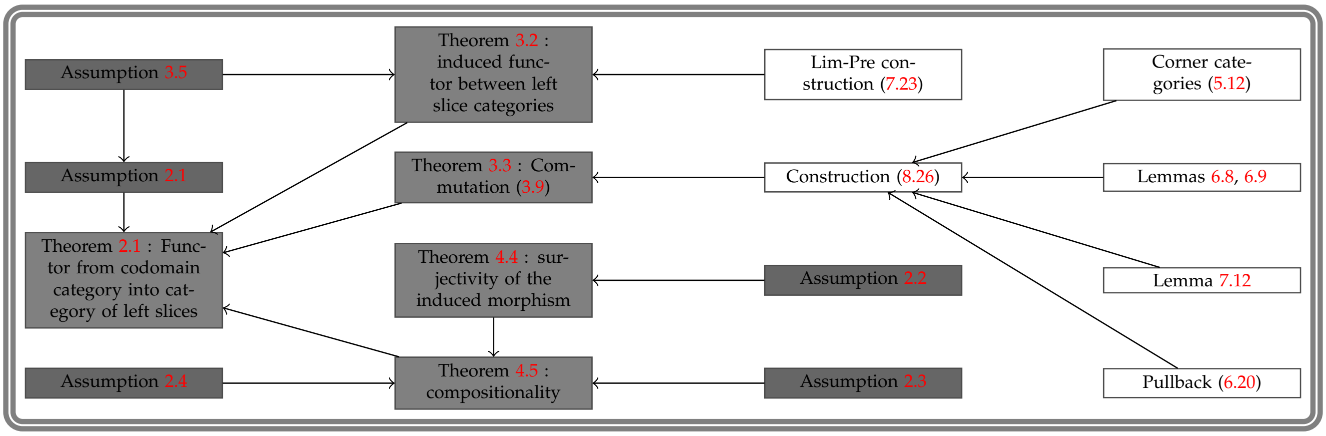
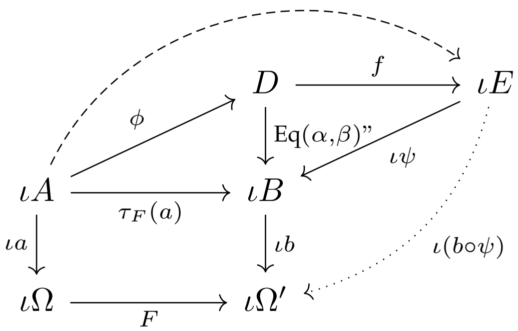


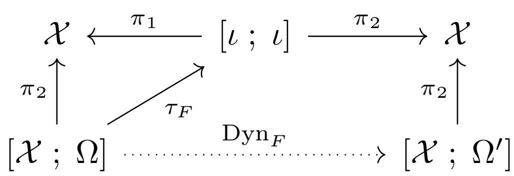
FIGURE 1. Outline of the results, assumptions, and their logical dependence. The white boxes display the various assumptions, and grey boxes display the main results.

This commutation diagram can be joined with the definition of $\tau_F(f)$ to get



By the universality of $\tau_F(f)$, the morphism ψ must be an isomorphism. This would mean that $\text{Eq}(\alpha, \beta)$ is an isomorphism too. This in turn implies that $\alpha = \beta$, which was our goal. This completes the proof of Theorem 4.4. □

We have been examining the particular instance of (1.1) when $\mathcal{A} = \mathcal{B} = \mathcal{X}$, and $\mathcal{C} = \mathcal{Y}$, and both functors α, β are $\iota : \mathcal{X} \to \mathcal{Y}$. In that case, the diagram (3.8) becomes



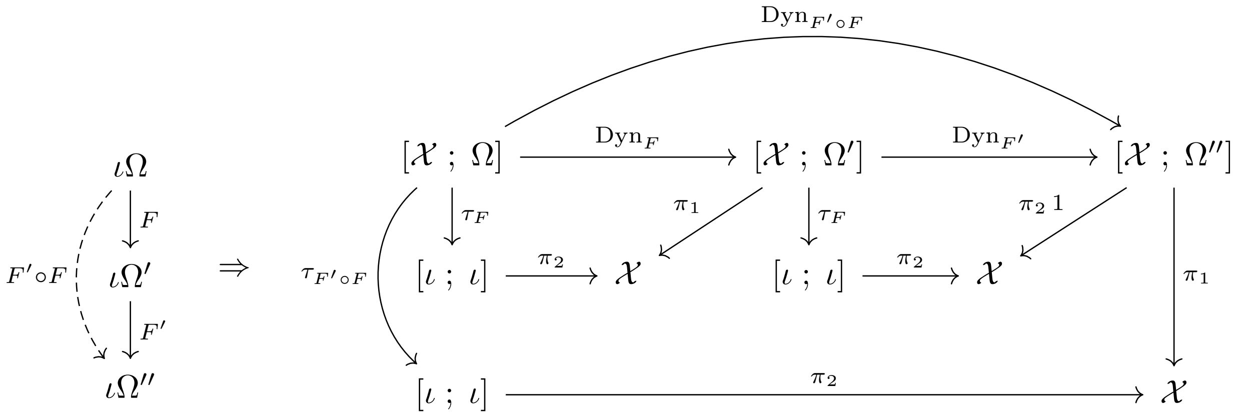
One of the consequences of equating $\mathcal{A}$ and $\mathcal{B}$ is that the functor described by Theorems 3.2 and 3.3 are between slices of the same category. Our goal is to investigate the composability of the horizontal arrows in the bottom row. To gain a precise footing, we assume

Theorem 4.5 (Compositionality of induced functors)**.** *Suppose Assumptions 2.1, 2.3 and 2.4 hold. Then there is a functor*

$$\iota(\mathcal{X}) \xrightarrow{\mathcal{T}} \mathit{LeftSlice}\,(\mathcal{X})\,,$$

which maps an morphism $\iota\Omega \xrightarrow{F} \iota\Omega'$ *into* $[\mathcal{X}\ ;\ \Omega] \xrightarrow{\tau_F} [\mathcal{X}\ ;\ \Omega']$.

Remark 4.6. *Theorem 4.5 essentially says that the correspondence of* Dyn_F *with* F *preserves composition. This leads to the following diagram :*



The upper commuting loop is the statement of Theorem 4.5. The outer commutating loop, along with the two smaller loops are a consequence of (3.8).

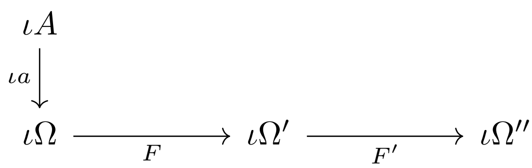
Lemma 4.3. *In any category* $\mathcal{C}$*, if* f, g *are two composable morphisms such that* $f, g \circ f$ *are surjective, then* g *is also surjective.*

Proof. We need to shown that for any morphisms α, β such that $\alpha g = \beta g$, α must equal β. Now note that

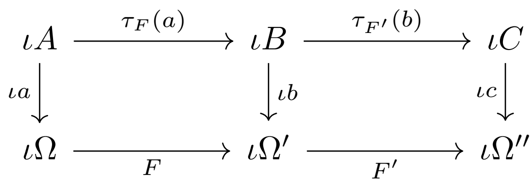
$$\alpha (gf) = (\alpha g) f = (\beta g) f = \beta (gf) .$$

Since gf is surjective, we must have $\alpha = \beta$, proving the claim. □

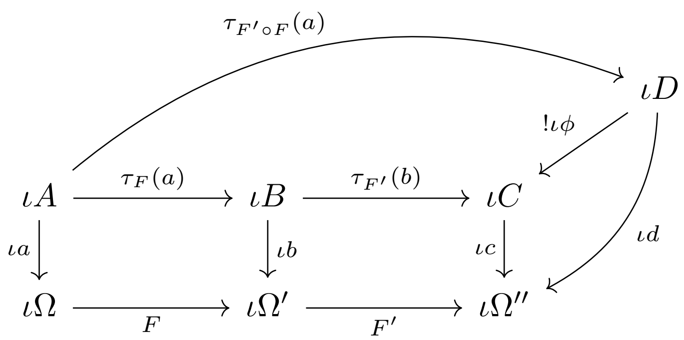
Proof of Theorem 4.5. We start with the following setup:



This contains an object $A \in [\mathcal{X} ; \Omega]$, and two composable morphisms $F, F' \in [\iota ; \iota]$. We can apply the functors τ_F and $\tau_{F'}$ in succession to get



To prove Theorem 4.5, it has to be shown that the composition along the morphisms in the upper row equals $\tau_{F' \circ F}$. The object $\tau_{F' \circ F}(a)$ itself can be drawn as shown below:



The connecting morphism $\phi : D \to C$ exists by the minimality of $\tau_{F' \circ F}(a)$. The upper commuting loop can be expressed as

$$\tau_{F'}(b) \circ \tau_F(a) = \iota\phi \circ \tau_{F' \circ F}(a).$$

By Theorem 4.4, all the three morphisms $\tau_F(a)$, $\tau_{F'}(b)$ and $\tau_{F' \circ F}(a)$ are surjective. Thus by Lemma 4.3, $\iota\phi$ must be surjective too. By Assumption 2.3, $\iota\phi$ is also injective. Thus by Assumption 2.4, $\iota\phi$ is an isomorphism. This implies that $\tau_{F'}(b) \circ \tau_F(a)$ and $\tau_{F' \circ F}(a)$ are equal up to isomorphism. This completes the proof of Theorem 4.5. □

This completes the statement of our main results. The proof of Theorem 2.1 can now be completed.

Proof of Theorem 2.1. Note that Assumption 2.1 in Theorem 2.1 is a special case of Assumption 3.5. As a result we can build the induced functors $\bar{\tau}_F$ and τ_F from Theorems 3.2 and 3.3 respectively. Since ι is assumed to be injective on objects, the objects of $\iota(\mathcal{X})$ are in bijection with the object of $\mathcal{X}$. Thus each object in $\iota(\mathcal{X})$ corresponds to a unique image $\iota\Omega$, for some $\Omega \in ob(\mathcal{X})$. The functoriality now follows from Theorem 4.5. This completes the proof of Theorem 2.1. □

Theorems 3.2 and 3.3 remain to be proven. The proofs require building a deeper insight into the inter-relations between comma, arrow, and slice categories. We build this insight over the course of three sections 5, 6 and 7. In the next section, we complete the proof of Theorem 2.1.

5. Comma and Arrow Categories

In this section, we take a deeper look into the commutation squares in comma categories. We assume throughout this section the general arrangement of (1.1), and the resultant comma category $[\alpha\ ;\ \beta]$. We have seen how an arrow category is a special instance of a comma category. In this section we are interested in the arrow category of the comma category: Arrow $[[\alpha\ ;\ \beta]]$. The objects of this category are commutations of the form

$$\begin{array}{ccc} \alpha a & \xrightarrow{\phi} & \beta b \\ {\scriptstyle \alpha f}\downarrow & & \downarrow{\scriptstyle \beta g} \\ \alpha a' & \xrightarrow{\phi'} & \beta b' \end{array}, \quad a, a' \in ob(\mathcal{A}),\ b, b' \in ob(\mathcal{B}). \tag{5.10}$$

The vertical morphisms lie in $\mathcal{C}$ while the horizontal morphisms are the images of morphisms in $\mathcal{A}$ and $\mathcal{B}$. The key to proving our results is the realization that the different pieces of (5.10) are also comma categories of various kinds. Let us consider the lower left and upper right corners of (5.10):

$$\begin{array}{ccc} \alpha a & & \\ {\scriptstyle \alpha f}\downarrow & & \\ \alpha a' & \xrightarrow{\phi'} & \beta b' \end{array}, \qquad \begin{array}{ccc} \alpha a & \xrightarrow{\phi} & \beta b \\ & & \downarrow{\scriptstyle \beta g} \\ & & \beta b' \end{array}$$

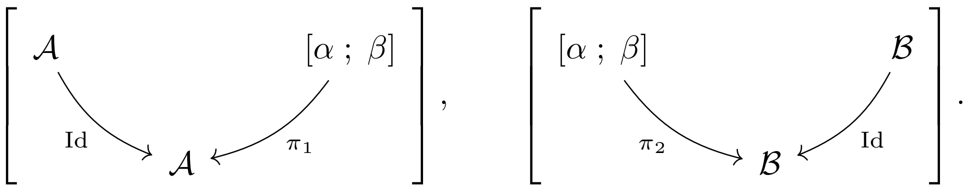
This first diagram is an object of the comma category

$$\mathcal{DL}(\alpha, \beta) := \left[\mathrm{Id}_{\mathcal{A}}\ ;\ \pi_1^{[\alpha\ ;\ \beta]}\right]$$

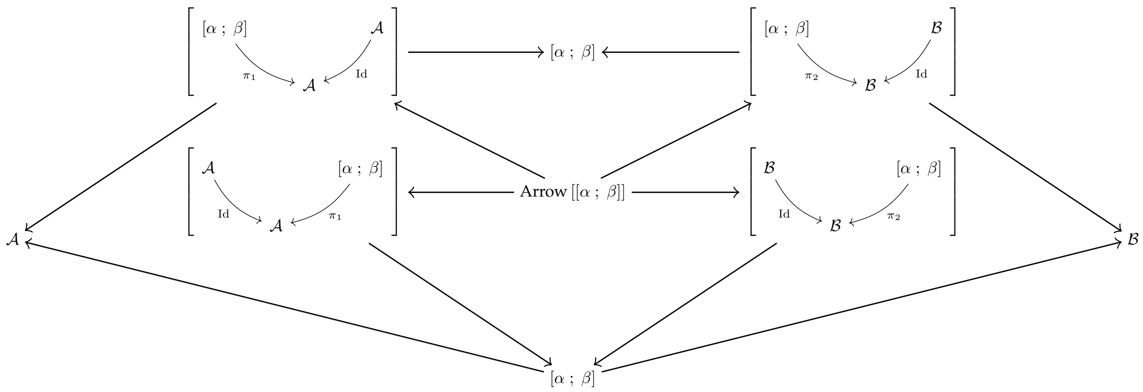
The initials DL indicates "down-left", the position of an object of this category relative to an object of Arrow $[[\alpha\ ;\ \beta]]$ (5.10). Similarly, the upper-right corner is an object of the category

$$\mathcal{UR}(\alpha, \beta) := \left[\pi_2^{[\alpha\ ;\ \beta]}\ ;\ \mathrm{Id}_{\mathcal{B}}\right].$$

Both the categories $\mathcal{DL}(\alpha, \beta)$ and $\mathcal{UR}(\alpha, \beta)$ can be written more expressively as

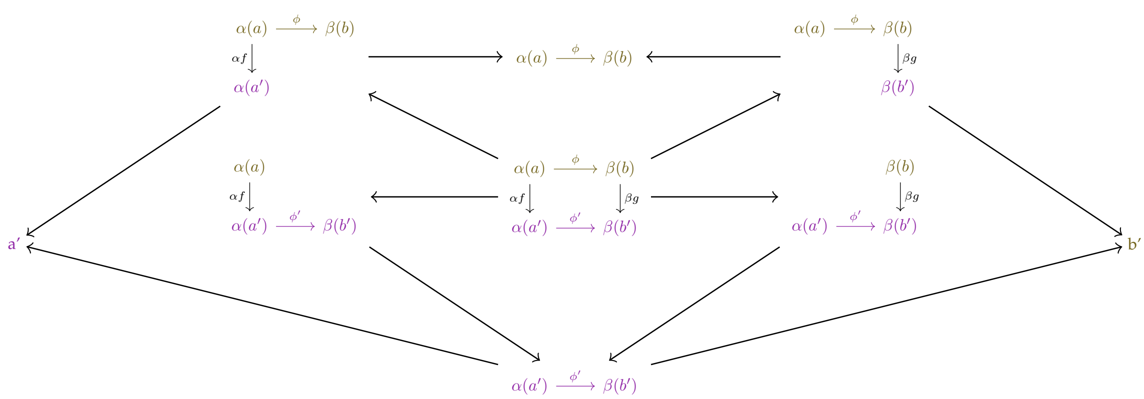


One can proceed similarly to describe each of the other two corners of (5.10) as categories. This leads to the following layout of the arrow category and its corner categories:

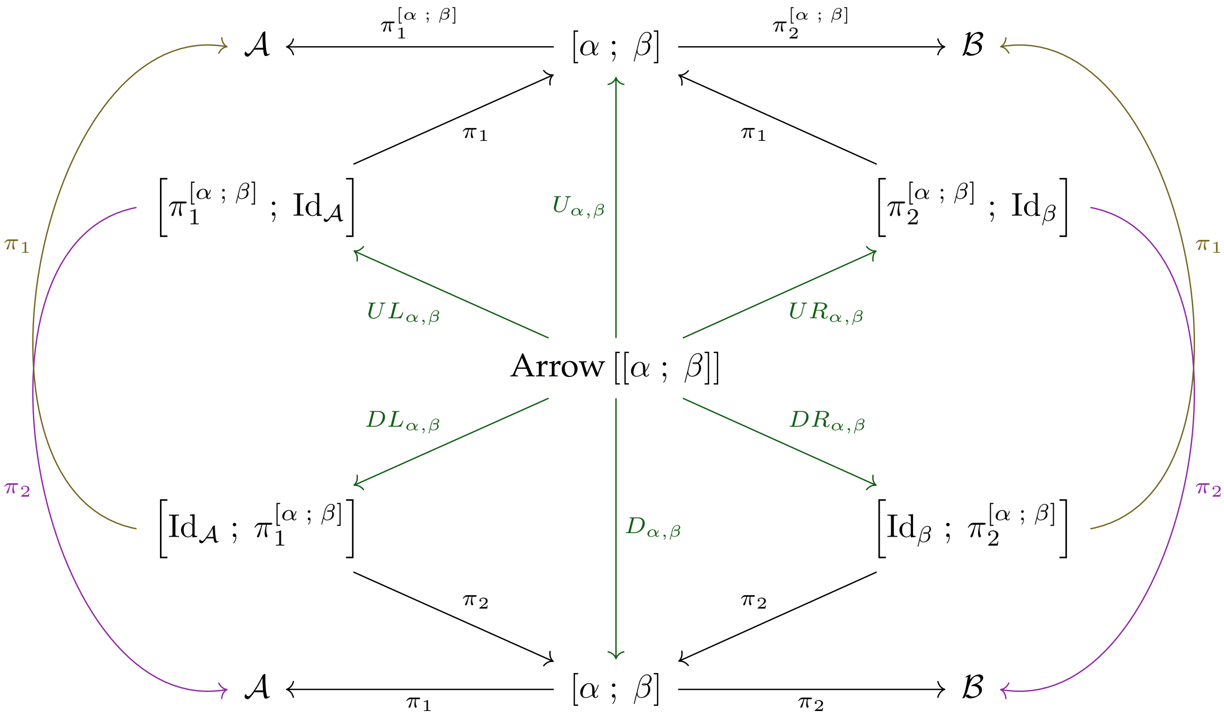


(5.11)

The arrows connecting the comma categories are functors, created from the forgetful functors associated with the arrow category. The commutative diagram in (5.10) is an object in the central category of this diagram. The image of (5.10) under the various functors of (5.11) are displayed below:



The corner categories, which have been presented pictorially, can be written more succinctly as comma categories:
(5.12)



The commutations in (5.12) will be one of the most important theoretical tools in our proofs. The green arrows labeled U, D, UR, UL, DL, DR respectively represent the upper, lower, upper-right, upper-left, lower left and lower-right corners of the object in (5.10). The categories $\mathcal{A}, \mathcal{B}$ also find their place in this diagram as the smallest ingredients of the arrow comma category Arrow $[[\alpha\ ;\ \beta]]$. We next shift our attention to transformations from between comma categories.
Comma transformations. Consider a commuting diagram

(5.13)
$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xleftarrow{G} & \mathcal{C} \\ \downarrow I & & J \downarrow & & \downarrow K \\ \mathcal{A}' & \xrightarrow[F']{} & \mathcal{B}' & \xleftarrow[G']{} & \mathcal{C}' \end{array}$$

in which functors I, J, K connect two comma arrangements F, G and $F'G'$. Then, we have

Proposition 5.1 (Functors between comma categories)**.** *Consider the arrangement of categories $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ and functors F, G, H, I, J from (5.13). Then there is an induced functor between comma categories*

(5.14)
$$\Psi_{I,J,K} : [F\ ;\ G] \to [F'\ ;\ G'] \,,$$

where the map between objects and morphisms is as follows:

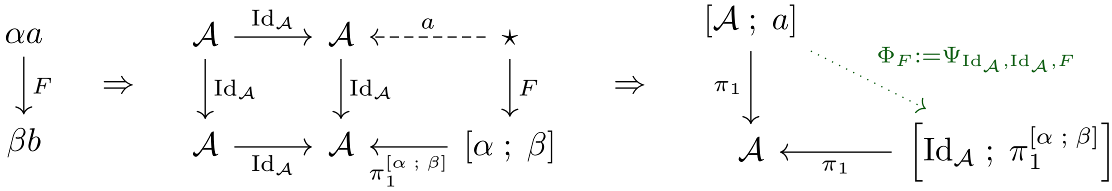
$$\begin{array}{c} (a,\phi,c) \\ \downarrow f,g \\ (a',\phi',c') \end{array} = \begin{array}{ccc} Fa & \xrightarrow{\phi} & Gc \\ Ff\downarrow & & \downarrow Gg \\ Fa' & \xrightarrow[\phi']{} & Gc' \end{array} \mapsto \begin{array}{ccccccc} F'Ia & \xrightarrow{=} & JFa & \xrightarrow{J\phi} & JGc & \xrightarrow{=} & IKc \\ & & {\scriptstyle JFf=F'If}\downarrow & & \downarrow{\scriptstyle JGg=G'Kf} & & \\ F'Ia' & \xrightarrow[=]{} & JFa' & \xrightarrow[J\phi']{} & JGc' & \xrightarrow[=]{} & IKc \end{array} = \begin{array}{c} (Ia, J\phi, Kc) \\ \downarrow If,Kg \\ (Ia', J\phi', Kc') \end{array}$$

Moreover, the following commutation holds with the marginal functors:

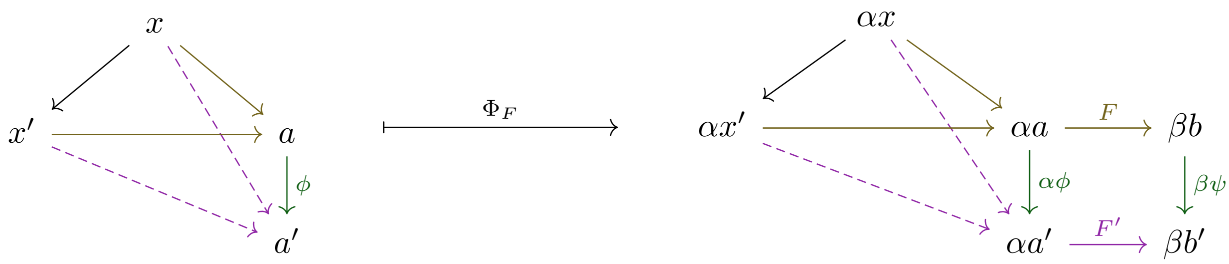
$$\begin{array}{ccccc} \mathcal{A} & \xleftarrow{\pi_1} & [F\ ;\ G] & \xrightarrow{\pi_2} & \mathcal{A} \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle \Psi_{I,J,K}} & & \downarrow{\scriptstyle K} \\ \mathcal{A}' & \xleftarrow[\pi_1]{} & [F'\ ;\ G'] & \xrightarrow[\pi_2]{} & \mathcal{C}' \end{array} \tag{5.15}$$

The proof of Proposition 5.1 will be omitted. A particular instance of (5.13) is shown in the center below,

(5.16)

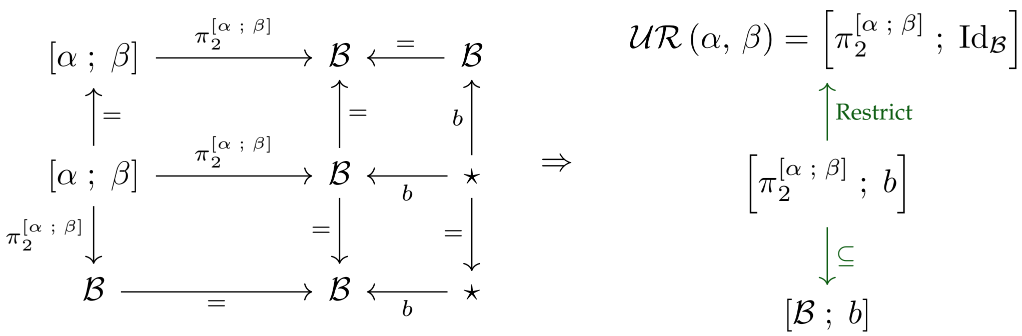


The leftmost figure in (5.16) is an object F in $[\alpha\ ;\ \beta]$. The middle diagram presents a simple commutation in which this object is re-interpreted as a functor. Finally, the leftmost figure presents an application of Proposition 5.1 to this commutation. The dashed arrow in the above diagram indicate that it is are defined via composition. Proposition 5.1 applied to the commutative diagram in the center leads to the functor $\Psi_{\mathrm{Id}_\mathcal{A},\mathrm{Id}_\mathcal{A},F}$ shown on the right. Composition with this functor leads to the functor Φ_F shown in green on the right. The top right commutation is a consequence of (5.15). The action of Φ_F can be explained simply as



The yellow and blue sub-diagrams on the actions of Φ_F, $\Phi_{F'}$ for different objects $F, F' \in [\alpha\ ;\ \beta]$. Two other examples of Proposition 5.1 can be found in the diagram on the left below:

(5.17)

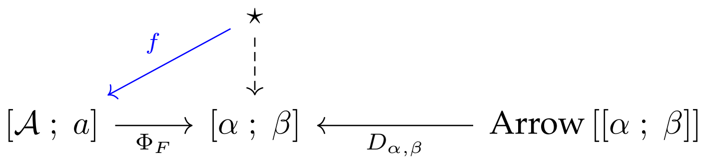


gain Proposition 5.1 yields the trivial transformations between three categories, as indicated on the right above. We next use this functorial relation Φ_F between categories to study more complicated arrangements.

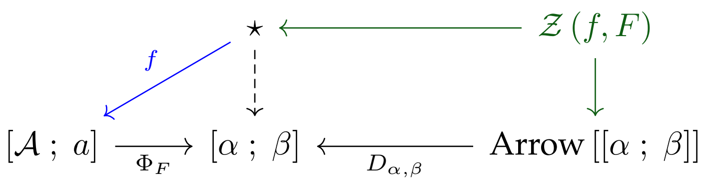
The dynamics map. The language of comma categories enable complex arrangements of spaces and transformations to be concisely depicted by the comma notation. Once a comma category is built, one has two forgetful or projection functors from the comma category. One can then use these functors to build comma categories of a higher level of complexity. We have already seen several examples of this constructive procedure over the course of the diagrams (5.11) and (5.12). Another important instance arises when the functor Φ_F from (5.16) is combined with a part of (5.12) to give

$$[\mathcal{A} \,;\, a] \xrightarrow[\Phi_F]{} [\alpha \,;\, \beta] \xleftarrow[D_{\alpha,\beta}]{} \text{Arrow}\,[[\alpha \,;\, \beta]]$$

Now, consider any element $a' \xrightarrow{f} a$ from the slice category $[\mathcal{A} \,;\, a]$, which we just represent as f. This can be represented as a functor from the one point category $\star$. This leads to



This arrangement is also a diagram in [Cat], the category of small categories. As a result we can construct its pull back, which is shown below in green:

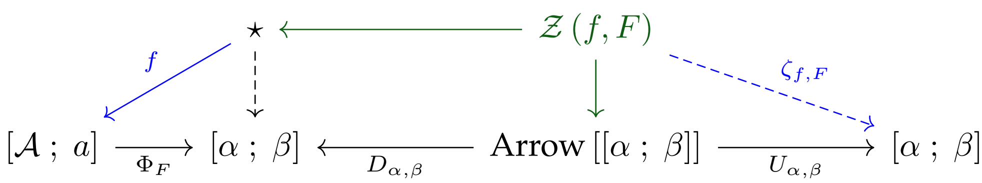


The category $\mathcal{Z}\,(f, F)$ is the full subcategory of Arrow $[[\alpha \,;\, \beta]]$ whose objects are pairs (F', g) such that

$$\begin{array}{ccc} \alpha a' & \xrightarrow{F'} & \beta b' \\ {\scriptstyle \beta f}\downarrow & & \downarrow{\scriptstyle \beta g} \\ \alpha a & \xrightarrow[F]{} & \beta b \end{array}$$

The upper horizontal arrow can be recovered via the functor $U_{\alpha,\beta}$. This functor can be added to our previous arrangement to get:

(5.18)



The functor $\zeta_{f,F}$ which is created via composition, directly yields the functor we are looking for:

$$\tau_F(f) := \lim \zeta_{f,F}. \tag{5.19}$$

Equation (5.19) is an alternative and easy route to construct the functor τ_F from Theorem 3.2. However, this construction does not reveal the functorial nature of the correspondence between f and $\lim \zeta_{f,F}$. In the following section, we describe a different route to establishing functoriality.

6. Adjointness in Commas

In this section the categorical properties of various comma categories and forgetful functors will be examined. The first is a classic result from category theory:

Lemma 6.4. [41, Theorem 3] *Let $\alpha : \mathcal{A} \to \mathcal{C}$ and $\beta : \mathcal{B} \to \mathcal{C}$ be functors with α (finitely) continuous. If $\mathcal{A}$ and $\mathcal{B}$ are (finitely) complete, then so is the comma category $[\alpha\ ;\ \beta]$.*

One immediate consequence of Lemma 6.4 is:

Lemma 6.5. *The comma category $[\alpha\ ;\ \beta]$ is complete.*

We also have:

Lemma 6.6. *The upper right category $\mathcal{UR}(\alpha, \beta)$ is complete.*

Lemma 6.6 follows directly from the construction of $\mathcal{UR}(\alpha, \beta)$ as a comma category in Section 5, the continuity of the identity functor, and Lemma 6.4. The following basic lemma is an useful tool in establishing the existence of right adjoints.

Lemma 6.7 (Right inverse as right adjoint)**.** *Suppose $F : \mathcal{P} \to \mathcal{Q}$ and $G : \mathcal{Q} \to \mathcal{P}$ are two functors such that $FG = \mathrm{Id}_{\mathcal{Q}}$ and $\mathrm{Id}_{\mathcal{P}} \Rightarrow GF$. Then F, G are left and right adjoints of each other.*

These insights lead to two technical results. The first one is:

Lemma 6.8. *Consider an arrangement of categories and functors $\mathcal{P} \xrightarrow{P} \mathcal{R} \xleftarrow{Q} \mathcal{Q}$. If $\mathcal{P}$ and $\mathcal{R}$ have initial elements $0_{\mathcal{P}}$ and $1_{\mathcal{R}}$ respectively, and $P(0_{\mathcal{P}}) = 0_{\mathcal{R}}$, then the functor $\pi_2 : [P\ ;\ Q]$ has a left adjoint given by*

$$(\pi_2)^{(L)} : \mathcal{Q} \to [P\ ;\ Q], \quad q \mapsto \begin{array}{c} P0_{\mathcal{P}} = 0_{\mathcal{R}} \\ \big\downarrow {\scriptstyle !_{Qq}} \\ Qq \end{array}$$

In fact, $(\pi_1)^{(L)}$ is a right inverse of π_1, i.e., $\pi_2 \circ (\pi_2)^{(L)} = \mathrm{Id}_{\mathcal{Q}}$.

Proof. Since $(\pi_1)^{(R)}$ is a right inverse of π_1 by Lemma 6.7, it only remains to be shown that there is a natural transformation $(\pi_2)^{(L)} \circ \pi_2 \Rightarrow \mathrm{Id}_{[P\ ;\ Q]}$, called the *counit*. The diagram on the left below traces the action of this composite functor on an object F (blue) of $[P\ ;\ Q]$ into an object (green) of $[P\ ;\ Q]$:

$$\begin{array}{c} Pp \\ {\scriptstyle F}\big\downarrow \\ Qq \end{array} \xmapsto{\quad \pi_2 \quad} q \xmapsto{\quad (\pi_2)^{(L)} \quad} \begin{array}{c} P0_{\mathcal{P}} = 0_{\mathcal{R}} \\ \big\downarrow {\scriptstyle !_{Qq}} \\ Qq \end{array} \quad ; \quad \begin{array}{ccc} P0_{\mathcal{P}} & \xrightarrow{!_{Qq}} & Qq \\ {\scriptstyle P(!_p)}\big\downarrow & & \big\downarrow {\scriptstyle Q\,\mathrm{Id}_q} \\ Pp & \xrightarrow{F} & Qq \end{array}$$

The diagram on the left demonstrates a commutation arising out of the initial element preserving property. The descending yellow morphisms together constitute the connecting morphism of the counit transformation we seek. This completes the proof. □

The second technical results is:

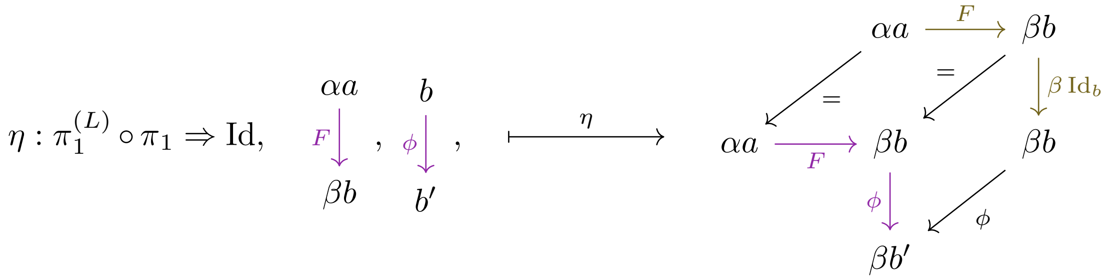
Lemma 6.9. *In the arrangement of (1.1), the forgetful functor*

$$\pi_1 : \mathcal{UR}(\alpha, \beta) := \left[\pi_2^{[\alpha\ ;\ \beta]}\ ;\ \mathrm{Id}_{\mathcal{B}}\right] \to [\alpha\ ;\ \beta],$$

has a left adjoint

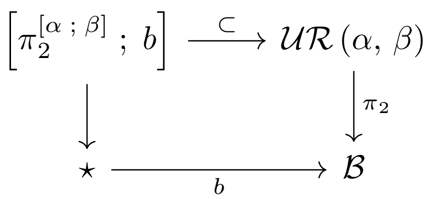
$$\pi_1^{(L)} : [\alpha\ ;\ \beta] \to \mathcal{UR}(\alpha, \beta), \quad \begin{array}{c} \alpha a \\ {\scriptstyle F}\big\downarrow \\ \beta b \end{array} \longmapsto \begin{array}{ccc} \alpha a & \xrightarrow[F]{} & \beta b \\ & & \big\downarrow {\scriptstyle \beta\,\mathrm{Id}_b} \\ & & \beta b \end{array}.$$

Proof. The functor $\pi_1^{(L)}$ is clearly a right inverse of π_1. Thus again by Lemma 6.7 , it is enough to show the existence of a natural transformation as shown on the left below:



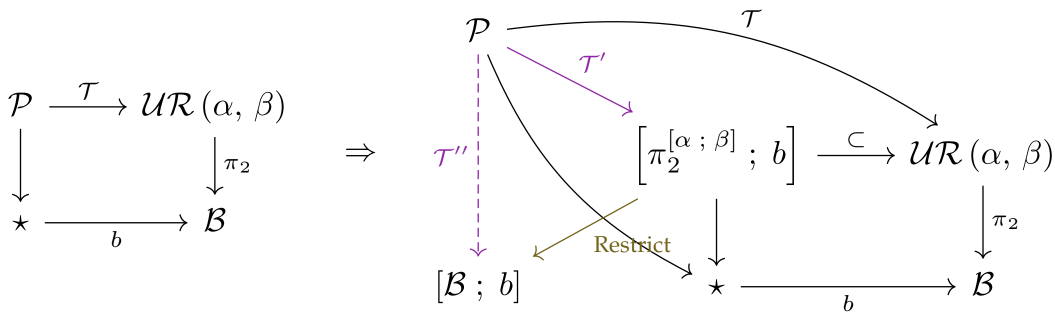
The connecting morphisms of this natural transformation is shown above on the right. □

The reader is once again referred to the outline presented in Figure 1. The main results that remain to be proven are Theorems 3.2 and 3.3. They are proved by a final, complicated diagram presented later in (8.26). Lemmas 6.8 and 6.9 help in the construction of this diagram. We end this section with one final observation about the upper-right corner category. Recall the inclusion functor from (5.17). It leads to a pull-back square:



Now consider any functor $\mathcal{T} : \mathcal{P} \to \mathcal{UR}(\alpha, \beta)$ such that $\pi_2 \circ \mathcal{T} \equiv b$, for some object b of $\mathcal{B}$. Then the commutation on the left below is satisfied:

(6.20)



This commutation must factor through the pullback square as shown by the blue arrow in the diagram in the middle. Thus given any functor $\mathcal{T}$ as above, (6.20) says that $\mathcal{T}$ factors through a map $\mathcal{T}'$ mapping into $\left[\pi_2^{[\alpha\,;\,\beta]}\,;\, b\right]$, a subcategory of $\mathcal{UR}(\alpha, \beta)$. Since the category $\left[\pi_2^{[\alpha\,;\,\beta]}\,;\, b\right]$ can be further restricted to the slice $[\mathcal{B}\,;\, b]$, $\mathcal{T}'$ extends to a functor mapping into $[\mathcal{B}\,;\, b]$.

7. The Lim-Pre Construction

At this stage we can start constructing the functors declared in Theorem 3.2 and (3.9). These will be constructed by taking various limits. For that purpose we need to established the completeness and continuity of various categories and functors involved. We start with a classic result from Category theory:

Lemma 7.10 (Right adjoints preserve limits)**.** [39, Thm 4.5.3] *If a functor $F : \mathcal{P} \to \mathcal{Q}$ has a left adjoint, then for any diagram $\Psi : J \to \mathcal{P}$, if $\lim \psi$ exists, then $\lim(F \circ \Psi) = F(\lim \Psi)$.*

Consider an arrangement $\mathcal{Y} \xleftarrow{F} \mathcal{X} \xrightarrow{G} \mathcal{Z}$. Given an object y of $\mathcal{Y}$, one can create the composite functor:

(7.21)
$$
\begin{array}{ccc}
[F\ ;\ y] & \xrightarrow{\mathrm{Pre}_{F,G}(y)} & Z \\
\downarrow{\scriptstyle \pi_{[F\ ;\ y]}} & & \uparrow{\scriptstyle G} \\
X \times \{\star\} & \xrightarrow{\cong} & X
\end{array}
$$

Recall that $[[\mathrm{Cat}]\ ;\ \mathcal{Z}]$ is the left-slice of $\mathcal{Z}$ in the category $[\mathrm{Cat}]$ of small categories. Its objects are thus all possible functors with codomain $\mathcal{Z}$. The construction (7.21) thus gives us a functor (7.22)

$$
\mathrm{Pre}_{F,G} : \mathcal{Y} \to [[\mathrm{Cat}]\ ;\ \mathcal{Z}]\,,\ y \mapsto ([F\ ;\ y]\,, \mathrm{Pre}_{F,G}(y))\,;\quad
\begin{array}{c} y \\ \downarrow{\scriptstyle \psi} \\ y' \end{array}
\ \mapsto\
\begin{array}{ccccc}
[F\ ;\ y] & & \xrightarrow{\mathrm{Pre}_{F,G}(y)} & & \\
\downarrow{\scriptstyle \psi\circ} & \searrow{\scriptstyle \pi_{[F\ ;\ y]}} & & & \\
& & X & \xrightarrow{G} & Z \\
& \nearrow{\scriptstyle \pi_{[F\ ;\ y']}} & & & \\
[F\ ;\ y'] & & \xrightarrow{\mathrm{Pre}_{F,G}(y')} & &
\end{array}
$$

Now suppose that $\mathcal{Z}$ is a complete category. The collection of all diagrams in $\mathcal{Z}$, which are functors $F : J \to \mathcal{Z}$ is the left slice of $\mathcal{Z}$ within $[\mathrm{Cat}]$ the category of small categories. We then have the following result from basic category theory:

Lemma 7.11. *Given a complete category $\mathcal{Z}$, there is a functor* $\lim : [[\mathrm{Cat}]\ ;\ \mathcal{Z}] \to \mathcal{Z}$ *which maps each diagram $F : J \to \mathcal{Z}$ into* $\lim F$.

For a complete category $\mathcal{Z}$, the $\lim$ functor can be used to extend the functor from (7.22) into the dashed green arrow as shown below:

(7.23)
$$
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\mathrm{Pre}_{F,G}} & [[\mathrm{Cat}]\ ;\ Z] \\
& \searrow{\scriptstyle \mathrm{LimPre}_{F,G}} & \downarrow{\scriptstyle \lim} \\
& & \mathcal{Z}
\end{array}
$$

This construction $\mathrm{LimPre}_{F,G}$ is one of the main innovations in this paper. The use of colimits instead of limits would have yielded the right Kan extension of F along G. for any object y of $\mathcal{Y}$, $\mathrm{LimPre}_{F,G}(y)$ is the limit point of the functor G restricted to the left slice $[F\ ;\ y]$.

Given two complete categories $\mathcal{Z}, \mathcal{Z}'$ a functor $\Psi : \mathcal{Z} \to \mathcal{Z}'$ is called *limit preserving* if the following commutation holds:

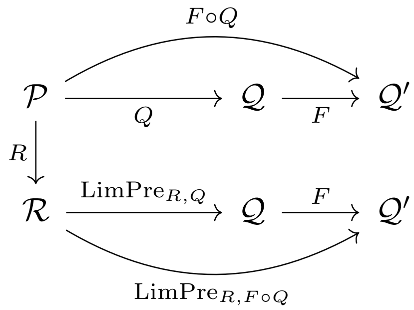
(7.24)
$$
\begin{array}{ccc}
[[\mathrm{Cat}]\ ;\ \mathcal{Z}] & \xrightarrow{F\circ} & [[\mathrm{Cat}]\ ;\ \mathcal{Z}'] \\
\downarrow{\scriptstyle \lim} & & \downarrow{\scriptstyle \lim} \\
\mathcal{Q} & \xrightarrow{F} & \mathcal{Z}'
\end{array}
$$

The LimPre construction involves two functors R, Q with the same domain category. The second functor Q may be extended to a different codomain by composition with some functor F. The next lemma presents a simple condition under which the LimPre applied to a composition of Q with F coincides with the composition of $\mathrm{LimPre}_{R,Q}$ with F.

Lemma 7.12. *Given an arrangement of functors*

$$
\mathcal{R} \xleftarrow{R} \mathcal{P} \xrightarrow{Q} \mathcal{Q} \xrightarrow{F} \mathcal{Q}'
$$

in which F is limit preserving, the following commutation holds between the LimPre *functors:*



Proof. Note that for every $y \in ob(\mathcal{Y})$,

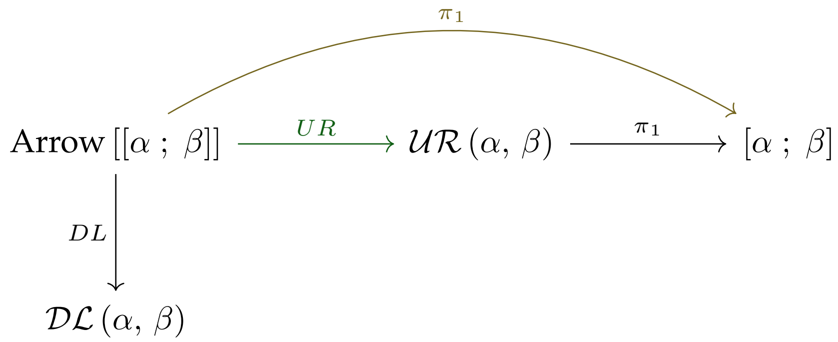
$$\begin{aligned} \mathrm{LimPre}_{R,F\circ Q}(y) &= \lim \mathrm{Pre}_{R,F\circ Q}(y) = \lim \mathrm{Pre}_{R,F\circ \mathrm{Pre}_{R,Q}}(y) \\ &= F \circ \lim \mathrm{Pre}_{R,\mathrm{Pre}_{R,Q}}(y) = F \circ \mathrm{LimPre}_{R,Q}(y), \end{aligned}$$

where the second last inequality holds from the limit preservation property. □

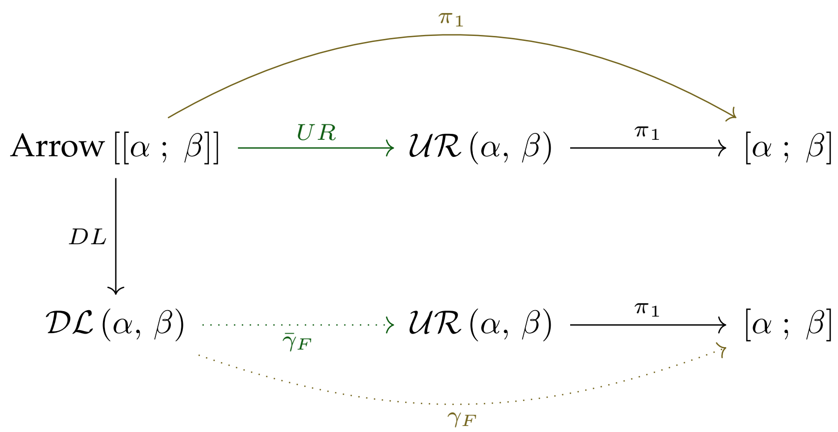
The reader is once again referred to the outline presented in Figure 1. The statement of Lemma 7.12 is essentially a commutation. This commutation will be seen to occur multiple times in the diagram presented later in (8.26). This diagram is the final step towards proving Theorems 3.2 and 3.3. At this point we are ready to begin the proof.

8. Proof of Theorems 3.2 and 3.3

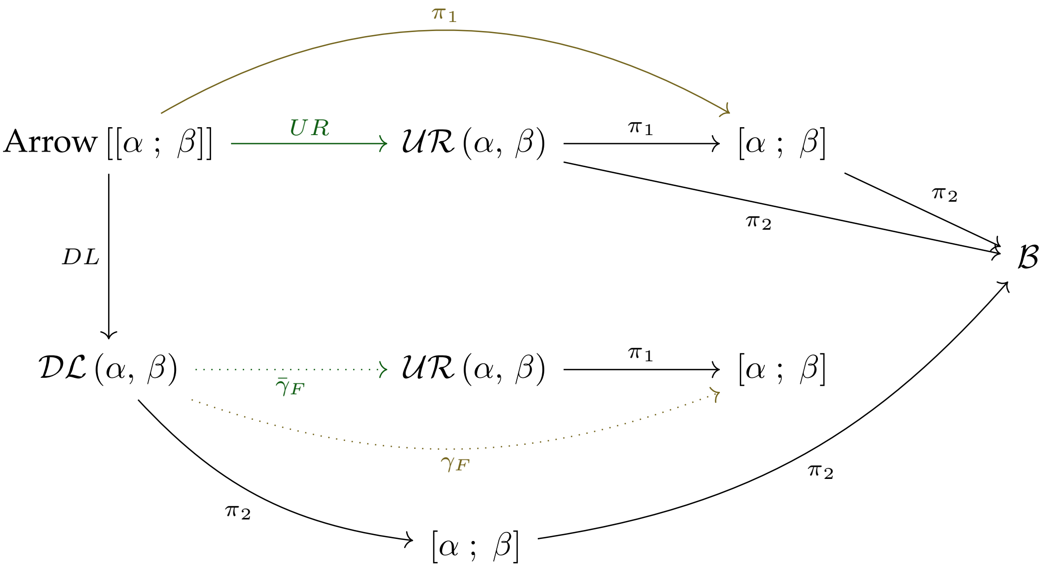
The proofs of the two theorems shall be derived simultaneously. We begin the proof by drawing a portion of (5.12).



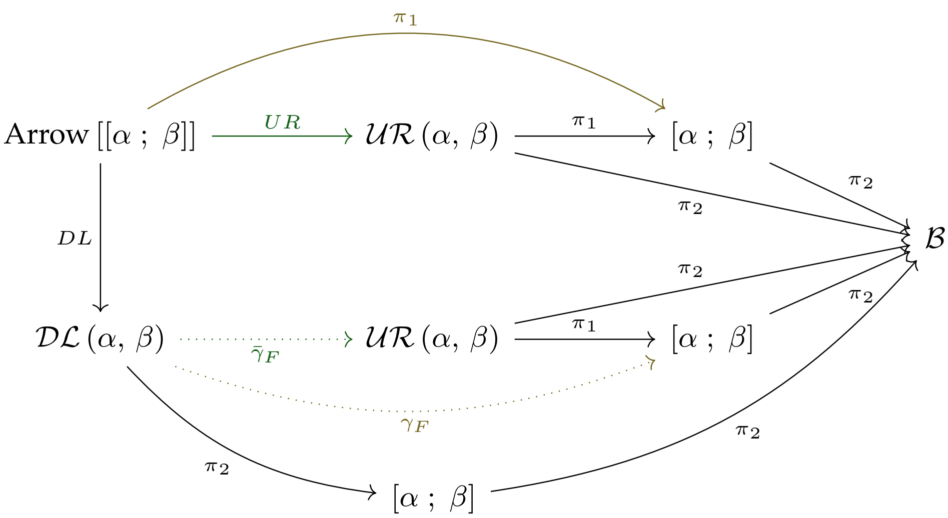
Since the categories $[\alpha\ ;\ \beta]$ and $\mathcal{UR}(\alpha, \beta)$ are complete by Lemmas 6.4 and 6.6 respectively, we can create the LimPre constructions of these functors, as shown below:



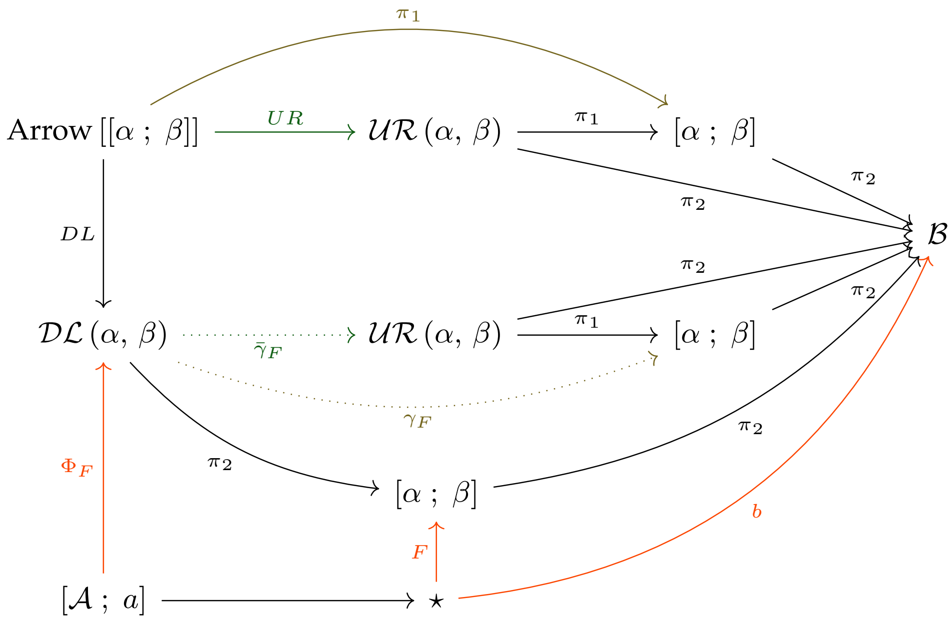
Each colored dotted arrow is the LimPre construction corresponding to the functor of the same color on the top row. The commutation between the LimPre constructions in the second row holds by Lemma 6.8, Lemma 6.9 and Lemma 7.12. We now add some of the peripheral commutations of (5.12) to get



Again, by Lemma 6.8 and Lemma 7.12, we can fill in:



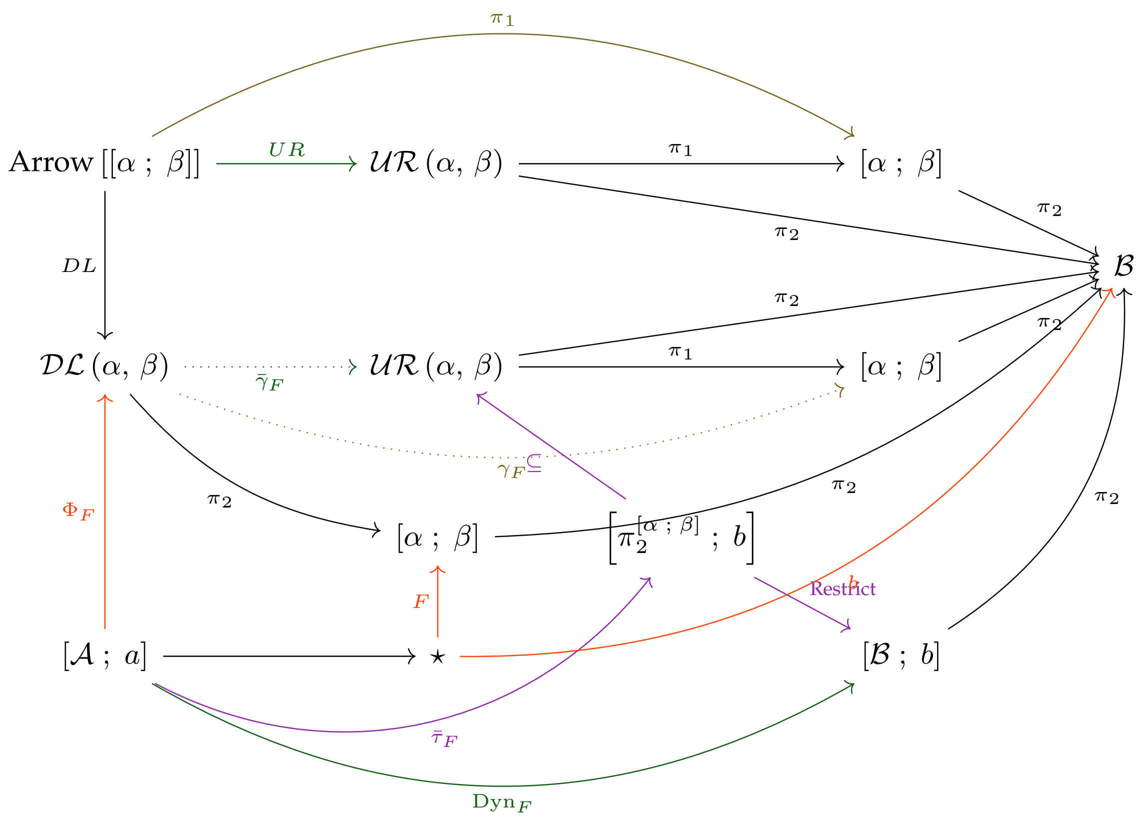
We now add the functor Φ_F from (5.16) to this diagram:



Now, set $\bar{\tau_F} = \bar{\gamma}_F \circ \Phi_F$ and $\tau_F = \gamma_F \circ \Phi_F$. Note that this creates a commutation:

(8.25)
$$\begin{array}{ccc} [\mathcal{A}\,;\, a] & \xrightarrow{\bar{\tau}_F} & \mathcal{UR}(\alpha, \beta) \\ \downarrow & & \downarrow \pi_2 \\ \star & \xrightarrow[b]{} & \mathcal{B} \end{array}$$

This is precisely the commutation described on the left of (6.20). Thus the conclusions of (6.20) along with (5.17) hold and we get:

(8.26)


The commutations in (3.9) are included within the commutation of (8.26). The claim of minimality in Theorem 3.2 follows from the construction of γ_F as a limit. The commutation diagram in (8.25) links this minimal comma object to minimal commutation squares completing

$$\begin{array}{ccc} \alpha a' & & \\ {\scriptstyle \alpha f}\downarrow & & \\ \alpha a & \xrightarrow[F]{} & \beta b \end{array} .$$

This completes the proof of Theorems 3.2 and 3.3. □

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SUDDHASATTWA DAS
TEXAS TECH UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS
LUBBOCK, TEXAS, USA
Email address: iamsuddhasattwa@gmail.com

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Research Article

A Browder-Petryshyn coincidence point theorem

Maher Berzig*

Abstract. Let C be a subset of a Hilbert space, and let f and g be self-maps on C such that the range of f is a convex, closed, and bounded subset of the range of g. If f does not increase distances more than g, we demonstrate that f and g have coincidence points. This result generalizes a fixed point theorem of Browder-Petryshyn and provide a new result for certain firmly nonexpansive-type mappings. As applications, we establish the existence of solutions to both matrix and integral equations.

Keywords: Coincidence point, matrix equation, integral equation.

2020 Mathematics Subject Classification: 54H25, 39B42, 45F05.

1. Introduction and preliminaries

The study of nonexpansive mappings, which are mappings that do not increase the distance between points, has been a central topic in fixed point theory since the mid-20th century. The Banach Contraction Principle [4] provides a foundational result for contractive mappings; however, nonexpansive mappings do not generally guarantee the existence of fixed points under the same conditions. Major improvements were achieved independently by Browder [8], Göhde [10], and Kirk [13], who established fixed point theorems for nonexpansive self-maps in uniformly convex Banach spaces, showing that such maps admit fixed points in closed, convex, and bounded subsets. These results marked a turning point and have since been extended to broader contexts, such as hyperconvex metric spaces (see Aronszajn and Panitchpakdi [3]) CAT(0) spaces (see Bridson and Haefliger [6]), and more general topological vector spaces. More recently, coincidence point theory, which initiated by Jungck [12], has provided a framework to investigate when two mappings share a coincidence point, enriching fixed point theory and leading to numerous extensions of classical results (see, for example, [1, 2, 5]).

Many important real-world problems can be framed as finding fixed or coincidence points of certain mappings. For instance, the existence of solutions to problems in semi-definite programming [14], digital signal processing [17] or fractional differential equations [18] frequently involves exploring the existence of coincidence points of matrix or integral equations. For more recent references on applications, see, for example, the study of convergence of the viscosity generalization of Halpern's iteration [16], the application to delay differential equations with finite constant delays [15], and applications in split feasibility problems [11]. When the mapping under consideration is nonexpansive, the associated problem can be resolved in Hilbert spaces via the Browder-Petryshyn fixed point theorem [7].

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*Corresponding author: Maher Berzig; maher.berzig@gmail.com
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In this work, we extend the Browder-Petryshyn theorem by exploring the existence of coincidence points for two self-mappings f and g defined on a suitable set C, under the condition that f does not increase distances more than g, or equivalently, that f is g-nonexpansive or g is f-expansive, meaning that

$$\|f(u) - f(v)\| \leq \|g(u) - g(v)\| \text{ for all } u, v \in C,$$

without requiring compactness or commutativity assumptions. Moreover, we provide a new result for firmly g-nonexpansive mappings, defined subsequently. As the first application, we establish the existence of a Hermitian solution to a matrix equation. As the second application, we investigate the existence of a solution to an integral equation. Before concluding the introduction, we recall the fixed point theorems of Browder and Petryshyn, and of Al-Thagafi and Shahzad.

Theorem 1.1 (Browder and Petryshyn [7])**.** *Let C be a closed, bounded, convex subset of a Hilbert space, and let $f\colon C \to C$ be a nonexpansive mapping, that is,*

$$\|f(u) - f(v)\| \leq \|u - v\| \text{ for all } u, v \in C.$$

Then f has a fixed point in C.

Theorem 1.2 (Al-Thagafi and Shahzad [2])**.** *Let (X, d) be a metric space and $f, g\colon X \to X$ be given mappings such that the closure of $f(X)$ is complete and subset of $g(X)$. If there exists $\lambda < 1$ such that*

$$d(f(x), f(y)) \leq \lambda\, d(g(x), g(y)) \text{ for all } x, y \in X,$$

then f and g have a coincidence point in X.

Remark 1.1. *Additional insight into the uniqueness of the coincidence point is provided by* [2, Theorem 2.1].

2. The Main Result

The main result is the following theorem.

Theorem 2.3. *Let C be a subset of a Hilbert space and let $f, g\colon C \to C$ be given maps such that $f(C)$ is closed, convex and bounded subset of $g(C)$, and f is g-nonexpansive. Then f and g have a coincidence point in C.*

Proof. Since $f(C)$ is convex, then for a fixed element v_0 of C and for every $\lambda \in (0, 1)$, the map $h_\lambda\colon C \to C$ given by

$$h_\lambda(x) = \lambda f(x) + (1 - \lambda) f(v_0),$$

is well defined. We also have

$$\|h_\lambda(x) - h_\lambda(y)\| = \lambda \|f(x) - f(y)\| \leq \lambda \|g(x) - g(y)\|.$$

According to Theorem 1.2, h_λ and g have a coincidence point $u_\lambda \in C$. Since $f(C)$ is closed, convex and bounded in a Hilbert space, it is weakly compact. Hence, we may find a sequence $\lambda_j \to 1$ as $j \to \infty$ such that $g(u_{\lambda_j}) = g(u_j) = h_{\lambda_j}(u_j) = f(w_j)$ converges weakly to an element y_0 of the Hilbert space, where $\{w_j\}$ is a sequence of C. Now, since $f(C)$ is closed, $y_0 \in f(C) \subseteq g(C)$, so there exists $x_0 \in C$ such that $y_0 = gx_0$. We shall prove that x_0 is a coincidence point of f and g. Let u be any point in C, then

$$\begin{aligned}\|g(u_j) - g(u)\|^2 &= \|(g(u_j) - g(x_0)) + (g(u_0) - g(u))\|^2 \\ &= \|g(u_j) - g(x_0)\|^2 + \|g(u_0) - g(u)\|^2 \\ &+ 2(g(u_j) - g(x_0), g(x_0) - g(u)),\end{aligned}$$

where $(g(u_j) - g(x_0), g(x_0) - g(u)) \to 0$ as $j \to \infty$, since $g(u_j) - g(x_0)$ converges weakly to 0. Now, since $f(C) \subseteq g(C)$, we can take u such that $g(u) = f(x_0)$, we deduce

$$\lim_{j\to\infty} \left(\|g(u_j) - f(x_0)\|^2 - \|g(u_j) - g(x_0)\|^2\right) = \|g(x_0) - f(x_0)\|^2.$$

Now, since f is g-nonexpansive, we have

$$\|f(u_j) - f(x_0)\| \leq \|g(u_j) - g(x_0)\|.$$

Thus, we obtain

$$\begin{aligned}\|g(u_j) - f(x_0)\| &\leq \|g(u_j) - f(u_j)\| + \|f(u_j) - f(x_0)\| \\ &\leq \|g(u_j) - f(u_j)\| + \|g(u_j) - g(x_0)\|.\end{aligned}$$

Observe that since $s_j \to 1$, then

$$\begin{aligned}f(u_j) - g(u_j) &= \lambda_j f(u_j) + (1-\lambda_j) f(v_0) - g(u_j) + (1-\lambda_j)\big(f(u_j) - f(v_0)\big) \\ &= h_{\lambda_j}(u_j) - g(u_j) + (1-\lambda_j)\big(f(u_j) - f(v_0)\big) \\ &= (1-\lambda_j)\big(f(u_j) - f(v_0)\big).\end{aligned}$$

Thus, $\limsup_{j\to\infty} \left(\|g(u_j) - f(x_0)\| - \|g(u_j) - g(x_0)\|\right) \leq 0$, and therefore

$$\limsup_{j\to\infty} \left(\|g(u_j) - f(x_0)\|^2 - \|g(u_j) - g(x_0)\|^2\right) \leq 0.$$

We conclude that $\|g(x_0) - f(x_0)\|^2 = 0$, so $f(x_0) = g(x_0)$. □

Remark 2.2. *Note that since $f(C) \subseteq g(C)$, Theorem 2.3 holds true even if the convexity of $f(C)$ is replaced by this of $g(C)$.*

The following corollaries follow immediately.

Corollary 2.1. *Let C be a subset of a Hilbert space, and let $f, g \colon C \to C$ be given maps such that $f(C)$ is closed, convex and bounded, and g is surjective and f-expansive. Then f and g have a coincidence point in C.*

Corollary 2.2. *Let C be a convex, closed and bounded subset of a Hilbert space, and let $f, g \colon C \to C$ be given surjective maps such that f is g-nonexpansive. Then f and g have a coincidence point in C.*

By selecting one of the mappings as the identity map, the following corollaries can be readily derived.

Corollary 2.3. *Let C be a closed and bounded subset of a Hilbert space, and $f \colon C \to C$ be a map such that $f(C)$ is convex and $\|f(u) - f(v)\| \leq \|u - v\|$ for all $u, v \in C$. Then f has a fixed point in C.*

Corollary 2.4. *Let C be a closed convex and bounded subset of a Hilbert space and $g \colon C \to C$ be a surjective map such that $\|u - v\| \leq \|g(u) - g(v)\|$ for all $u, v \in C$. Then g has a fixed point in C.*

Remark 2.3. *The Corollary 2.3 extends Theorem 1.1. Since, it requires convexity only for $f(C)$, meaning that C itself need not be convex.*

At the end of this section, we study the existence of coincidence point for certain firmly nonexpansive-type mappings. Let C be a convex set, let $f, g \colon C \to C$ be a given mapping, and for $x, y \in C$ consider the function $\varphi_{x,y}$ defined by

$$\varphi_{x,y}(t) = \left\|t\big(f(x) - f(y)\big) + (1-t)(g(x) - g(y))\right\|, \text{ for all } t \in [0,1].$$

Definition 2.1. *Let C be a convex set, a mapping $f \colon C \to C$ is said to be firmly g-nonexpansive if for all $x, y \in C$ the function $\varphi_{x,y}$ is nonincreasing on $[0,1]$.*

Notice that $\varphi_{x,y}$ is a convex function of t. To compute the derivative of $\varphi_{x,y}$ at $t = 1$, set

$$u := g(x) - g(y), \qquad v := f(x) - f(y), \qquad w := v - u.$$

Then $\varphi_{x,y}(t) = \|u + tw\|$. Whenever $u + tw \neq 0$, the function $\varphi_{x,y}$ is differentiable and

$$\varphi'_{x,y}(t) = \frac{(u + tw,\, w)}{\|u + tw\|}.$$

In particular, if $v = f(x) - f(y) \neq 0$, then $\varphi_{x,y}$ is differentiable at $t = 1$ and

$$\varphi'_{x,y}(1) = \frac{(v,\, w)}{\|v\|} = \frac{\big(f(x) - f(y),\, (f(x) - f(y)) - (g(x) - g(y))\big)}{\|f(x) - f(y)\|}.$$

Thus $\varphi'_{x,y}(1) \leq 0$ is equivalent to the numerator being nonpositive

$$\big(f(x) - f(y),\, (f(x) - f(y)) - (g(x) - g(y))\big) \leq 0,$$

or equivalently,

$$\big(f(x) - f(y),\, g(x) - g(y)\big) \geq \|f(x) - f(y)\|^2.$$

Thus, we obtain the following equivalence

$$\varphi'_{x,y}(1) \leq 0 \quad \Longleftrightarrow \quad \big(f(x) - f(y),\, g(x) - g(y)\big) \geq \|f(x) - f(y)\|^2.$$

The equivalence remains valid in case $f(x) = f(y)$.

Let us present some examples of firmly g-nonexpansive mapping based on the following definition of projection.

Definition 2.2 (g-projection). *Let H be a real Hilbert space and $C \subseteq H$ a nonempty closed convex set. Let $g\colon H \to H$ be a mapping satisfying:*

(i) The restriction $g|_C\colon C \to g(C)$ is injective $\big($so that the inverse is well-defined on $g(C)\big)$,
(ii) $g(C)$ is a nonempty closed and convex subset of H.

*Then, for every $x \in H$, the g-*projection *of x onto C is the mapping $P_C^g\colon H \to C$ defined by*

$$P_C^g(x) := (g|_C)^{-1}\big(P_{g(C)}(g(x))\big),$$

where $P_{g(C)}$ denotes the standard metric projection onto the closed convex set $g(C)$.

Remark 2.4. *Under (i)–(ii), $P_{g(C)}(g(x))$ exists and is unique for all $x \in H$, and injectivity of $g|_C$ ensures that $P_C^g(x) \in C$ is uniquely defined. Moreover, a point $y = P_C^g(x)$ if and only if it satisfies the variational inequality*

$$y \in C \quad \text{and} \quad \big(g(x) - g(y),\, g(z) - g(y)\big) \leq 0 \quad \forall\, z \in C, \tag{2.1}$$

or equivalently, the minimization problem

$$P_C^g(x) = \arg\min_{z \in C} \big\|g(x) - g(z)\big\|. \tag{2.2}$$

Thus, $P_C^g(x)$ is the point in C whose image under g is closest to $g(x)$ in the Hilbert norm.

Definition of g-projection unifies several important notions of projection used in optimization, variational inequalities, and numerical analysis. Below are some examples of firmly g-nonexpansive mappings:

1. ***Metric projection.** Take $g = \mathrm{id}_H$, the identity mapping on H. Then g is injective, and $g(C) = C$ is closed and convex. Hence the conditions of the definition are satisfied, and*

$$P_C^{\mathrm{id}}(x) = \mathrm{id}^{-1}\big(P_C(\mathrm{id}(x))\big) = P_C(x),$$

the classical nearest point projection. Moreover, (2.2) reduces to $\min_{z \in C} \|x - z\|$.

2. ***Weighted projection.*** *Let $M: H \to H$ be a bounded linear isomorphism (bijective with bounded inverse), self-adjoint, and positive definite, i.e., $(Mx, x) \geq \alpha\|x\|^2$ for all $x \in H$ and some $\alpha > 0$. Define $g(x) = Mx$. Then g is injective and $g(C) = M(C)$ is closed and convex. The g-projection is*
$$P_C^g(x) = M^{-1}\big(P_{M(C)}(Mx)\big),$$
equivalently solving
$$\min_{z\in C} \|Mx - Mz\| = \min_{z\in C} \|x - z\|_M,$$
where $\|u\|_M := \sqrt{(Mu, u)} = \|M^{1/2}u\|$. This construction is widely used in variable-metric and preconditioned optimization algorithms; see, e.g., [9].
3. ***Bregman-type projection.*** *Let $\phi: H \to \mathbb{R}$ be strictly convex and Fréchet differentiable, and assume that $\nabla\phi(C)$ is closed and convex. Define $g = \nabla\phi$. Then the g-projection is*
$$P_C^g(x) = (\nabla\phi|_C)^{-1}\big(P_{\nabla\phi(C)}(\nabla\phi(x))\big) = \arg\min_{z\in C}\big\|\nabla\phi(x) - \nabla\phi(z)\big\|.$$
The g-projection satisfies
$$\big(P_C^g(x) - P_C^g(y),\, \nabla\phi(x) - \nabla\phi(y)\big) \geq \|P_C^g(x) - P_C^g(y)\|^2$$
if and only if ϕ is quadratic, that is, $\phi(x) = \frac{1}{2}(Mx, x) + (b, x) + c$ with M is positive definite. In this case, $\nabla\phi(z) - \nabla\phi(x) = M(z - x)$ and the g-projection is a firmly nonexpansive operator in the M-inner product. For general strictly convex ϕ, the g-projection need not be firmly g-nonexpansive.

Next, we have the following result:

Corollary 2.5. *Let C be a subset of a Hilbert space and let $f, g: C \to C$ be given maps such that $h(C)$ is closed, convex and bounded subset of $g(C)$ where $h = 2f - g$, and f is firmly g-nonexpansive. Then f and g have a coincidence point in C.*

Proof. For $x, y \in C$,
$$\begin{aligned}\|h(x) - h(y)\|^2 &= \|\,(2f(x) - g(x)) - (2f(y) - g(y))\,\|^2\\ &= \|\,2(f(x) - f(y)) - (g(x) - g(y))\,\|^2\\ &= 4\|f(x) - f(y)\|^2 - 4\big(f(x) - f(y),\, g(x) - g(y)\big) + \|g(x) - g(y)\|^2\\ &\leq \|g(x) - g(y)\|^2.\end{aligned}$$
Hence, by Theorem 2.3, h and g have a coincidence point, and so f and g. □

Example 2.1. *Let $C = [0, 1] \subset \mathbb{R}$. Choose the constants*
$$r = \tfrac{1}{5}, \qquad A = \tfrac{1}{20}, \qquad B = \tfrac{1}{4}.$$
Define the mappings
$$g(x) = \tfrac{1}{2} + r\cos(2\pi x), \qquad \phi(s) = \tfrac{1}{2}s + A\sin(2\pi s) + B, \qquad f(x) = \phi\big(g(x)\big).$$
1. Well-definedness: *Since $\cos(2\pi x) \in [-1, 1]$, we obtain*
$$g(C) = \big[\tfrac{1}{2} - r,\ \tfrac{1}{2} + r\big] = \big[\tfrac{1}{2} - \tfrac{1}{5},\ \tfrac{1}{2} + \tfrac{1}{5}\big] = \big[\tfrac{3}{10},\ \tfrac{7}{10}\big] \subset C.$$
The derivative of ϕ satisfies
$$\phi'(s) = \tfrac{1}{2} + 2\pi A\cos(2\pi s) \in \big[\tfrac{1}{2} - 2\pi A,\ \tfrac{1}{2} + 2\pi A\big].$$
Since $2\pi A = \frac{\pi}{10} < \frac{1}{2}$, it follows that
$$0 < \phi'(s) < 1 \quad \forall\, s \in g(C),$$

so ϕ is strictly increasing. Hence

$$f(C) = \phi(g(C)) \subset [0,1] = C.$$

Thus $f, g : C \to C$ are well defined.

2. Firm g-nonexpansiveness: *By the mean-value theorem, for any $s, t \in g(C)$ there exists ξ such that*

$$\phi(s) - \phi(t) = \phi'(\xi)(s-t).$$

Multiplying by $(s-t)$

$$(\phi(s) - \phi(t))(s-t) = \phi'(\xi)(s-t)^2.$$

Since $0 < \phi'(\xi) < 1$, we have

$$\phi'(\xi)(s-t)^2 \geq (\phi'(\xi))^2(s-t)^2 = (\phi(s) - \phi(t))^2.$$

Thus,

$$(\phi(s) - \phi(t))(s-t) \geq (\phi(s) - \phi(t))^2.$$

Setting $s = g(x)$ and $t = g(y)$, we obtain

$$(f(x) - f(y))(g(x) - g(y)) \geq (f(x) - f(y))^2,$$

so f is firmly g-nonexpansive.

3. The mapping $h = 2f - g$: *Using the definition of f,*

$$h(x) = 2\phi(g(x)) - g(x) = 2A\sin(2\pi g(x)) + 2B.$$

Observe that h is continuous and C is an interval, $h(C)$ is a connected interval in $\mathbb{R}$, hence convex. Precisely, since $\sin(2\pi g(x)) \in [-1,1]$, thus

$$h(C) = [\,2B - 2A,\ 2B + 2A\,] = \left[\tfrac{1}{2} - \tfrac{1}{10},\ \tfrac{1}{2} + \tfrac{1}{10}\right] = \left[\tfrac{2}{5},\ \tfrac{3}{5}\right].$$

Finally, since

$$h(C) = \left[\tfrac{2}{5}, \tfrac{3}{5}\right] \subset \left[\tfrac{3}{10}, \tfrac{7}{10}\right] = g(C)$$

the inclusion $h(C) \subset g(C)$ holds. We conclude that the mappings f, g and h satisfy all the hypotheses of Corollary 2.5, and therefore f and g have a coincidence point in C.

3. Existence of Hermitian Solutions to a Class of Matrix Equations

Let $\mathcal{M}(n), \mathcal{H}(n), \mathcal{P}(n)$ and $\overline{\mathcal{P}}(n)$ be respectively the sets of all $n \times n$ arbitrary, Hermitian, positive-definite and positive semi-definite matrices. The spectral norm of a matrix A is the largest singular value of A and it is denoted by $\|A\|_2$. The notation $X \leq Y$ means that $Y - X \in \overline{\mathcal{P}}(n)$. If $X, Y \in \mathcal{H}(n)$ such that $X \leq Y$, then the order interval is defined by

$$[X, Y] := (X + \overline{\mathcal{P}}(n)) \cap (Y - \overline{\mathcal{P}}(n)).$$

Then $\mathcal{H}(n)$, endowed with Hilbert-Schmidt inner product $\langle X, Y\rangle = tr(Y^*X)$ is a real Hilbert space, whereas $\mathcal{H}(n)$ with the spectral norm $\|\cdot\|_2$ is not a Hilbert space, since this norm is not induced by any inner product. The associated norm is the Hilbert-Schmidt norm, which coincides with the Frobenius norm

$$\|X\|_F^2 = tr(X^*X).$$

It is worthy to note that $\|XY\|_F \leq \|X\|_2\|Y\|_F$ for all $X, Y \in \mathcal{H}(n)$.

We next provide sufficient conditions for the existence of a Hermitian non trivial solution to the following matrix equation

$$(AB)^*XAB = A^*XA, \tag{3.3}$$

where A and B are $n \times n$ commuting matrices.

Let $M, N \in \overline{\mathcal{P}}(n)$ such that $M \leq N$ ($M \neq N$) and $C = [M, N]$ an order interval. Consider the following assumptions:

(A1) $M \leq (AB)^* MAB$ and $(AB)^* NAB \leq N$,
(A2) $M \leq B^* MB$ and $B^* NB \leq N$,
(A3) $\|B\|_2 = 1$.

Proposition 3.1. *Under the assumptions (A1)-(A3), the matrix equation (3.3) has a solution in C.*

Proof. Firstly, observe that C is a compact convex set. Take

$$f(X) = (AB)^* XAB \text{ and } g(X) = A^* XA.$$

From (A1) and (A2), we deduce easily that $f, g \colon C \to C$ are well defined.

We shall show that $f(C)$ is convex, closed, bounded, and subset of $g(C)$. Clearly, from (A1), we deduce that $g(C)$ is bounded. Let $Y_1, Y_2 \in f(C)$ such that $(Y_1, Y_2) = (f(X_1), f(X_2))$ for some $X_1, X_2 \in C$, we deduce by definition of f and the convexity of C that

$$\begin{aligned} \lambda Y_1 + (1-\lambda) Y_2 &= \lambda f(X_1) + (1-\lambda) f(X_2) \\ &= f(\lambda X_1 + (1-\lambda) X_2) \in f(C), \end{aligned}$$

for all $\lambda \in (0, 1)$ which proves that $f(C)$ is convex. Now, since f is linear between finite-dimensional vector spaces, it is continuous and thus maps the compact C to a compact set, which implies that $f(C)$ is closed and bounded.

We have that $f(C) \subseteq g(C)$, since from the commutativity of A and B, we have for every $X \in C$, $f(X) = g(h(X))$, where $h(X) = B^* XB$ with $h(C) \subset C$ comes from (A2).

Finally, from (A3), we have

$$\begin{aligned} \|f(X) - f(Y)\|_F &= \|B^*(g(X) - g(Y))B\|_F \\ &\leq \|B\|_2^2 \, \|g(X) - g(Y)\|_F \\ &= \|g(X) - g(Y)\|_F, \end{aligned}$$

for all $X, Y \in C$, which proves that f is g-nonexpansive.

We conclude by Theorem 2.3, that f and g have a coincidence point in C, that is, the matrix equation (3.3) has a solution in C. □

Example 3.2. *Consider the following matrices:*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \quad B = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix},$$

$$M = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \quad N = 2 \begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Since (A1)-(A3) hold, we conclude by Proposition 3.1 that (3.3) has a solution in $C = [M, N]$. Note that $X = M$ is a solution of (3.3) in C.

Consider now the following matrices:

$$A = B = M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Again, since (A1)-(A3) hold, we conclude by Proposition 3.1 that (3.3) has a solution in $C = [M, N]$. Observe that $X = \frac{1}{2}(M + N)$ is a solution of (3.3) in C.

4. Existence of solutions to a class of integral equations

Let a, b be two real constants. The set of square integrable functions of $L^2([a, b])$ is a Hilbert space endowed with the inner product $\langle f, g\rangle := \int_a^b f(s)g(s)\mathrm{d}s$, where f and g are real functions. The L^2-norm is given by $\|f\|_{L^2}^2 := \langle f, f\rangle$. We study the existence of a nontrivial solution x of the following integral equation:

$$\int_a^b K(t,s)x(s)\,\mathrm{d}s = \int_a^b \int_a^b K(t,s)K(s,r)x(r)\,\mathrm{d}r\,\mathrm{d}s, \tag{4.4}$$

for all $t \in [a, b]$, where K is a continuous non-negative function defined on $[a, b]^2$.

Let

$$(fx)(t) = \int_a^b \int_a^b K(t,s)K(s,r)x(r)\mathrm{d}r\,\mathrm{d}s,$$
$$(gx)(t) = \int_a^b K(t,s)x(s)\mathrm{d}s.$$

A mapping $h\colon C \to C$ is said to be nondecreasing, if $x \le y$ implies $hx \le hy$ where the notation $x \le y$ means $x(t) \le y(t)$ for all $t \in [a, b]$.

Assume now that there exist two functions $u, v \in C[a, b]$ $(u \neq v)$ such that $u \le v$. Define a set C to be the closure of $C_{u,v}$, where

$$C_{u,v} := \{x \in C[a, b] : u \le x \le v\},$$

and consider the following assumptions:

(B1) $u \le fu$ and $fv \le v$,
(B2) $u \le gu$ and $gv \le v$,
(B3) $\int_a^b \int_a^b K(t,s)^2 \mathrm{d}s\,\mathrm{d}t = 1$.

Proposition 4.2. *Under the assumptions (B1)-(B3), the integral equation (4.4) has a solution in C.*

Proof. Firstly, observe that f and g are nondecreasing, then from (B1) and (B2) the maps $f, g\colon C \to C$ are well defined.

We now claim that $f(C)$ is closed, convex and bounded. To see this, observe that the set $C_{u,v}$ is convex, then so is its closer which is obviously closed.

Let now $y_1, y_2 \in f(C)$, so there exist $x_1, x_2 \in C$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Let $\lambda \in (0, 1)$, then by definition of g it follows that

$$\begin{aligned}\lambda y_1 + (1-\lambda)y_2 &= \lambda f(x_1) + (1-\lambda)f(x_2)\\ &= f(\lambda x_1 + (1-\lambda)x_2),\end{aligned}$$

which implies $\lambda y_1 + (1-\lambda)y_2 \in f(C)$, so $f(C)$ is convex. Moreover, from the fact that $u, v \in C[a, b]$, it follows by the extreme value theorem that $C_{u,v}$ is bounded, and so it is its closer. We deduce that our claim holds.

Next, by using the Cauchy-Schwarz inequality and (B3), we get for all $x, y \in C$,

$$\begin{aligned}\|fx - fy\|_{L^2}^2 &= \int_a^b \left| \int_a^b K(t,s) \left(\int_a^b K(s,r)(x(r) - y(r))\mathrm{d}r \right) \mathrm{d}s \right|^2 \mathrm{d}t\\ &\le \int_a^b \int_a^b K(t,s)^2 \mathrm{d}s\,\mathrm{d}t \int_a^b \left| \int_a^b K(s,r)(x(r) - y(r))\mathrm{d}r \right|^2 \mathrm{d}s\\ &\le \|gx - gy\|_{L^2}^2,\end{aligned}$$

which implies that f is g-nonexpansive.

Finally, note that $f = g \circ g$, which implies by (B2) that $f(C) \subseteq g(C)$. We conclude the result by Theorem 2.3. $\square$

Example 4.3. *Consider the following integral equation*

$$8 \int_a^b s\, x(s) \mathrm{d}s = 3 \int_a^b \int_a^b s^2 r\, x(r)\, \mathrm{d}r\, \mathrm{d}s, \tag{4.5}$$

where $a = 0$ and $b = 2$ with $K(t, s) = \frac{3}{8} ts$ for all $t, s \in [a, b]$. Clearly, K is non-negative, continuous and satisfies (B3). It is easy to see that the conditions (B1) and (B2) hold for the functions u and v defined by $u(t) = 0$ and $v(t) = \frac{3}{2} t$ for all $t \in [a, b]$. Hence, according to Proposition 4.2 the integral equation (4.5) has a solution in C. It is not difficult to see that the function x defined by $x(t) = \frac{3}{4} t$ for all $t \in [a, b]$ is in C and it is a solution of the integral equation (4.5).

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Maher Berzig
Université de Tunis
École Nationale Supérieure d'Ingénieurs de Tunis
Département de Mathématiques
5 avenue Taha Hussein Montfleury, 1008 Tunis, Tunisia
Email address: maher.berzig@gmail.com

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Survey Article

A survey on the distance functions

ERDAL KARAPINAR* AND RAVI P. AGARWAL

ABSTRACT. This survey aims to express different distance functions and consider their relationships, if any. Indeed, in the literature, numerous different and interesting distance functions with distinct properties have been introduced. Presenting and discussing distance functions with their original motivations can open a new window for researchers working in various disciplines. The distance functions mentioned here and the corresponding abstract spaces may offer alternative solutions for the existing problems.

Keywords: Abstract metric space, fixed point, metric spaces, perturbed metric spaces, self-mappings.

2020 Mathematics Subject Classification: 46T99, 47H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

Throughout the history of mathematics, the idea of distance is among the first concepts that have been recognized. In contemporary mathematics, the idea of distance is generalized through the concept of a metric [37, 38]. It would not be an overstatement to claim that the idea of metric serves as the essential driving force of contemporary mathematics. In reality, it has been applied not just in mathematics, but also to address issues in other sciences and fields that can be described mathematically; for instance, the idea of metrics has been widely utilized in imaging challenges and computer science. Consequently, the idea of metrics has captured the interest of numerous researchers. This idea has been examined through various methods and viewpoints, with efforts made to broaden, enhance, and refine it.

In academic publications, various types of metrics can be discovered. It is not an easy task to mention all these abstract structures since there are several mixed version of the following abstract distance notions: 2-metric [40], D-metric [29], G-metric [66], S-metric [84], A-metric [1], a quasi-metric [64, 67, 72, 73, 74, 75, 83], ultra-metric, symmetric metric, bipolar metric [59], modular metric [24, 80], fuzzy metric, b-metric [13, 15, 27, 71, 77], strong b-metric [63], partial metric [42, 65], cone metric (Banach-valued metric) [44], b-cone metric [43], TVS-valued metric [35], complex-valued metric [8, 12], C^*-algebra valued metric space [10], quaternion-valued metric [5], generalized metric [62], Branciari distance function [20], supra-metric [17, 18], super metric [53], and interpolative metric [63], along with numerous others, as seen in [16, 55, 56]. In these mentioned structures, it is possible to put some of them in the same class. More precisely, we can consider three-point structures and put the following abstract spaces together: 2-metric [40], D-metric [29], G-metric [66], S-metric [84], and even A-metric [1]. In the classical distance notion, for any two points, the definition of the distance function assigns a non-negative real number. See also, e.g. [7, 9, 21, 22, 23, 25, 51, 54, 57, 60, 68, 69,

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*Corresponding author: Erdal Karapınar; erdalkarapinar@yahoo.com
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76, 78, 79, 81, 88, 89]. On the other hand, in three-point structures, the considered distance function needs three points to assign a non-negative real number. Another class of the distance function can be considered as extended range metrics: cone metric (Banach-valued metric) [44], b-cone metric [52], TVS-valued metric [35], complex-valued metric [8, 12], C^*-algebra valued metric space [10], quaternion-valued metric [5] are the straightforward examples of this class. In the classical metric structure, for any two points, the metric function assigns a non-negative real number. In other words, the range of the metric function is the set of non-negative reals. For an extended range of metric functions, the range, non-negative real numbers, has been changed by a positive cone of the given Banach space or TVS or quaternion space or C^*-algebra space. Necessarily, in this class, considered new distance function assigns not a nonnegative real number but, instead, a Banach value, complex value, TVS-value, quaternion value, or C^*-algebra value, and so on. On the other hand, there are some other abstract structures that can not be comparable or embedded in another structure. For instance, Branciari distance is not comparable with the standard metric or any other metric mentioned above. There are also some equivalent structure with different disguises. For example, a multiplicative metric space can be considered as an equivalent structure to the standard metric space.

This short survey aims to fill one of the literature's gaps by compiling all significant abstract distance functions as much as possible. In the upcoming sections, we shall consider significant abstract structures with some examples. We start this section with one of the most interesting structures, so-called, ultra metric spaces.

2. Quasi-Metric Spaces

The definition of a quasi-metric is given as follows:

Definition 2.1. *Let X be a non-empty and let $d : X \times X \to [0, \infty)$ be a function which satisfies:*
(d1) $d(x, y) = 0$ *if and only if* $x = y$,
(d2) $d(x, y) \leq d(x, z) + d(z, y)$.
Then d is called a quasi-metric and the pair (X, d) is called a quasi-metric space.

Remark 2.1. *Any metric space is a quasi-metric space, but the converse is not true, in general.*

Now, we give convergence, completeness, and continuity on quasi-metric spaces.

Definition 2.2. *Let (X, d) be a quasi-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if*

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0. \tag{2.1}$$

Example 2.1 ([6]). *Let X be a subset of $\mathbb{R}$ containing $[0, 1]$ and define, for all $x, y \in X$,*

$$q(x, y) = \begin{cases} x - y, & \text{if } x \geq y, \\ 1, & \text{otherwise.} \end{cases}$$

Then (X, q) is a quasi-metric space. Notice that $\{q(1/n, 0)\} \to 0$ but $\{q(0, 1/n)\} \to 1$. Therefore, $\{1/n\}$ converges to 0 on the right but does not converge from the left. We also point out that this quasi-metric verifies the following property: if a sequence $\{x_n\}$ has a right limit x, then it is unique.

Remark 2.2. *A quasi-metric space is Hausdorff, that is, we have the uniqueness of limit of a convergent sequence.*

Definition 2.3. *Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is left-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n \geq m > N$.*

Definition 2.4. *Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is right-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n > N$.*

Definition 2.5. *Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > N$.*

Remark 2.3. *A sequence $\{x_n\}$ in a quasi-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.*

Definition 2.6. *Let (X, d) be a quasi-metric space. We say that*
(1) (X, d) is left-complete if and only if each left-Cauchy sequence in X is convergent.
(2) (X, d) is right-complete if and only if each right-Cauchy sequence in X is convergent.
(3) (X, d) is complete if and only if each Cauchy sequence in X is convergent.

Definition 2.7. *Let (X, d) be a quasi-metric space. The map $f : X \to X$ is continuous if for each sequence $\{x_n\}$ in X converging to $x \in X$, the sequence $\{fx_n\}$ converges to fx, that is,*

$$\lim_{n\to\infty} d(fx_n, fx) = \lim_{n\to\infty} d(fx, fx_n) = 0. \tag{2.2}$$

3. Ultrametric Spaces

In this section, we shall recall one of the most interesting abstract space structures, so-called, ultrametric spaces.

Definition 3.8 (Ultrametric)**.** *Let X be a non-empty set. A distance function*

$$d : X \times X \to [0, \infty)$$

is called an ultrametric on X if for all $x, y, z \in X$, the following conditions hold:
(U_1) $d(x, y) \geq 0$ (Non-negativity),
(U_2) $d(x, y) = 0 \iff x = y$ (Identity of indiscernible),
(U_3) $d(x, y) = d(y, x)$ (Symmetry),
(U_4) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ (Ultrametric inequality, or strong triangle inequality).

Example 3.2 (p-adic metric on $\mathbb{Q}_p$)**.** *Let p be a prime number. The p-adic absolute value $|\cdot|_p$ on the field of rational numbers $\mathbb{Q}$ is defined as follows: for any non-zero rational numbers $x = \frac{a}{b}p^n$, where a, b are integers not divisible by p, define*

$$|x|_p = p^{-n}, \quad |0|_p = 0.$$

The corresponding metric is

$$d_p(x, y) = |x - y|_p.$$

This metric satisfies the ultrametric inequality

$$d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z)\},$$

and hence defines an ultrametric on $\mathbb{Q}$. Its completion gives the field $\mathbb{Q}_p$ of p-adic numbers.

Example 3.3 (Space of infinite sequences over a finite alphabet)**.** *Let $A = \{a_1, a_2, \ldots, a_k\}$ be a finite alphabet. Let $X = A^{\mathbb{N}}$ be the set of infinite sequences over A. Define a metric on X by:*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-n} & \text{if } x \neq y \text{ and } n \text{ is the first index where } x_n \neq y_n. \end{cases}$$

This metric satisfies the ultrametric inequality, because if x and y agree up to position n, y and z agree up to position m, then x and z must agree at least up to $\min(n, m)$, so

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

Thus, (X, d) is an ultrametric space.

Example 3.4 (Hierarchical clustering/dendrograms). *Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set of objects (e.g., data points). Suppose we are given a hierarchical clustering of X, represented by a dendrogram (tree structure). Define a distance between two points as the height at which the two points merge in the dendrogram*

$$d(x_i, x_j) = \text{height at which clusters containing } x_i \text{ and } x_j \text{ merge.}$$

This distance satisfies the ultrametric inequality and gives rise to an ultrametric on X. Such metrics are used in phylogenetics, taxonomy, and cluster analysis.

Remark 3.4. *In general, every ultrametric space is a standard metric space, but not every standard metric space is an ultrametric space.*

(1) *Every ultrametric space is a standard metric space:*

If (X, d) is an ultrametric space, then (X, d) is a metric space.

(2) *Not every standard metric space is an ultrametric space:*

There exist standard metric spaces that are not ultrametric spaces.

Example 3.5. *Consider the standard Euclidean metric on $\mathbb{R}^2$:*

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

This metric does not satisfy the ultrametric condition. For example, with points $x = (0, 0)$, $y = (1, 0)$, and $z = (0, 1)$

$$d(x, z) = \sqrt{(0-0)^2 + (0-1)^2} = 1, \quad d(x, y) = 1, \quad d(y, z) = \sqrt{2}.$$

Here, $d(x, z) \not\leq \max(d(x, y), d(y, z))$, since $1 \not\leq \max(1, \sqrt{2})$.

Theorem 3.1 (All triangles are isosceles or equilateral). *In an ultrametric space, every triangle is either isosceles or equilateral.*

Proof. Let $x, y, z \in X$. Without loss of generality, assume $d(x, y) \leq d(x, z)$. By the ultrametric inequality

$$d(y, z) \leq \max\{d(y, x), d(x, z)\} = \max\{d(x, y), d(x, z)\} = d(x, z).$$

Also applying the inequality in reverse:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

Since $d(x, y) \leq d(x, z)$, this implies

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} \leq \max\{d(x, z), d(y, z)\} = d(x, z),$$

so equality must hold throughout. Thus, at least two sides have equal length: either $d(x, y) = d(x, z)$ or $d(y, z) = d(x, z)$. Therefore, all triangles are isosceles or equilateral. □

Theorem 3.2 (Every interior point is a center). *If $y \in B(x, r)$, then $B(x, r) = B(y, r)$, where $B(x, r) = \{z \in X : d(x, z) < r\}$.*

Proof. Let $y \in B(x,r)$, so $d(x,y) < r$. We show that $B(x,r) = B(y,r)$. First, take any $z \in B(x,r)$, i.e., $d(x,z) < r$. Then

$$d(y,z) \leq \max\{d(y,x), d(x,z)\} < \max\{r,r\} = r,$$

so $z \in B(y,r)$. Hence, $B(x,r) \subseteq B(y,r)$. Now take any $z \in B(y,r)$, so $d(y,z) < r$. Then

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} < \max\{r,r\} = r,$$

so $z \in B(x,r)$. Thus, $B(y,r) \subseteq B(x,r)$, and therefore $B(x,r) = B(y,r)$. This shows that every point inside a ball is also a center of the ball. □

Theorem 3.3 (Balls are clopen)**.** *In an ultrametric space, all balls $B(x,r) = \{y \in X : d(x,y) < r\}$ are both open and closed (clopen).*

Proof. We already know from general metric space theory that open balls are open sets. To show that they are also closed, we will prove that their complement is open.

Suppose $y \notin B(x,r)$, so $d(x,y) \geq r$. Consider the ball $B(y,r')$ with radius $r' = d(x,y)$. Take any $z \in B(y,r')$, so $d(y,z) < r' = d(x,y)$. Then by the ultrametric inequality:

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} = d(x,y),$$

and since $d(y,z) < d(x,y)$, we get

$$d(x,z) \leq d(x,y).$$

But also

$$d(x,y) \leq \max\{d(x,z), d(z,y)\} = \max\{d(x,z), d(y,z)\} = d(x,z),$$

since $d(y,z) < d(x,y) \leq d(x,z)$. So $d(x,z) = d(x,y) \geq r$, which means $z \notin B(x,r)$. Hence, $B(y,r') \cap B(x,r) = \emptyset$, and so the complement of $B(x,r)$ is open. Therefore, $B(x,r)$ is closed. Thus, all open balls are clopen. □

4. Partial Metric Spaces

Definition 4.9 (Partial metric space [65])**.** *Let X be a nonempty set. A function $p : X \times X \to [0,\infty)$ is called a partial metric if it satisfies the following conditions for all $x,y,z \in X$:*

(P1) $x = y \iff p(x,x) = p(x,y) = p(y,y)$
(P2) $p(x,x) \leq p(x,y)$
(P3) $p(x,y) = p(y,x)$
(P4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$

The pair (X,p) is called a partial metric space.

Example 4.6. *If $X := \{[a,b] |\ a,b \in \mathbb{R},\ a \leq b\}$ then $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,b\}$ defines a partial metric p on X.*

Example 4.7. *Let $X := \mathbb{R}^{\mathbb{N}_0} \cup \bigcup_{n \geq 1} \mathbb{R}^{\{0,1,\ldots,n-1\}}$, where $\mathbb{N}_0$ is the set of nonnegative integers. By $L(x)$ denote the set $\{0,1,\ldots,n\}$ if $x \in \mathbb{R}^{\{0,1,\ldots,n-1\}}$ for some $n \in \mathbb{N}$, and the set $\mathbb{N}_0$ if $x \in \mathbb{R}^{\mathbb{N}_0}$. Then a partial metric is defined on X by*

$$p(x,y) = \inf\left\{\frac{1}{2^i} | i \in L(x) \cap L(y) \text{ and } \forall j \in \mathbb{N}_0\ (j < i \implies x(j) = y(j))\right\}.$$

Let (X,p) be a partial metric space. Then, the functions $d_p, d_m : X \times X \to [0,\infty)$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

and

$$d_m(x,y) = \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\}$$

are (usual) metrics on X. It is easy to check that d_p and d_m are equivalent. Note that each partial metric p on X generates a T_0-topology τ_p with a base of the family of open $p-$balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$.

Definition 4.10 ([2, 45]). *Let (X,p) be a partial metric space.*

(1) A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x)$.
(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and finite).
(3) (X,p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$.
(4) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0,\delta)) \subset B(f(x_0),\varepsilon)$.

Referring to [70], we say that a sequence $\{x_n\}$ in (X,p) is called the sequence *0-Cauchy* if $\lim_{n,m\to\infty} p(x_n,x_m) = 0$. Also, we say that (X,p) is *0-complete* if every 0-Cauchy sequence in X converges, with respect to the partial metric p, to a point $x \in X$ such that $p(x,x) = 0$. Notice that if (X,p) is complete, then it is 0-complete, but the converse does not hold. Moreover, every 0-Cauchy sequence in (X,p) is Cauchy in (X,d_p).

Example 4.8 ([65, 70]). *(1) Let $X = [0,+\infty)$ and define $p(x,y) = \max\{x,y\}$, for all $x,y \in X$. Then (X,p) is a complete partial metric space. It is clear that p is not a (usual) metric.*
(2) Let $X = [0,+\infty) \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. Define $p(x,y) = \max\{x,y\}$, for all $x,y \in X$. Then (X,p) is a 0-complete partial metric space which is not complete.

Proposition 4.1 ([2, 45]). *Let (X,p) be a partial metric space.*

(1) A sequence $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if $\{x_n\}$ is a Cauchy sequence in (X,d_p).
(2) (X,p) is complete if and only if (X,d_p) complete. Moreover,

$$\lim_{n\to\infty} d_p(x_n,x) = 0 \Leftrightarrow \lim_{n\to\infty} p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_m,x_n).$$

The following lemmas play a crucial role in the proof of the theorems.

Lemma 4.1 ([2, 50]). *Assume $x_n \to z$ as $n \to \infty$ in a partial metric space (X,p) such that $p(z,z) = 0$. Then, $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$.*

Lemma 4.2 ([2, 50]). *Let (X,p) be a complete partial metric space. Then*

(1) If $p(x,y) = 0$ then $x = y$.
(2) If $x \neq y$, then $p(x,y) > 0$.

Lemma 4.3 ([2, 50]). *Let (X,p) be a partial metric spaces. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ for all $x_n, y_n, x, y \in X$ then $p(x_n,y_n) \to p(x,y)$ as $n \to \infty$.*

5. Interpolative Metric Spaces

In what follows, we shall state the definition of (α,c)-interpolative metric.

Definition 5.11. *Let X be a nonempty set. We say that $d : X \times X \to [0,+\infty)$ is (α,c)-interpolative metric if*

($m1$) $d(x,y)=0$, *if and only if*, $x=y$ *for all* $x,y\in X$,
($m2$) $d(x,y)=d(y,x)$, *for all* $x,y\in X$,
($m3$) *there exist an* $\alpha\in(0,1)$ *and* $c\geq 0$ *such that*

$$d(x,y)\leq d(x,z)+d(z,y)+c\left[(d(x,z))^{\alpha}\,(d(z,y))^{1-\alpha}\right]$$

for all $(x,y,z)\in X\times X\times X$.

Then, we call (X,d) *an* (α,c)*-interpolative metric space.*

Note that each metric space can be considered an (α,c)-interpolative metric space with $c=0$. In the following example, we shall clarify that the converse is invalid.

Example 5.9. *Let* (X,ρ) *be a standard metric space. Define a function* $d:X\times X\to[0,\infty)$ *as follows*

$$d(x,y):=\rho(x,y)(\rho(x,y)+A),$$

where $A>0$. *Since* ρ *is a metric on* X, *the conditions* ($m1$) *and* ($m2$) *are straightforward. For* ($m3$), *it is enough to consider* $c\geq 2$ *for any* $\alpha\in(0,1)$. *Thus,* (X,d) *is* $(\frac{1}{2},2)$*-interpolative metric space. Indeed, we have*

$$\begin{aligned}
d(x,y)&=\rho(x,y)(\rho(x,y)+A)\\
&\leq(\rho(x,z)+\rho(z,y))(\rho(x,z)+\rho(z,y)+A)\\
&\leq(\rho(x,z)+\rho(z,y))(\rho(x,z)+\rho(z,y)+A)\\
&\leq[\rho(x,z)(\rho(x,z)+A)+\rho(x,z)\rho(z,y)]+[\rho(z,y)(\rho(z,y)+A)+\rho(z,y)\rho(x,z)]\\
&\leq[\rho(x,z)(\rho(x,z)+A)]+[\rho(z,y)(\rho(z,y)+A)]+2\rho(x,z)\rho(z,y)\\
&\leq d(x,z)+d(z,y)+2\,(\rho(x,z))^{\frac{1}{2}}\,(\rho(x,z))^{\frac{1}{2}}\,(\rho(z,y))^{\frac{1}{2}}\,(\rho(z,y))^{\frac{1}{2}}\\
&\leq d(x,z)+d(z,y)+2\,(\rho(x,z))^{\frac{1}{2}}\,[\rho(x,z)+A]^{\frac{1}{2}}\,(\rho(z,y))^{\frac{1}{2}}\,[\rho(z,y)+A]^{\frac{1}{2}}\\
&\leq d(x,z)+d(z,y)+2(d(x,z))^{\frac{1}{2}}(d(z,y))^{\frac{1}{2}}.
\end{aligned}$$

The function $d(x,y)$ *does not form a metric. Note also that the above estimation for the pair* $(\frac{1}{2},2)$ *is very rough and can be improved in several ways.*

Example 5.10. *Let* X *be a non-empty set and define a function* $d:X\times X\to[0,\infty)$ *as follow*

$$d(x,y):=|x-y|^3,\ \text{for all } x,y\in X.$$

Regarding the notion of the absolute value function, we conclude that the conditions ($m1$) *and* ($m2$) *are satisfied trivially. For* ($m3$), *it is enough to consider* $c\geq 6$ *for any* $\alpha=\frac{1}{3}\in(0,1)$. *Then,* (X,d) *is* $(\frac{1}{3},6)$*-interpolative metric space. More precisely, by a simple calculation and manipulation, we derive that*

$$\begin{aligned}
d(x,y)=|x-y|^3&=|x-z+z-y|^3\\
&=|x-z|^3+|z-y|^3+3|x-z|^2|z-y|+3|x-z||z-y|^2\\
&\leq d(x,z)+d(z,y)+3\left[(d(x,z))^{\frac{2}{3}}\,(d(z,y))^{\frac{1}{3}}\right]+3\left[(d(x,z))^{\frac{1}{3}}\,(d(z,y))^{\frac{2}{3}}\right]
\end{aligned}\tag{5.3}$$

without loss of generality, we assume $d(x,z)\geq d(z,y)$,

$$d(x,y)\leq d(x,z)+d(z,y)+6\left[(d(x,z))^{\frac{2}{3}}\,(d(z,y))^{\frac{1}{3}}\right].$$

Consequently, we conclude that ($m3$) *is fulfilled. Hence,* (X,d) *is* $(\frac{1}{3},6)$*-interpolative metric space.*

Lemma 5.4 ([58]). *For each* $p,q\in[0,\infty)$ *and each* $\alpha\in(0,1)$, $p^{\alpha}q^{1-\alpha}\leq p+q$.

Proof. If $p = 0$ or $q = 0$, then the inequality follows trivially. Thus, we assume that $p > 0$ and $q > 0$. In this case, we find

$$p^{\alpha}q^{1-\alpha} \leq (\max\{p,q\})^{\alpha}(\max\{p,q\})^{1-\alpha} = \max\{p,q\} \leq p+q.$$

□

Employing the technical lemma above, we can state the following theorem:

Theorem 5.4. *Let (X,d) an (α,c)-interpolative metric space. Then, it lies between the standard metric and the b-metric.*

Proof. Trivially, we have the following,

$(m1)$ $d(x,y) = 0$, if and only if, $x = y$ for all $x, y \in X$,
$(m2)$ $d(x,y) = d(y,x)$, for all $x, y \in X$.

To prove the assertion of the theorem above, we shall use Lemma 5.4. Indeed, for any $\alpha \in (0,1)$,

$$p^{\alpha}q^{1-\alpha} \leq (p+q),$$

is equivalent to

$$[d(x,z)]^{\alpha}[d(z,y)]^{1-\alpha} \leq d(x,z) + d(z,y),$$

by letting $p = d(x,z)$ and $q = d(z,y)$, for all $x, y, z \in X$. Since d is an (α,c)-interpolative metric space, for any $x, y, z \in X$ there is $c \in X$ there exist an $\alpha \in (0,1)$ and $c \geq 0$ such that

$$\begin{aligned} d(x,y) &\leq d(x,z) + d(z,y) + c\left[(d(x,z))^{\alpha}\,(d(z,y))^{1-\alpha}\right] \\ &\leq d(x,z) + d(z,y) + c[d(x,z) + d(z,y)] \\ &\leq (c+1)[d(x,z) + d(z,y)] \\ &\leq s[d(x,z) + d(z,y)] \end{aligned}$$

where $s = c + 1$. Indeed, it is clear to see that

$$d(x,z) + d(z,y) \leq d(x,z) + d(z,y) + c\left[(d(x,z))^{\alpha}\,(d(z,y))^{1-\alpha}\right] \leq s[d(x,z) + d(z,y)].$$

□

6. b-METRIC

Definition 6.12 (b-metric)**.** *Let X be a nonempty set and let $d : X \times X \to [0,\infty)$ be a function satisfying:*

(b_1) $d(x,y) = 0$ if and only if $x = y$,
(b_2) $d(x,y) = d(y,x)$ for all $x, y \in X$,
(b_3) There exists a constant $s \geq 1$ such that

$$d(x,y) \leq s[d(x,z) + d(z,y)], \quad \text{for all } x, y, z \in X.$$

Then (X,d) is called a b-metric space with coefficient s.

Example 6.11. *Let $X = \mathbb{R}$ and define a function $d : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ by:*

$$d(x,y) = |x-y|^2.$$

Then (X,d) is a b-metric space with coefficient $s = 2$. This satisfies the inequality:

$$d(x,y) \leq 2[d(x,z) + d(z,y)] \quad \text{for all } x, y, z \in X.$$

Example 6.12. *Let $X = \{0,1,2\}$ and define $d : X \times X \to \mathbb{R}_+$ as follows*

$$d(0,1) = d(1,0) = d(0,2) = d(2,0) = 1,$$
$$d(1,2) = d(2,1) = \alpha \geq 2,$$
$$d(0,0) = d(1,1) = d(2,2) = 0.$$

Then (X,d) is a b-metric space with coefficient $s = \frac{\alpha}{2}$.

Example 6.13. *Let $X = \mathbb{R}^m$ be the space of all ordered m-tuples of real numbers. For any two points $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$, define*

$$d_p(x,y) = \left(\sum_{k=1}^{m} |y_k - x_k|^p\right)^{1/p}, \quad p \geq 1.$$

Then (X, d_p) is a b-metric space with constant $s = 2^{1/p}$.

Example 6.14. *Let $X = \ell^\infty$, the space of all bounded sequences $x = \{x_k\}_{k=1}^{\infty}$, with the metric*

$$d_\infty(x,y) = \sup_{k \in \mathbb{N}} |x_k - y_k|.$$

Then (X, d_∞) is a complete b-metric space.

Example 6.15. *Let $X = L^2([-1,1])$, the space of square-integrable functions on $[-1,1]$, with the metric*

$$d(f,g) = \left(\int_{-1}^{1} (f(t) - g(t))^2 dt\right)^{1/2}.$$

Then (X,d) is not a standard metric space but is a b-metric space with $s = \sqrt{2}$.

Example 6.16. *Let $X = C([0,1], \mathbb{R})$, the space of continuous real-valued functions on $[0,1]$. Define*

$$d(f,g) = \left(\int_{0}^{1} |f(t) - g(t)|^2 dt\right)^{1/2}.$$

Then (X,d) is a b-metric space with $s = \sqrt{2}$.

Example 6.17. *Let $X = \{a,b,c\}$ and $d : X \times X \to [0,\infty)$ be defined by*

$$d(a,b) = 1, \quad d(a,c) = \frac{1}{2} \quad \text{and} \quad d(b,c) = 2,$$

with $d(x,x) = 0$ and $d(x,y) = d(y,x)$ for all $x,y \in X$. Notice that d is not a metric since $d(b,c) > d(b,a) + d(a,c)$. However, it is easy to see that d is a b-metric space with $s \geq \frac{4}{3}$.

7. Strong b-Metric Space

Kirk and Shahzad [63] defined the strong b-metric space as follows:

Definition 7.13 (Strong b-metric)**.** *Let X be a nonempty set and let $d : X \times X \to [0,\infty)$ be a function satisfying:*

(b_1) $d(x,y) = 0$ if and only if $x = y$,
(b_2) $d(x,y) = d(y,x)$ for all $x,y \in X$,
(b_3) There exists a constant $K \geq 1$ such that

$$d(x,y) \leq d(x,z) + Kd(z,y), \quad \text{for all } x,y,z \in X.$$

Then $(X,d,)$ is called a strong b-metric space with coefficient K.

Example 7.18. *Let* $X = \{x_1, x_2, x_3\}$, *the function* $d : X \times X \to [0, \infty)$ *be defined by* $d(x_i, x_i) = 0,$ for each $i = 1, 2, 3.$

$$d(x_1, x_2) = d(x_2, x_1) = 2,$$
$$d(x_2, x_3) = d(x_3, x_2) = 1,$$
$$d(x_1, x_3) = d(x_3, x_1) = 6.$$

It is clear that (X, d) *is a strong b-metric with* $K = 4$.

8. Extended b-Metric Space

Definition 8.14 (Extended b-metric space). *A function* $d_\theta : X \times X \to [0, \infty)$ *is called an extended b-metric if there exists a function* $\theta : X \times X \to [1, \infty)$ *such that:*

(1) $d_\theta(x, y) = 0 \iff x = y$,
(2) $d_\theta(x, y) = d_\theta(y, x)$ *for all* $x, y \in X$,
(3) $d_\theta(x, y) \le \theta(x, z) d_\theta(x, z) + \theta(z, y) d_\theta(z, y)$, *for all* $x, y, z \in X$.

Then (X, d_θ) *is called an extended b-metric space.*

Example 8.19. *Let* $X = [0, 1]$, *define* $d_\theta(x, y) = |x - y|^3$, *and let* $\theta(x, y) = 2$ *for all* x, y. *Then* (X, d_θ) *is an extended b-metric space.*

9. 2-Metric Spaces

During the 1960s, Gahler [39, 40] introduced the notation of 2-metric spaces as an extension of the standard metric space.

Definition 9.15. *Let* X *be a nonempty set. A function* $d : X \times X \times X \to [0, \infty)$, *satisfying the following properties:*

(d1) *For distinct* $x, y \in X$, *there exists* $z \in X$ *such that* $d(x, y, z) \neq 0$,
(d2) $d(x, y, z) = 0$ *if two of the triple* $x, y, z \in X$ *are equal,*
(d3) $d(x, y, z) = d(x, z, y) = \cdots$ *(symmetry in all three variables),*
(d4) $d(x, y, z) \le d(x, y, a) + d(x, a, z) + d(a, y, z)$, *for all* $x, y, z, a \in X$ *(rectangle inequality)*

is called a 2- *metric on* X. *The set* X *together with such a* 2*-metric,* d, *is called a* 2*-metric space and denoted by* (X, d).

Gahler asserted that a 2-metric extends the standard concept of a metric, yet it is not. It was demonstrated that 2-metric is not a continuous function, while a typical metric is. Gahler noted that geometrically $d(x, y, z)$ signifies the area of a triangle constructed by the points $x, y,$ and z in X, although not required. For more details, see e.g. [41, 87].

10. D-Metric Spaces

In 1992, Dhage [29, 30, 31, 32, 33] attempted to develop a 2-metric by introducing a new concept of a generalized metric:

Definition 10.16. *Let* X *be a nonempty set, a function* $D : X \times X \times X \to \mathbf{R}^+$ *satisfying the following axioms:*

(D1) $D(x, y, z) \ge 0$ *for all* $x, y, z \in X$,
(D2) $D(x, y, z) = 0$ *if and only if* $x = y = z$,
(D3) $D(x, y, z) = D(x, z, y) = \cdots$ *(symmetry in all three variables),*
(D4) $D(x, y, z) \le D(x, y, a) + D(x, a, z) + D(a, y, z)$, *for all* $x, y, z, a \in X$ *(rectangle inequality)*

is called a generalized metric or a D-metric on X. *An additional property was proposed on a D-metric (see* [30]*) is*

(D5) $D(x,y,y) \leq D(x,z,z) + D(z,y,y)$ *for all* $x,y,z \in X$.

The set X *together with such a generalized metric,* D*, is called a generalized metric space or D-metric space, and denoted by* (X,D)*. It is called symmetric if* $D(x,x,y) = D(x,y,y)$ *for all* $x,y \in X$*.*

In a D–metric space (X,D), three possible notions for the convergence of a sequence (x_n) to a point x present themselves:

$(C1)$ $x_n \to x$ if $D(x_n,x,x) \to 0$ as $n \to \infty$,
$(C2)$ $x_n \to x$ if $D(x_n,x_n,x) \to 0$ as $n \to \infty$,
$(C3)$ $x_n \to x$ if $D(x_m,x_n,x) \to 0$ as $m,n \to \infty$.

Clearly, $(C3) \Rightarrow (C2)$ and if D is symmetric then $(C1) \Leftrightarrow (C2)$.

In generalized metric spaces, the demonstrations for many fixed point theorems proposed by Dhage and others depended, either explicitly or implicitly, on the persistence of D regarding convergence in the sense of $(C3)$ or - regarding the convergence in the context of $(C2)$. Nonetheless, there are opposing examples provided by Mustafa and Sims [66].

Example 10.20 ([66])**.** *Let* $A = \{\frac{1}{n} : n = 1,2,3,\ldots\}$ *and* $X = A \cup \{0\}$ *and let* p *be the semi-metric on* X *defined by:*

$$p(x,x) := 0, \text{ for all } x \in X,$$
$$p(0,\frac{1}{n}) := p(\frac{1}{n},0) = \frac{1}{n}, \text{ for } n = 2,3,\ldots,$$
$$p(x,y) := 1, \text{ for } x,y \in A \text{ with } x \neq y.$$

Then, for $D(x,y,z) := \max\{p(x,y),p(x,z),p(y,z)\}$ *we have* $D(x_n,x_m,0) = 1$ *for all* n,m *with* $n \neq m$*, thus the sequence* $(\frac{1}{n})$ *does not converge in the sense of* $(C3)$*.*

Example 10.21 ([66])**.** *For* X *as above, define* D *by:*

$$D(x,y,z) := \begin{cases} 0, & \text{if } x = y = z \\ \frac{1}{n}, & \text{if one of } x,y,z \text{ is equal to } 0 \text{ and the other two are equal to } \frac{1}{n} \\ 1, & \text{otherwise.} \end{cases}$$

Then it is readily seen that D *is a generalized metric which satisfies* $C2(D5)$*. Further,* $(\frac{1}{n})$ *does not converge in the sense of* $(C1)$ *or* $(C3)$*.*

Example 10.22 ([66])**.** *For* X *as above, define* D *by:*

$$D(x,y,z) := \begin{cases} 0, & \text{if } x = y = z \\ \frac{1}{n}, & \text{if two of } x,y,z \text{ are equal to } 0 \text{ and the other is equal to } \frac{1}{n} \\ 1, & \text{otherwise.} \end{cases}$$

Then it is readily seen that D *is a generalized metric satisfying* $(D5)$*. The sequence* $(\frac{1}{n})$ *does not converge in the sense of* $(C2)$ *or* $(C3)$*.*

Example 10.23 ([66])**.** *For* X *again as in Example 10.20, but with semi-metric* p *defined by:*

$$p(0,1) := p(1,0) = 1,$$
$$p(1,\frac{1}{n}) := p(\frac{1}{n},1) = \frac{1}{2}, \text{ for } n = 2,3,\ldots,$$
$$p(1,1) := 0,$$
$$p(x,y) := |x-y|, \text{ for } x,y \in X \setminus \{1\}.$$

Then $D(x,y,z) := p(x,y) + p(x,z) + p(y,z)$ *converges to* 0 *in each of the senses* $(C1),(C2),(C3)$*.*

11. G-Metric Spaces

Definition 11.17 ([66]). *A generalized metric (or a G-metric) on X is a mapping $G : X \times X \times X \to [0, \infty)$ verifying, for all $x, y, z \in X$:*

- (G_1) $G(x, x, x) = 0$,
- (G_2) $G(x, x, y) > 0$ *if* $x \neq y$,
- (G_3) $G(x, x, y) \leq G(x, y, z)$ *if* $y \neq z$,
- (G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ *(symmetry in all three variables),*
- (G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *(rectangle inequality).*

Let (X, G) be a G-metric space, let $\{x_n\} \subseteq X$ be a sequence and let $x \in X$. Then the following conditions are equivalent.

(a) $\{x_n\}$ G-converges to x.
(b) $\lim_{n\to\infty} G(x_n, x_n, x) = 0 \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : x_n \in B_G(x, \varepsilon)\, \forall n \geq n_0$.
(c) $\lim_{n\to\infty} G(x_n, x, x) = 0$.
(d) $\lim_{n,m\to\infty,\ m\geq n} G(x_n, x_m, x) = 0$.
(e) $\lim_{n\to\infty} G(x_n, x_n, x) = 0$ and $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0$.
(f) $\lim_{n\to\infty} G(x_n, x, x) = 0$ and $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0$.
(g) $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$ and $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0$.
(h) $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$ & $\lim_{n,m\to\infty,\ m>n} G(x_n, x_m, x) = 0$.
(i) $\lim_{n,m\to\infty,\ m>n} G(x_n, x_m, x) = 0$.

Every G-metric on X defines a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \text{ for all } x, y \in X. \tag{11.4}$$

Example 11.24. *Let (X, d) be a metric space. The functions $G_m(d), G_s(d) : X \times X \times X \to [0, +\infty)$, defined by*

$$G_m(d)(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}, \tag{11.5}$$

$$G_s(d)(x, y, z) = \frac{1}{3}[d(x, y) + d(y, z) + d(z, x)], \tag{11.6}$$

for all $x, y, z \in X$, are standard G-metrics on X.

Definition 11.18. *Let (X, G) be a G-metric space, and let $x_0 \in X$, given $\varepsilon > 0$, define the sets*

$$B_G(x_0, \varepsilon) := \{y \in X; G(x_0, y, y) < \varepsilon\}$$

and

$$\overline{B}_G(x_0, \varepsilon) := \{y \in X, G(x_0, y, y) \leq \varepsilon\}$$

Then, $B_G(x_0, \varepsilon)$ and $\overline{B}_G(x_0, \varepsilon)$ are called the open and closed balls, with centers x_0 and radius ϵ, respectively.

Each G-metric G on X generates a topology τ_G on X whose base is a family of open G-balls $\{B_G(x, \varepsilon) : x \in X, \varepsilon > 0\}$. A nonempty set A in the $G-$metric space (X, G) is G-closed if $\overline{A} = A$. Moreover,

$$x \in \overline{A} \Leftrightarrow B_G(x, \varepsilon) \cap A \neq \emptyset, \quad \text{for all } \varepsilon > 0.$$

Definition 11.19. *A sequence $\{x_n\}$ in a G-metric space X is said to converge if there exists $x \in X$ such that $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and one say that the sequence (x_n) is G-convergent to x. We call x the limit of the sequence $\{x_n\}$ and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.*

Theorem 11.5. *Every G-convergent sequence in a G-metric space (X, G) has a unique limit.*

Definition 11.20. *In a G-metric space X, a sequence (x_n) is said to be G-Cauchy if given $\epsilon > 0$, there is $N \in \mathbf{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$, that is $G(x_n, x_m, x_l) \longrightarrow 0$ as $n, m, l \longrightarrow \infty$.*

Definition 11.21. *A G-metric space X is said to be complete if every G-Cauchy sequence in X is G-convergent in X.*

Theorem 11.6. *Let (X, G) be a G-metric space. Let $d : X \times X \to [0, \infty)$ be the function defined by $d(x, y) = G(x, y, y)$. Then*

(1) (X, d) is a quasi-metric space,
(2) $\{x_n\} \subset X$ is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X, d),
(3) $\{x_n\} \subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X, d),
(4) (X, G) is G-complete if and only if (X, d) is complete.

Every quasi-metric induces a metric, that is, if (X, d) is a quasi-metric space, then the function $\delta : X \times X \to [0, \infty)$ defined by $\delta(x, y) = \max\{d(x, y), d(y, x)\}$ is a metric on X.

12. S-Metric

Another such generalization of three point was given by S. Sedghi, N. Shobe and A. Aliouche [84] in 2012 as follows:

Definition 12.22. *Let X be a nonempty set, a function $S : X \times X \times X \to \mathbf{R}^+$ satisfying the following axioms:*

(S1) $S(x, y, z) = 0$ if and only if $x = y = z$,
(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$, for all $x, y, z, a \in X$ (rectangle inequality),

is called a S-metric on X. The pair (X, S) is called as S-metric space.

Definition 12.23. *Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$, we define the open ball $B_S(x, r)$ and the closed ball $B_S[x, r]$ with center x and radius r as follows:*

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\},$$
$$B_S[x, r] = \{y \in X : S(y, y, x) \leq r\}.$$

The topology induced by the S-metric is the topology generated by the base of all open balls in X.

Definition 12.24. *Let (X, S) be an S-metric space.*

(1) A sequence $\{x_n\} \subset X$ converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \epsilon$. We write $x_n \to x$ for brevity.
(2) A sequence $\{x_n\} \subset X$ is a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \epsilon$.
(3) The S-metric space (X, S) is complete if every Cauchy sequence is a convergent sequence.

Lemma 12.5. *In an S-metric space, we have $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

Lemma 12.6. *Let (X, S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ then $S(x_n, x_n, y_n) \to S(x, x, y)$.*

Example 12.25 ([84]). *Let $X = \mathbb{R}^n$ and $\|\cdot\|$ be a norm on X. Then, $S(x, y, z) = \|y+x-2z\|+\|y-z\|$ is an S-metric on X. In general, if X is a vector space over $\mathbb{R}$ and $\|\cdot\|$ is a norm on X, then it is easy to see that*

$$S(x, y, z) = \|y - z\| + \|x - z\|$$

where $x + \lambda y = z$ for every $\lambda \geq 1$, is an S-metric on X.

Theorem 12.7 ([86]). *Let (X, S) be a S-metric space. Let $d : X \times X \to [0, \infty)$ be the function defined by $d(x, y) = S(x, x, y)$. Then*

(1) (X, d) is a b-metric space,
(2) $\{x_n\} \subset X$ is S-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X, d),
(3) $\{x_n\} \subset X$ is S-Cauchy if and only if $\{x_n\}$ is Cauchy in (X, d).

13. Cone Metric Space

Let E be a real Banach space. A subset P of E is called a cone if and only if the following hold:

(a_1) P is closed, nonempty, and $P \neq \{0\}$,
(a_2) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply that $ax + by \in P$,
(a_3) $x \in P$ and $-x \in P$ imply that $x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$, if and only if, $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal, if there exist a number $K > 1$ such that, $0 \leq x \leq y$ implies $||x|| \leq K||y||$, for all $x, y \in E$. The least positive number satisfying this, called the normal constant. E denotes a real Banach space, P denotes a cone in E with $intP \neq \emptyset$, and $\leq$ denotes partial ordering with respect to P.

Definition 13.25 ([44]). *Let X be a nonempty set. A function $d : X \times X \to E$ is called a cone metric on X, if it satisfies the following conditions:*

(b_1) *$d(x, y) \geq 0$, $\forall x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$,*
(b_2) *$d(x, y) = d(y, x)$, for all $x, y \in X$,*
(b_3) *$d(x, y) \leq d(x, z) + d(y, z)$, for all $x, y, z \in X$.*

Then, (X, d) is called a cone metric space.

Definition 13.26. *Let (X, d) be a cone metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. If for all $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_0) \ll c$, then $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent and $\{x_n\}_{n\in\mathbb{N}}$ converges to x and x is the limit of $\{x_n\}_{n\in\mathbb{N}}$.*

Definition 13.27. *Let (X, d) be a cone metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. If for all $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence in X.*

Definition 13.28. *Let (X, d) be a cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.*

Definition 13.29. *Let (X, d) be a cone metric space. A self-map T on X is said to be continuous, if $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} T(x_n) = T(x)$ for all sequence $\{x_n\}_{n\in\mathbb{N}}$ in X.*

Lemma 13.7. *Let (X, d) be a cone metric space and P be a cone. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then, $\{x_n\}_{n\in\mathbb{N}}$ converges to x, if and only if,*

$$\lim_{n\to\infty} d(x_n, x) = 0. \tag{13.7}$$

Lemma 13.8. *Let (X,d) be a cone metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. If $\{x_n\}_{n\in\mathbb{N}}$ is convergent, then it is a Cauchy sequence.*

Lemma 13.9. *Let (X,d) be a cone metric space and P be a cone in E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, if and only if, $\lim_{m,n\to\infty} d(x_m,x_n)=0$.*

Example 13.26 ([44]). *Let $E=\mathbb{R}^2$, $P=\{(x,y)\in E \mid y>0\}\subset\mathbb{R}^2$, $X=\mathbb{R}$ and $d: X\times X\to E$ such that*

$$d(x,y)=|(x-y),\alpha|x-y||\,,$$

where $\alpha>0$ is a constant. Then (X,d) is a cone metric space.

The cone P is called normal if there exists a constant $K>0$ such that for all $a,b\in E$, $0\le a\le b$ implies $|a|\le K|b|$. The cone $[0,\infty)\subset\mathbb{R}^2$ is a normal cone with constant $K=1$. However, there are examples of non-normal cones.

Example 13.27 ([28]). *Let $E=\mathbb{R}^2$ with the norm $||\cdot||$ and consider the cone $P=\{(x,y)\in E \mid y>0\}$. For each $k\ge 1$, let $f(x)=x$ and $g(x)=2^k$. Then $0\le g(x)\le 2k$. Since $||f||=2$ and $||g||=2k+1$. There are no normal cones with normal constant $K<1$. Indeed, if P were a normal cone and if $|x|<K|y|$, then $0<K<1$. For each k, consider the real vector space E with the supremum norm and the cone $P=\{x=a+b : a,b\ge 0\}$. Since P is regular and normal, it can be shown that the normal constant for this cone is greater than one. This shows that we can construct cones with different normal constants $K>1$.*

14. Cone b-Metric Space

Definition 14.30 ([43]). *Let X be a nonempty set and $p>1$ be a given real number. A mapping $d: X\times X\to E$ is said to be a cone b-metric if, for all $x,y,z\in X$, the following conditions hold:*

(1) $d(x,y)\ge 0$ and $d(x,y)=0$ if and only if $x=y$,
(2) $d(x,y)=d(y,x)$,
(3) $d(x,z)\le d(x,y)+d(y,z)$.

The pair (X,d) is called a cone b-metric space.

Remark 14.5. *The class of cone b-metric spaces is larger than the class of cone metric spaces since any cone metric space must be a cone b-metric space. However, the converse is not always true.*

Example 14.28 ([43]). *Let $E=\mathbb{R}^2$, $p>1$, $X=R$ and $d(x,y)=|y-x|^p$. We see that the mapping d satisfies the conditions for a cone b-metric. Let $x,y\in X$ and $x\ne y$. From the inequality*

$$|y-x|^p\ge 0,$$

which implies that $d(x,y)\ge 0$. It is impossible for all x,y to be equal. Indeed, taking account of the inequality

$$|y-x|^p>0,$$

for all $x\ne y$. Thus, (d_3) in Definition 14 is not satisfied, i.e., (X,d) is not a cone metric space.

Example 14.29 ([43]). *Let $X=[0,1]$, $E=\mathbb{R}^2$ and p be a constant. Take*

$$P=\{(x,y)\in E : x,y>0\}.$$

We define $d: X\times X\to E$ as

$$d(x,y)=|(x-y)^p|^{\frac{1}{p}}\,.$$

Then (X,d) is a complete cone b-metric space.

15. Complex Valued Metric Spaces

The concept of complex valued metric space which is given by Azam et al. [12]. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

It follows that

$$z_1 \precsim z_2$$

if one of the following conditions is satisfied

(h_1) $Re(z_1) = Re(z_2); Im(z_1) < Im(z_2)$
(h_2) $Re(z_1) < Re(z_2); Im(z_1) = Im(z_2)$
(h_3) $Re(z_1) < Re(z_2); Im(z_1) < Im(z_2)$
(h_4) $Re(z_1) = Re(z_2); Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(h_1), (h_2)$ and (h_3) is satisfied and we will write $z_1 \prec z_2$ if only (h_3) is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \implies |z_1| < |z_2|$$

where $|\cdot|$ represent modulus or magnitude of z, and

$$z_1 \precsim z_2,\ z_2 \prec z_3 \implies z_1 \prec z_3.$$

Definition 15.31. *Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X, if it satisfies the following conditions:*

(b_1) *$0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$, if and only if, $x = y$,*
(b_2) *$d(x, y) = d(y, x)$, for all $x, y \in X$,*
(b_3) *$d(x, y) \precsim d(x, z) + d(y, z)$, for all $x, y, z \in X$.*

Here, the pair (X, d) is called a complex valued metric space.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space.

Lemma 15.10. *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 15.11. *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

Let $(X, d_{\mathbb{C}})$ be a complex-valued metric space where $\mathbb{C}$ is the skew field of complex number z, i.e.,

$$\mathbb{C} = \{x + yi : (x, y) \in \mathbb{R}^2\}.$$

Define

$$\mathcal{P}_{\mathbb{C}} = \{x + yi : x \geq 0, y \geq 0\}.$$

It is apparent that $\mathcal{P}_{\mathbb{C}} \subset \mathbb{C}$. Assume $\mathbf{0}_{\mathbb{C}}$ be the zero of $\mathbb{C}$ from now on. Note that $(\mathbb{C}, |.|)$ is a real Banach space.

Lemma 15.12. *$\mathcal{P}_{\mathbb{C}}$ is a normal cone in real Banach space $(\mathbb{C}, |.|)$.*

Lemma 15.13. *Any complex-valued metric space $(X, d_{\mathbb{C}})$ is a cone metric space.*

16. Quaternion-Valued Metric Space

The skew field of quaternion, denoted by $\mathbb{H}$ means to write each element $q \in \mathbb{H}$ in the form

$$q = x_0 + x_1 i + x_2 j + x_3 k,$$

$x_n \in \mathbb{R}$; where $1, i, j, k$ are the basis elements of $\mathbb{H}$ and $n = 1, 2, 3$. For these elements, we have the multiplication rules

$$\begin{aligned} & i^2 = j^2 = k^2 = -1, \\ & ij = -ji = k, \\ & kj = -jk = -i, \\ & ki = -ik = j. \end{aligned}$$

The conjugate element $\bar{q}$ is given by

$$\bar{q} = x_0 - x_1 i - x_2 j - x_3 k.$$

The quaternion modulus has the form of

$$|q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

A quaternion q may be viewed as a four-dimensional vector (x_0, x_1, x_2, x_3). Define a partial order $\precsim$ on $\mathbb{H}$ as follows:

$$q_1 \precsim q_2 \iff \begin{cases} Re(q1) \leq Re(q2) \\ Im_s(q_1) \leq Im_s(q_2), \; q_1, q_2 \in \mathbb{H}, s = i, j, k. \end{cases}$$

where $Im_i = x_1, Im_j = x_2$ and $Im_k = x_3$. It follows that $q_1 \precsim q_2$ if one of the following conditions hold:

$$\begin{array}{llll}
(I) & Re(q_1) = Re(q_2); & Im_{s_1}(q_1) = Im_{s_1}(q_2) & where\ s_1 = j, k; Im_i(q_1) < Im_i(q_2) \\
(II) & Re(q_1) = Re(q_2); & Im_{s_2}(q_1) = Im_{s_2}(q_2) & where\ s_2 = i, k; Im_j(q_1) < Im_j(q_2) \\
(III) & Re(q_1) = Re(q_2); & Im_{s_3}(q_1) = Im_{s_3}(q_2) & where\ s_3 = i, j; Im_k(q_1) < Im_k(q_2) \\
(IV) & Re(q_1) = Re(q_2); & Im_{s_1}(q_1) < Im_{s_1}(q_2); & Im_i(q_1) = Im_i(q_2) \\
(V) & Re(q_1) = Re(q_2); & Im_{s_2}(q_1) < Im_{s_2}(q_2); & Im_j(q_1) = Im_j(q_2) \\
(VI) & Re(q_1) = Re(q_2); & Im_{s_3}(q_1) < Im_{s_3}(q_2); & Im_k(q_1) = Im_k(q_2) \\
(VII) & Re(q_1) = Re(q_2); & Im_s(q_1) < Im_s(q_2) & \\
(VIII) & Re(q_1) < Re(q_2); & Im_s(q_1) = Im_s(q_2) & \\
(IX) & Re(q_1) < Re(q_2); & Im_{s_1}(q_1) = Im_{s_1}(q_2); & Im_i(q_1) < Im_i(q_2) \\
(X) & Re(q_1) < Re(q_2); & Im_{s_2}(q_1) = Im_{s_2}(q_2); & Im_j(q_1) < Im_j(q_2) \\
(XI) & Re(q_1) < Re(q_2); & Im_{s_3}(q_1) = Im_{s_3}(q_2); & Im_k(q_1) < Im_k(q_2) \\
(XII) & Re(q_1) < Re(q_2); & Im_{s_1}(q_1) < Im_{s_1}(q_2); & Im_i(q_1) = Im_i(q_2) \\
(XIII) & Re(q_1) < Re(q_2); & Im_{s_2}(q_1) < Im_{s_2}(q_2); & Im_i(q_1) = Im_i(q_2) \\
(XIV) & Re(q_1) < Re(q_2); & Im_{s_3}(q_1) < Im_{s_3}(q_2); & Im_i(q_1) = Im_i(q_2) \\
(XV) & Re(q_1) < Re(q_2); & Im_s(q_1) < Im_s(q_2) & \\
(XVI) & Re(q_1) = Re(q_2); & Im_s(q_1) = Im_s(q_2). &
\end{array}$$

Remark 16.6. *In particular, we write $q_1 \precnsim q_2$ if $q_1 \neq q_2$ and one from (I) to (XVI) is satisfied. Also, we will write $q_1 < q_2$ if only (XV) is satisfied. It should be remarked that*

$$q_1 \precsim q_2 \;\Rightarrow\; |q_1| \leq |q_2|.$$

Ahmed et al. [4], introduced the definition of the quaternion-valued metric space as follows:

Definition 16.32. *Let X be a nonempty set. A function $d_{\mathbb{H}} : X \times X \to \mathbb{H}$ is called a quaternion valued metric on X, if it satisfies the following conditions:*

(d_1) $0 \lesssim d_{\mathbb{H}}(x,y)$ *for all* $x, y \in X$ *and* $d_{\mathbb{H}}(x,y) = 0$, *if and only if*, $x = y$,
(d_2) $d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(y,x)$, *for all* $x, y \in X$,
(d_3) $d_{\mathbb{H}}(x,y) \lesssim d_{\mathbb{H}}(x,z) + d_{\mathbb{H}}(y,z)$, *for all* $x, y, z \in X$.

Then, $(X, d_{\mathbb{H}})$ *is called a quaternion valued metric space.*

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 \lesssim c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d_{\mathbb{H}}(x_n, x) \lesssim c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \lesssim c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d_{\mathbb{H}}(x_n, x_{n+m}) \lesssim c$, then $\{x_n\}$ is called a Cauchy sequence in $(X, d_{\mathbb{H}})$. If every Cauchy sequence is convergent in $(X, d_{\mathbb{H}})$, then $(X, d_{\mathbb{H}})$ is called a complete quaternion valued metric space.

Lemma 16.14. *Let* (X, d) *be a quaternion valued metric space and let* $\{x_n\}$ *be a sequence in* X *. Then* $\{x_n\}$ *converges to* x *if and only if* $|d(x_n, x)| \to 0$ *as* $n \to \infty$.

Lemma 16.15. *Let* $(X, d_{\mathbb{H}})$ *be a quaternion valued metric space and let* $\{x_n\}$ *be a sequence in* X *. Then* $\{x_n\}$ *is a Cauchy sequence if and only if* $|d_{\mathbb{H}}(x_n, x_{n+m})| \to 0$ *as* $n \to \infty$.

Let $(X, d_{\mathbb{H}})$ be a quaternion-valued metric space where $\mathbb{H}$ is the skew field of quaternion number q, i.e.,

$$\mathbb{H} = \{x_0 + x_1 i + x_2 j + x_3 k : (x_0, x_1, x_2, x_3) \in \mathbb{R}^4\}.$$

Define

$$\mathcal{P}_{\mathbb{H}} = \{x_0 + x_1 i + x_2 j + x_3 k : x_0 \geq 0, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}.$$

It is apparent that $\mathcal{P}_{\mathbb{H}} \subset \mathbb{H}$. Assume $\mathbf{0}_{\mathbb{H}}$ be the zero of $\mathbb{H}$ from now on. Note that $(\mathbb{H}, |.|)$ is a real Banach space.

Lemma 16.16. $\mathcal{P}_{\mathbb{H}}$ *is a normal cone in real Banach space* $(\mathbb{H}, |.|)$.

Lemma 16.17. *Any quaternion valued metric space* $(X, d_{\mathbb{H}})$ *is a cone metric space.*

Lemma 16.18. *A sequence* $\{x_n\}$ *in* $(X, d_{\mathbb{H}})$ *be convergent in the context of quaternion valued metric space if and only if* $\{x_n\}$ *be convergent in the setting of cone metric space.*

17. Multiplicative Metric Spaces

Definition 17.33 ([14]). *Let* X *be a nonempty set. An operator* $d* : X \times X \to [1, \infty)$ *is a multiplicative metric (MM for short) on* X*, if it satisfies:*

(m_1*) $d*(x,y) \geq 1 \ \forall x, y \in X$, *and*
$d*(x,y) = 1$ *if and only if* $x = y$,
(m_2*) $d*(x,y) = d*(y,x)$ *for all* $x, y \in X$,
(m_3*) $d*(x,z) \leq d*(x,y) \cdot d*(y,z)$ *for all* $x, y, z \in X$, *(multiplicative triangle inequality).*

If the operator $d*$ *satisfies* $(m_1*) - (m_3*)$ *then the pair* $(X, d*)$ *is called a multiplicative metric space (MMS).*

$$\ln(\max\{a, b\}) = \max\{\ln a, \ln b\}$$

for all $a, b > 0$ as well as

$$e^{\max\{a,b\}} = \max\{e^a, e^b\}$$

for all $a, b \in \mathbb{R}$.

Theorem 17.8 ([34]). $(X, d*)$ *is an MMS if and only if* $(X, \ln d*)$ *is an S-MS, that is,* (X, d) *is an S-MS if and only if* (X, e^d) *is an MMS.*

18. Perturbed Metric Space

Definition 18.34 ([48]). *For a nonempty set X, a function $D : X \times X \to [0, \infty)$ is called perturbed metric (PM) with respect to $R : X \times X \to [0, \infty)$ if*

$$d = D - R : X \times X \quad \text{such that} \quad d(x, u) \mapsto D(x, u) - R(x, u)$$

forms a usual metric over X. More precisely, for any $u, z \in X$, the upcoming statements are provided:

- *(i)* $(D - R)(q, u) \geq 0$,
- *(ii)* $(D - R)(q, u) = 0$ *if and only if* $q = u$,
- *(iii)* $(D - R)(q, u) = (D - R)(u, q)$,
- *(iv)* $(D - R)(q, u) \leq (D - R)(q, z) + (D - R)(z, u)$.

In addition, $R : X \times X$ shall be named perturbed mapping, where $d = D - R$ is a standard metric. The triple (X, D, R) is reserved for perturbed metric space.

The topological basics of perturbed metric space are given in the next definition.

Definition 18.35 ([48]). *Let $\{x_n\}$ be a sequence in perturbed metric space (X, D, R) T be a self-mapping on perturbed metric space (X, D, R).*

- *(i) A sequence $\{x_n\}$ is a perturbed convergent or p-convergent in perturbed metric space (X, D, R) if $\{x_n\}$ converges in the corresponding standard metric space (SMS) (U, d), with $d = D - R$.*
- *(ii) A sequence $\{x_n\}$ in (X, D, R) is called perturbed Cauchy or p-Cauchy if $\{x_n\}$ forms a Cauchy sequence with respect to standard metric space (X, d).*
- *(iii) A triple (X, D, R) is called a complete perturbed metric space (in short, Cperturbed metric space) if the corresponding MS (X, d) is complete.*
- *(iv) A map T is called perturbed continuous or p-continuous with respect to perturbed metric space (X, D, R) if the same mapping T is continuous within the standard metric space (X, d).*

Example 18.30 ([48]). *Let $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by*

$$D(x, y) = |x - y| + \frac{x^2 y^4}{2}, \quad x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping

$$P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x, y) = \frac{x^2 y^4}{2}, \quad x, y \in \mathbb{R}.$$

In this case, the exact metric is the mapping $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by

$$d(x, y) = |x - y|, \quad x, y \in \mathbb{R}.$$

Remark that D is not a metric on X. This can be easily seen observing that $D(1, 1) = 1 \neq 0$.

Example 18.31 ([48]). *Let $D : C([0, 1]) \times C([0, 1]) \to [0, \infty)$ be the mapping defined by*

$$D(f, g) = \int_0^1 |f(t) - g(t)|\, dt + (f(0) - g(0))^2, \quad f, g \in C([0, 1]),$$

where $C([0, 1]) = \{f : [0, 1] \to \mathbb{R} \mid f \text{ is continuous on } [0, 1]\}$. Then D is a perturbed metric on $C([0, 1])$ with respect to the perturbed mapping

$$P : C([0, 1]) \times C([0, 1]) \to [0, \infty)$$

given by

$$P(f, g) = (f(0) - g(0))^2, \quad f, g \in C([0, 1]).$$

In this case, the exact metric is the mapping $d : C([0,1]) \times C([0,1]) \to [0,\infty)$ *defined by*

$$d(f,g) = \int_0^1 |f(t) - g(t)|\,dt, \quad f,g \in C([0,1]).$$

Remark that D *is symmetric and* $D(f,g) = 0$ *if and only if* $f = g$*. However,* D *is not a metric on* $C([0,1])$*. Namely, consider three constant functions* $f_1 \equiv C_1$, $f_2 \equiv C_2$, $f_3 \equiv C_3$*. Then*

$$D(f_1,f_3) = |C_1 - C_3| + (C_1 - C_3)^2,$$

$$D(f_1,f_2) = |C_1 - C_2| + (C_1 - C_2)^2,$$

and

$$D(f_2,f_3) = |C_2 - C_3| + (C_2 - C_3)^2.$$

In particular, for $(C_1, C_2, C_3) = (0, \frac{1}{2}, 1)$*, we get*

$$D(f_1,f_3) = 2, \quad D(f_1,f_2) = \frac{3}{4}, \quad D(f_2,f_3) = \frac{3}{4},$$

which yields

$$D(f_1,f_3) > D(f_1,f_2) + D(f_2,f_3).$$

Example 18.32 ([48])**.** *Let* $D : \mathbb{N} \times \mathbb{N} \to [0,\infty)$ *be the mapping defined by*

$$D(n,m) = (n-m)^2, \quad n,m \in \mathbb{N}.$$

Then D *is a perturbed metric on* $\mathbb{N}$*, where the perturbed mapping*

$$P : \mathbb{N} \times \mathbb{N} \to [0,\infty)$$

is given by

$$P(n,m) = (n-m)^2 - |n-m|, \quad n,m \in \mathbb{N},$$

and the exact metric $d : \mathbb{N} \times \mathbb{N} \to [0,\infty)$ *is given by*

$$d(n,m) = |n-m|, \quad n,m \in \mathbb{N}.$$

Remark that D *is not a metric on* $\mathbb{N}$*, but it is a b-metric on* $\mathbb{N}$*.*

19. Suprametric

Very recently, another new distance notion, so-called suprametric, was defined by Berzig [17, 18].

Definition 19.36. *Let* X *be a nonempty set. A function* $d : X \times X \to [0,\infty)$ *is called suprametric if for all* $x,y,z \in X$ *the following properties hold:*

- (d_1^*) $d(x,y) = 0$ *if and only if* $x = y$,
- (d_2^*) $d(x,y) = d(y,x)$,
- (d_3^*) $d(x,y) \le d(x,z) + d(z,y) + \rho d(x,z)d(z,y)$ *for some constant* $\rho \in \mathbb{R}^+$.

A suprametric space is a pair (X,d)*, where* X *is a nonempty set and* d *is a suprametric.*

Example 19.33 ([26])**.** *Let* $X = \{1,2,3\}$ *and* $d : X \times X \mapsto [0,\infty)$ *fulfilling* (d_1) *and* (d_2) *such that* $d(1,2) = 1$, $d(1,3) = 3$ *and* $d(2,3) = 1$*. Then* (X,d) *is not a metric space since* $d(1,3) > d(1,2) + d(2,3)$*, but is a suprametric space for* $\rho = 1$*.*

Clearly, every metric serves as a suprametric; however, there are various methods for constructing a suprametric from a metric without including the triangle inequality overall.

Example 19.34 ([17, 26]). *If (X, d) is a metric space, then the functions*

$$d_\alpha(x, y) = d(x, y)(d(x, y) + \alpha),$$
$$d_\beta(x, y) = \beta(e^{d(x,y)} - 1)$$

for any $x, y \in X$ are suprametrics on X with $\rho = \frac{2}{\alpha}$ and $\rho = \frac{1}{\beta}$, respectively. While the function

$$d_\gamma(x, y) = e^{-\gamma d(x,y)^2} - 1$$

for any $x, y \in X$ is a suprametric with a constant $\rho = 1$.

Observe that if (d_3^*) is satisfied for a particular $\rho > 0$, it will also be true for any larger value of ρ. The concepts of convergent and Cauchy sequences, as well as the continuity of mappings, are presented similarly to their corresponding terms in metric space

Definition 19.37 ([17, 18]). *Let (X, d) be a suprametric space and (x_n) be a sequence in X. The sequence (x_n) converges to $x \in X$ if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_0) < \varepsilon$ for all $n \geq n_0$. Furthermore, if for all $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n \geq n_0$, then (x_n) is a Cauchy sequence in X. If any Cauchy sequence is convergent in a suprametric space (X, d), then (X, d) is a complete suprametric space.*

Definition 19.38 ([17, 18]). *Let (X, d) be a suprametric space. A mapping $T : X \mapsto X$ is continuous at a point $x \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(Tx, Ty) < \varepsilon$ whenever $d(x, y) < \delta$. If T is continuous at every point of X, then it is a continuous mapping on a suprametric space (X, d).*

Definition 19.39 ([19]). *Let X be a nonempty set, $d\colon X \times X \to [0, +\infty)$ be a function and $b \geq 1$ and $c \geq 0$ be real constants. Then d is called strong b-suprametric if for all $x, y, z \in X$ the following properties hold:*

- $(s1)$ $d(x, y) = 0$ *if and only if* $x = y$,
- $(s2)$ $d(x, y) = d(y, x)$,
- $(s3)$ $d(x, y) \leq d(x, z) + bd(z, y) + cd(x, z)d(z, y)$.

A pair (X, d) is called strong b-suprametric (or sb-suprametric) space.

Proposition 19.2 ([17]). *The topology τ is Hausdorff.*

Proof. Let $x, y \in X$ with $x \neq y$ and $r = d(x, y)$. Denote $U = B(x, \frac{r}{2})$ and $V = B(y, \frac{r}{2+\rho r})$. We shall show that $U \cap V = \emptyset$. If not, there exists $z \in U \cap V$, so from $d(x, z) < \frac{r}{2}$ and $d(y, z) < \frac{r}{2+\rho r}$, we obtain

$$r = d(x, y) \leq d(x, z) + d(z, y) + \rho d(x, z)d(z, y) < \frac{r}{2} + \frac{r}{2 + \rho r} + \rho \frac{r}{2}\frac{r}{2 + \rho r} = r,$$

which is absurd, so $U \cap V = \emptyset$ and therefore X is Hausdorff. □

20. Conclusion

Listing all abstract distance structures in this survey is quite challenging. Why? The extent of our understanding is determined by the creativity of the researchers. There have consistently been fresh methods and innovative concepts to enhance, refine, and broaden the current ones. Conversely, there are numerous unique hybrid structures. For example, a b-metric may be regarded as a partial b-metric, cone b-metric, quasi b-metric, G_b-metric, etc. This method generates numerous different combinations, and it is not necessary to list all since we can deduce how one can obtain such hybrid spaces.

The evolution of the 2-metric occurred as follows: First, the D-metric was introduced to make the 2-metric function continuous. Later, the G-metric was defined by criticizing the lack of some technical topological properties in the D-metric concept. As a result, the notion of G-metric became the most advanced version of 2-metric. Indeed, G-metric was the perfect version of this trend. But, it was understood that under certain condition it is equivalent to quasi-metric [47, 82]. After these discussion, the notion of S-metric was defined but the destiny of it was not different from G-metric [85].

The historical advances of cone metric space or Banach-valued spaces is in the following way. First of all, TVS-valued metric was defined as a natural generalization [35]. It was realized that its composition with a scalarization function can be equivalent to the standard metric [3, 11, 35, 36, 46, 49, 61]. As a continuation of this trend, the notion of complex-valued metric, quaternion valued metric, b-cone metric, C^*-algebra valued metric were defined. Naturally, the novelty of these notions were discussed in [5, 8, 10, 52] and some related references therein. For the multiplicative metric, the equivalence of it with the standard metric with a simple transformation was given in [34]. Another discussion for the novelty of bipolar metric was given in [59]. Very clearly, quasi-metric space is contained by standard metric space and standard metric space lies in an strong b-metric space and it is covered by an interpolative metric space. It was clear from [58] that interpolative metric space was covered by b-metric space. Clearly, b-metric space lies in extended b-metric space. One can easily show that extended metric space and supra-metric space lie in perturbed metric space. It is also clear that ultra-metric space lies in the standard metric space. The relation of ultra-metric space and quasi-metric space is needed to be investigated. It can be considered as an open problem.

As a result, there are a number of abstract distance structures. We aim to put some significant examples of them with known relations. This may helps to researchers to use the proper one for solving their own problems.

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Erdal Karapinar
Atilim University
Department of Mathematics
06830, İncek, Ankara, Türkiye
Email address: erdalkarapinar@yahoo.com

Ravi P. Agarwal
Florida Institute of Technology
Department of Mathematics and Systems Engineering
Melbourne, FL, USA
Email address: agarwalr@fit.edu

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Research Article

On intrinsic dimension of point clouds by a persistent homology approach: Computational tips

CINZIA BANDIZIOL, STEFANO DE MARCHI*, AND MICHELE ALLEGRA

ABSTRACT. We present new results on estimating the intrinsic dimension (ID) of point clouds using persistent homology. In particular, we compare topological ID estimators with different approaches, comprehensively assessing their strengths and weaknesses. We show that a combination of the so-called i-dimensional persistent homology fractal dimension estimator and the persistent homology dimension, which we termed i-dimensional α persistent homology fractal dimension, is a suitable choice for obtaining an effective estimation of the ID in many benchmark datasets.

Keywords: Topological data analysis, persistent homology, intrinsic dimension.

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1. INTRODUCTION

Due to the increasing availability of large datasets, developing principled and effective approaches to compress the information they contain is becoming a central issue in all applied sciences. In this context, a key role is played by dimensionality reduction approaches that reduce the number of variables (coordinates) in the data with little or no loss of information [8, 33, 60]. This step is justified because, although the data are initially defined in a high-dimensional space, they typically lie on a lower-dimensional manifold. This dimension, corresponding to the minimum number of local coordinates needed to describe the data, is called Intrinsic Dimension (shortly, ID). Knowing the value of the ID is critical to ensure the reliability of low-dimensional data visualization [61] and the validity of dimensionality reduction as a data preprocessing step [43]. In addition, the ID is often a valuable metric in its own right, allowing the analyst to capture key information about the data geometry [2, 6, 39, 46], compare data and models [23], and track temporal variations in complexity [4, 9]. Yet, the ID is generally not known a priori, calling for methods to obtain ID estimates directly from the data.

A wide variety of ID estimation techniques have been advanced in the literature [13, 14]. A majority of the proposed methods fall into one of two categories: projective and geometric-statistical. Projective methods try to project the data onto a space of dimension D, and assess the quality of the projection (in terms of its ability to retain key characteristics of the original dataset) as a function of D [62]. The prototype for these methods is principal component analysis (PCA) [34], which projects the data onto the linear subspace spanned by the first D eigenvectors of the covariance matrix, and uses the fraction of variance within this subspace as a quality metric. A major limitation of projective methods is that they yield a clear ID estimate only when the quality metric exhibits a clear drop below a given D. Typically, this does

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*Corresponding author: Stefano De Marchi; stefano.demarchi@unipd.it
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not occur, and the ID is fixed (rather than estimated) by searching for an optimal compromise between quality and compression (e.g., by retaining the D components that explain 95% of the variance in PCA).

Geometric-statistical methods build on the fact that, under broad assumptions, distances between points in the dataset follow statistical relations that depend parametrically on the ID. The prototypes of this class are methods developed in the 1980s to characterize the dimension of strange attractors in dynamical systems. For instance, the correlation dimension [28] is based on the fact that the number of points within small neighborhoods of radius r around a given point scales as $N_r \sim r^D$ for $r \ll 1$, with D being the intrinsic dimension, so that D can be estimated as $\lim_{r\to 0} \frac{N_r}{r}$. More advanced methods in this class follow a similar logic, but make specific assumptions about the probability distribution of the data, such as local uniformity [22, 37] or isotropy [14, 21]. The main limitation of these methods is that they typically require a large number of data points when the ID is high (a facet of the well-known curse of dimensionality problem), and may fail in the presence of highly non-uniform or non-isotropic distributions.

Following the recent surge of topological data analysis [15, 16, 17], several authors have explored topological approaches to ID estimation [1, 63], that do not fall within the two classes defined above, as they are insensitive to the particular metric chosen to define distances in the data set. In principle, these methods might provide more robust estimates by overcoming some of the difficulties previously listed. Topological ID estimators are based on persistent homology, a popular method for computing topological features of a space at different resolutions [18, 19, 20].

In this work, we compare topological ID estimators with alternative methods to provide a comprehensive assessment of their relative strengths and weaknesses. Upon comparing well-known and well-characterized benchmark datasets, we will focus on real data. One of the fields where ID estimation has risen to prominence in recent years is neuroscience, where it can be used to estimate the dimension of neural activity. The responses of N recorded neurons across time span a neural manifold embedded in the N-dimensional configuration space of all neurons [32]. While artificial neural networks trained to replicate real brains often exhibit low-dimensional activity [56], biological neural activity is typically high-dimensional [5, 55], which has sparked a wide debate. Here, we will consider an artificial network trained on simple tasks mimicking those performed by macaques in decision-making experiments [64], and compare the ID of the neural manifold as assessed by traditional methods and topological methods.

The paper is organized as follows: in Section 2, we introduce the basic definitions related to persistent homology, in Section 3, we describe the meaning of Intrinsic Dimension (ID), its importance, and common ways to compute, and we introduce and explain the actual estimators of ID using PH Section 4 collects all numerical tests that we have run. Finally, in Section 5, we draw some conclusions and discuss some future developments.

2. Persistent Homology: Basic Definitions

Persistent homology (PH) is now widely known and used. Comprehensive treatments are covered in recent textbooks on topological data analysis, such as [16, 17]. Here, we limit ourselves to a brief recapitulation.

Let X be a topological space (for all practical purposes, this can be assumed to be a manifold). The k-th homology group X, $H_k(X)$, consists of the k-dimensional holes of X. The number of connected components (0-dimensional holes), cycles (1-dimensional holes), cavities/voids (2-dimensional holes), and higher-order holes characterize the intrinsic topology space X, providing a qualitative summary of it. In practical applications, one does have access to X, but only to a set of points $\mathcal{X} = \{\mathbf{x}_i\}_{i=1,\dots,N} \subset X$. Persistent homology tries to characterize the X by analyzing the homology of simplicial complexes built on $\mathcal{X}$. Let us briefly recall the basic concepts of simplicial homology.

Definition 2.1. *A simplicial complex $\mathcal{K}$ consists of a set of simplices of different dimensions so that every face of a simplex $\Delta \in \mathcal{K}$ belongs to $\mathcal{K}$ and the non-empty intersection of any two simplices $\Delta_1, \Delta_2 \in \mathcal{K}$ is a face of both Δ_1 and Δ_2.*

Given a simplicial complex $\mathcal{K}$, let $\mathcal{S}_k(\mathcal{K})$ denote the set of k-dimensional simplices of $\mathcal{K}$.

Definition 2.2. *An integer valued k-dimensional chain is a linear combination of k-simplices of $\mathcal{K}$ with coefficients in $\mathbb{Z}$,*

$$c = \sum_i a_i \Delta_i\,, \quad \Delta_i \in \mathcal{S}_k(\mathcal{K}), \quad a_i \in \mathbb{Z}\,. \tag{2.1}$$

Let $\mathcal{C}_k(\mathcal{K})$ denote the set of integer-valued k-dimensional chains of $\mathcal{K}$, which is a group under the operation of addition.

Definition 2.3. *The boundary operator $\partial_k : \mathcal{S}_k \to \mathcal{C}_{k-1}$ maps an oriented simplex $\Delta \in S_k(K)$ into the $(k-1)$-dimensional chain*

$$\partial_k \Delta = \sum_{i=0}^{k} (-1)^k \sigma_i, \tag{2.2}$$

where σ_i is the $(k-1)$-face obtained by removing the i-the vertex of the simplex (with vertex order fixed by the orientation. The boundary operator can be extended by linearity to a general element of $C_k(\mathcal{K})$, obtaining a map $\partial_k : C_k(\mathcal{K}) \to C_{k-1}(\mathcal{K})$.

Definition 2.4. *The kernel of ∂_k is the group of k-cycles, $Z_k(\mathcal{K}) := \ker(\partial_k)$. The image of ∂_{k+1} is the group of k-dimensional boundaries, $B_k(\mathcal{K}) := \mathrm{im}(\partial_{k+1})$. Finally, the quotient group $H_k(\mathcal{K}) = Z_k(\mathcal{K})/B_k(\mathcal{K})$ is the k-homology group of $\mathcal{K}$. The generators of H_k are called homology classes.*

Simplicial complexes can be built on $\mathcal{X}$ by forming simplexes with all points below a certain distance, as follows.

Definition 2.5. *Let $(\mathcal{X}, d)$ denote a finite metric space. The Vietoris-Rips complex for $\mathcal{X}$, associated to a parameter ϵ and denoted by $\mathcal{V}_\epsilon(\mathcal{X})$, is the simplicial complex where the following hold.*

i) *$\mathcal{X}$ forms the vertex set,*
ii) *any subset $\{\boldsymbol{x}_0, \dots, \boldsymbol{x}_k\} \in \mathcal{X}$ spans a k-simplex if and only if $d(\boldsymbol{x}_i, \boldsymbol{x}_j) \leqslant 2\epsilon$ for all $0 \leqslant i, j \leqslant k$.*

Persistent homology analyzes nested sequences of simplicial complexes arising at increasing values of ϵ, trying to identify the topological features (homology classes) that persist across a wide range of values of ϵ.

Definition 2.6. *Let $0 < \epsilon_1 < \cdots < \epsilon_l$ be an increasing sequence of real numbers. A filtration is the sequence of sets*

$$\emptyset \subset \mathcal{K}_1 \subset K_2 \subset \cdots \subset \mathcal{K}_l \tag{2.3}$$

with $\mathcal{K}_i = \mathcal{V}_{\epsilon_i}(\mathcal{X})$.

Definition 2.7. *The p-persistent homology group of K_i is the group*

$$H_k^{i,p} = Z_k^i/(B_k^{i+p} \cap Z_k^i).$$

This group contains all stable homology classes in the interval $[i, i+p]$, that is, classes born before step i which are still alive after p steps.

Definition 2.8. *Let γ be a homology class in $H_p(\mathcal{K}_i)$. We say that γ is born at the instant i if $\gamma \notin H_k^{i-1,1}$, i.e. it cannot be identified with a previously existing class in $H_p(\mathcal{K}_{i-1})$. We say that a homology class born at $\mathcal{K}_i$ dies at $\mathcal{K}_{i+p}$ if $\gamma \in H_k^{i,p-1}$ and $\gamma \notin H_k^{i,p}$. Then p is called the persistence of γ.*

Notice that highly persistent homology classes typically correspond to topological features of $\mathcal{X}$ (cf. [3]). Hence, during the filtration process, homology classes thus appear and disappear. We can represent classes in $\mathbb{R}_+^2 = \mathbb{R}_{\geq 0} \times \{\mathbb{R}_{\geq 0} \cup \{+\infty\}\}$ by assigning the point (i, j) to a class born at $\mathcal{K}_i$ and died at $\mathcal{K}_j$ (j can take the value $+\infty$, since some features can be alive up to the end of the filtration). Since there can be several independent classes born at $\mathcal{K}_i$ and died at $\mathcal{K}_j$, then each point (i, j) has a multiplicity, say $\mu_{i,j}$.

The collections of points (i, j) together with their multiplicity is called a Persistence Diagram (PD). Figure 1 is an example of a PD collecting features (homology classes) of dimension 0 (in blue), 1 (in orange), and 2 (in green). Points close to the diagonal represent features with a short lifetime (usually associated with noise) while features away from the diagonal are stable topological features.

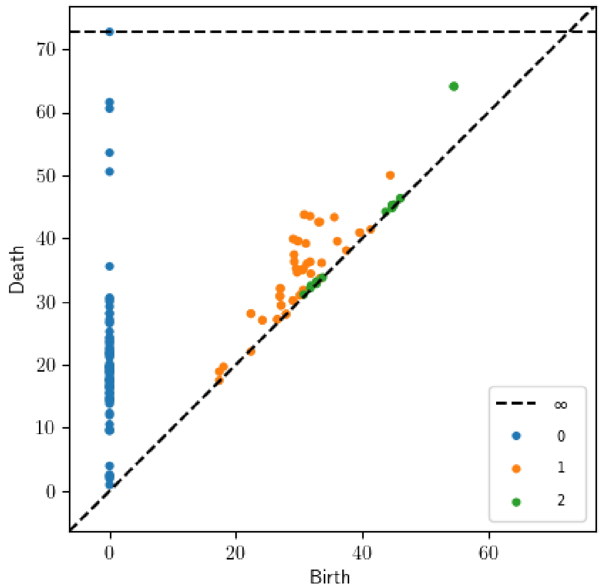

FIGURE 1. A persistence diagram with classes of dimension 0, 1 and 2. At the top of the plot, there is a dashed line that indicates infinity and allows for plotting also couples as $(i, +\infty)$

3. INTRINSIC DIMENSION ESTIMATION

The minimum number of parameters required to account for the observed properties of data is the intrinsic (or effective) dimension of the data set. The concept of Intrinsic Dimension (ID) has become central in all fields dealing with a large amount of data characterized by a high number D of features, such as machine learning [13]. Estimating the ID, which counts the number of essential and fundamental features $d \ll D$, is crucial for uncovering the data structure and simplifying analysis, enabling data compression. Following [13], we can define the ID of a dataset as the minimal dimension of a manifold preserving the information contained in the dataset.

Definition 3.9. *A dataset $\Omega \subset \mathbb{R}^D$ is said to have Intrinsic Dimension (ID) d if its elements lie entirely, without information loss, within a d-dimensional manifold $\mathcal{M}$ of $\mathbb{R}^D$, where $d \ll D$.*

3.1. **Common Fractal ID Estimators.** Estimating the ID from data is a challenging task. While the first ID estimation algorithm dates back to 1969 [13], the literature now includes a wide variety of estimators (cf. e.g. [14, 21, 22]). Among popular methods, we recall linear estimators, as the PCA, and the non-linear ones, such as the kernel PCA. While linear estimators are based on linear mappings to a lower-dimensional space of the data, they are fast and suitable for data that are "quasi" linearly distributed, but they often overestimate the ID of the manifold. The non-linear ones work better with more complex data, providing learning techniques that are particularly efficient because they preserve the geometric structure of the original feature space. Unfortunately, they are usually more computationally expensive (see, for instance, the recent paper [44]). A new class is the so-called fractal-based estimators which rely on the key idea that distances between points lying on a fractal or a low-dimensional manifold follow scaling laws that depend on the ID [1]. That is, the volume of the d-dimensional balls of radius ϵ grows proportional to ϵ^d.

The most famous fractal methods are the box-counting and the correlation dimension. The box-counting dimension is based on the number of boxes needed to cover a data set. Let $\mathcal{X}_N \subset \mathbb{R}^D$ be a set of N points of $\mathbb{R}^D$, considered as a metric space. Let $N(\epsilon)$ be the number of boxes (hypercubes) of size ϵ needed to cover $\mathcal{X}_N$. By scaling the box size ϵ, one generally obtains a power-law scaling of $N(\epsilon)$ [1]:

$$N(\epsilon) \propto \epsilon^{-d_{BC}}. \tag{3.4}$$

Definition 3.10. *Let $\mathcal{X}_N$ be a subset of $\mathbb{R}^D$, and let N_ϵ denote the infimum of the number of closed balls of radius ϵ required to cover $\mathcal{X}_N$. The Box-Counting Dimension of Ω is*

$$d_{BC} = \lim_{\epsilon \to 0} \frac{\log(N_\epsilon)}{\log(1/\epsilon)} \tag{3.5}$$

provided this limit exists.

If $\mathcal{X}_N$ is an I.I.D. sample of N points from a regular metric μ on $\mathbb{R}^D$, then $\lim_{N\to\infty} d_{BC} = d$. Another fractal ID estimator that has become one of the most common and widely used is the correlation dimension (CD). It describes how the number of points within a certain distance (or radius) scales with the radius as it increases.

Mathematically, it is defined using the correlation integral of a given measure μ, which is the probability that pairs of points in a dataset are within a certain distance, say ϵ, of each other [1]:

Definition 3.11. *Let $\Omega \subseteq \mathbb{R}^D$ be equipped with a measure μ. Given $\epsilon > 0$, the correlation integral of μ is defined as:*

$$C(\epsilon) := \mathbb{E}_{X\sim\mu, Y\sim\mu}\Big[\int_0^\epsilon dr\delta(r - ||X - Y||)\Big] = \mathbb{E}_{X\sim\mu, Y\sim\mu}\big[H(\epsilon - ||X - Y||)\big]. \tag{3.6}$$

where $||\cdot||$ is a norm (typically, the Euclidean norm), $\delta(x)$ is the delta function, and H is the Heaviside step function ($H(x) = 0$ for $x < 0$, $H(x) = 1$ for $x \geq 0$). The CD is defined as:

$$d_C := \lim_{\epsilon \to 0} \frac{\log(C(\epsilon))}{\log(\epsilon)}. \tag{3.7}$$

The idea behind this definition is that, as $\epsilon \to 0$, the correlation integral scales as

$$C(\epsilon) \propto \epsilon^{d_C} \tag{3.8}$$

If μ is an absolutely continuous measure on Ω, then $d_C \approx d$. The CD can be estimated from a finite set of points. Let $\mathcal{X}_N$ be an I.I.D. sample of N points from μ. Let us count the number of pairs of points within distance ϵ,

$$\tilde{C}(N,\epsilon) := \sum_{\substack{x_i, x_j \in \mathcal{X}_N \\ x_i \neq x_j}} H(\epsilon - \|x_i - x_j\|). \tag{3.9}$$

We then have (cf. e.g. [57]):

$$C(\epsilon) = \lim_{N\to\infty} \tilde{C}(N,\epsilon). \tag{3.10}$$

In applications, given a finite (large) set of points χ_N, the CD can be estimated as the slope of the log-log plot of $\tilde{C}(N,\epsilon)$ versus ϵ in the limit of small ϵ.

3.2. **ID Estimators Using Persistent Homology.** Several PH-based estimators have been proposed in the literature. For instance, the authors in [1] introduced the i-Dimensional Persistent Homology Fractal Dimension. In contrast, in [31] they introduced the Persistent Homology Dimension, and the Persistent Homology Complexity.

Assume we have a probability measure μ with support on $\mathcal{M} \subset \mathbb{R}^D$. Let $\mathcal{X}_N$ denote an I.I.D. sample from μ. Let $\mathcal{K}(\mathcal{X}_N)$ denote a Vietoris-Rips complex on $\mathcal{X}_N$. Let γ be a k-dimensional topological feature (persistent homology group generator), i.e., $\exists i,\ \gamma \in H_k^i(\mathcal{K}(\mathcal{X}_N)$, and let $|\gamma|$ denote its persistence.

Definition 3.12. *For any $\alpha > 0$,*

$$E_\alpha^k(\mathcal{X}_N) := \sum_{\exists i,\ \gamma\in H_k^i(\mathcal{K}(\mathcal{X}_N))} |\gamma|^\alpha. \tag{3.11}$$

The first PH-based ID estimator proposed in the literature is the k-dimensional Persistent Homology Fractal Dimension (cf. [1]):

Definition 3.13. *The k-dimensional Persistent Homology Fractal Dimension (k-PHFD) of μ is given by*

(3.12)

$$dim_{PH}^k(\mu) := \inf_{d>0} \left\{ \exists\, C(k,\mu,d)\ :\ E_1^k(\mathcal{X}_N) \le CN^{(d-1)/d} \text{ with probability 1 as } N \to +\infty \right\}.$$

This definition says to us that the dimension may depend on the choice of the filtered simplicial complex (in our case, the Vietoris-Rips) and on the choice of the coefficients for homology computations. Although a stringent analytical proof is still lacking, numerical tests brought the authors to the following

Conjecture 3.1. *Let μ be a nonsingular probability measure on a compact set $X \subseteq \mathbb{R}^D,\ D \ge 2$. Then, for all $0 \le k < D$, there is a constant $C(k,\mu,\ D) \ge 0$ such that*

$$E_1^k(\mathcal{X}_N) = CN^{(d-1)/d} \tag{3.13}$$

with probability 1 as $N \to \infty$.

Assuming the validity of this conjecture, taking the logarithm in (3.13), we get

$$\log(E_1^i(\mathcal{X}_N)) = \log(C) + \frac{d-1}{d}\log(N)\,, \tag{3.14}$$

which suggests that we can estimate d from the scaling of $\log(E_1^i(\mathcal{X}_N))$. Practically, an estimate of d is obtained by performing a linear regression of $(\log(E_1^i(\mathcal{X}_N))$ as a function of $\log(N))$, taking the slope as an estimate of $\frac{d-1}{d}$, and finally obtaining d through a simple inversion. In applications, the value d can then be inferred from this log-log plot. Building on this work, in [31] has been introduced the Persistent Homology Dimension (PHD).

Definition 3.14. *Let X and μ be as above. For each $k \in \mathbb{N}$ and $\alpha > 0$, the Persistent Homology Dimension (PHD) is*

$$dim_{PH_k^\alpha}(\mu) = \frac{\alpha}{1-\beta} \tag{3.15}$$

where

$$\beta = \limsup_{N \to +\infty} \frac{\log(\mathbb{E}(E_\alpha^k(\mathcal{X}_n)))}{\log(N)}. \tag{3.16}$$

In practice, it is not so obvious how to treat the expectation value. But analyzing this estimator more carefully, it appears to be an "extension" of k-PHFD. Observing the numerical results in [31], it is clear that the parameter α generally increases the global performance and, consequently, the estimation.

Inspired by the fact that the parameter α gives to the PHD some advantages over i-PHFD, we propose to combine both definitions, providing an estimator that we call i-dimensional α Persistent Homology Fractal Dimension, that can easily be defined.

Precisely, the i-dimensional α Persistent Homology Fractal Dimension (i-α-PHFD) of μ is given by

(3.17)
$$dim_{PH}^{i,\alpha}(\mu) = \inf_{d>0} \left\{ d \mid \exists C(i,\mu,D) : E_\alpha^i(X_N) \leq CN^{(d-\alpha)/d} \text{ with probability 1 as N} \to +\infty \right\}.$$

Finally, we recall the definition of the complexity of data, known as Persistent Homology Complexity (cf. [31]). This quantity indicates when the dimension is difficult to compute using the methods presented above. The starting point is the cumulative PH_i curve F_i

$$F_i(X,\epsilon) = \#\{I \in PH_i(X) | |I| > \epsilon\}.$$

Hence, the PH_i complexity of X is

$$comp_{PH_i}(X) = \lim_{\epsilon \to 0} \frac{-\log(F_i(X,\epsilon))}{\log \epsilon}. \tag{3.18}$$

We notice that $comp_{PH_i}(\mathbb{R}^n) = 0$ for all i. For more details, see the survey [31].

4. Numerical Tests

In the present literature, no comparison has been made of the available ID estimators based on PH. Here, we systematically test and compare them across benchmark datasets and a dataset from a computational neuroscience study.

- **Benchmark Datasets**. They are commonly tested in the context of ID estimators. They consist of data sampled from "regular" manifolds. We collect all the related information in the Table below (see e.g. [14]).

Dataset	d	Description
Helix	2	2-dimensional helix in $\mathbb{R}^3$
Swiss	2	Swiss-Roll in $\mathbb{R}^3$
Sphere	3	3-dimensional sphere linearly embedded in $\mathbb{R}^4$
NonLinear	4	Nonlinear Manifold in $\mathbb{R}^8$
Affine3d5d	3	Affine space in $\mathbb{R}^5$
Mist	4	Conc. figure, mistakable with a 3-dim. one in $\mathbb{R}^6$
CurvedManifold	12	Nonlinear (highly curved) manifold in $\mathbb{R}^{72}$
NonLinear6d36d	6	Nonlinear manifold in $\mathbb{R}^{36}$

- **Fractal Dataset**. This dataset comes from the world of fractals and dynamical systems, in particular the Sierpinski and Ikeda Attractor (see Figure 4), both taken from [31]. Because of the computational burden of computing topological features, we computed 4000 points for the Sierpinski Triangle and 5000 for the Ikeda attractor.

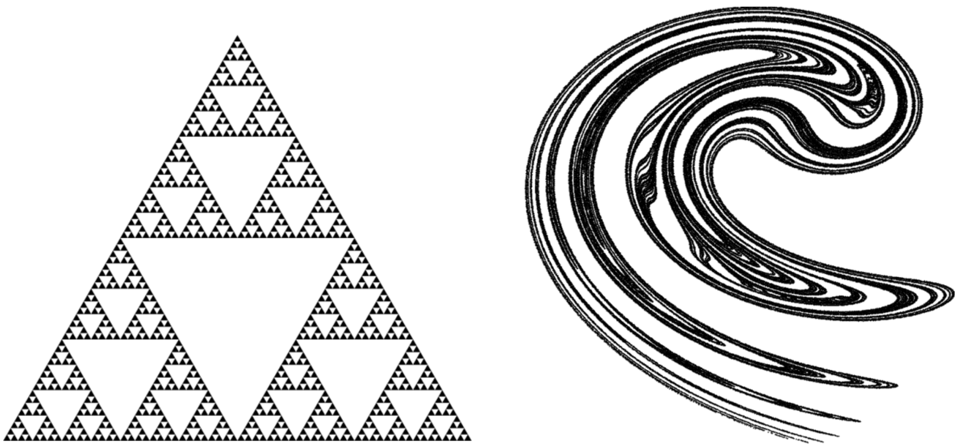

FIGURE 2. Sierpinski triangle (left) and Ikeda map (right)

- **Neural Activity Dataset**. To validate PH-based ID estimation methods, we test them on a real dataset that was extensively characterized with other ID estimation methods. In [64], the authors characterized the ID of manifolds generated by simulated neural activity. To this aim, they considered artificial recurrent neural networks (RNNs), which are frequently taken as simplified models of real brain networks. In particular, they trained RNNs to perform 20 interrelated tasks mimicking typical tasks in experiments with non-human primates and useful to understand basic cognitive processes such as working memory, inhibition, and context-dependent integration. In each task, the network receives one or two inputs or 'stimuli', both representing an angular variable or direction, and should produce an 'output', also representing a direction. In real experiments, animals are shown dots moving in a specific direction in their left visual field (input 1) and right visual field (input 2) and they should produce a motor response, typically a gaze movement (output) in a specific direction that is a function of the received stimuli. The figure illustrates how to use an RNN. We have considered only 3 stimuli, whose data are stored in .csv file with the names Fdgo, Context, and Reactgo_filtered with 25200, 10400 and 5200, respectively, in $\mathbb{R}^{256}$.

For the last examples, coming from an open field of research, the ID is a priori completely unknown. To have only a probable decent idea of which ID is expected, we have decided to compute the related Correlation Dimension that, nowadays, is indeed such a good indicator

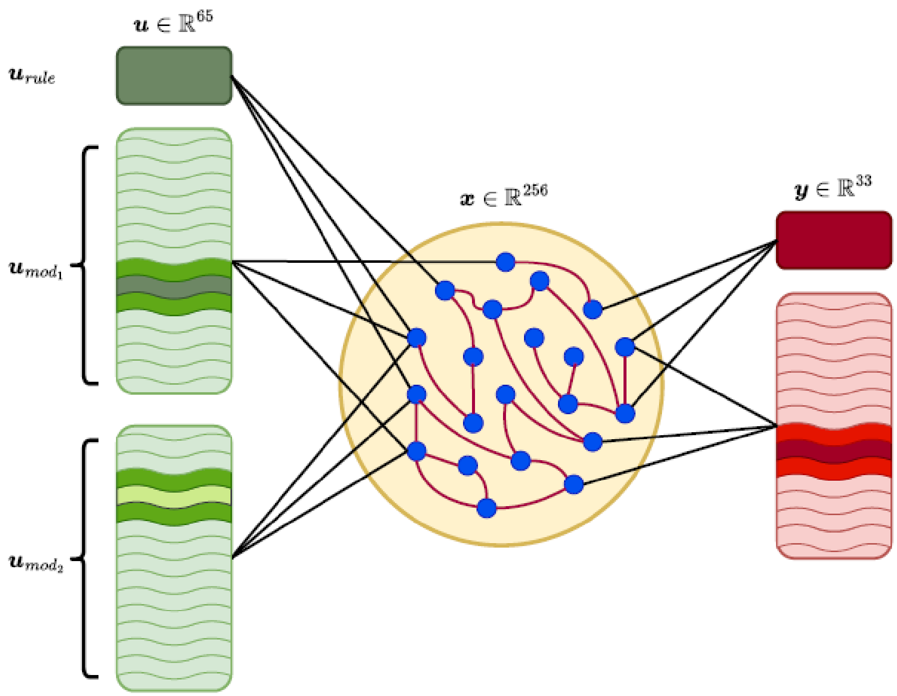

FIGURE 3. Scheme of the source of data

Dataset	d	Corr. dim.
Helix	2	1.99
Swiss	2	1.98
Sphere	3	2.98
NonLinear	4	3.87
Affine3d5d	3	3.01
Mist	4	3.54
CurvedManifold	12	11.66
NonLinear6d36d	6	5.82
Sierpinski Triangle	1.585	1.585
Ikeda	unk	1.68
Fdgo	unk	1.07
Contextdm1	unk	1.14
Reactgo	unk	2.15

TABLE 1. Correlation Dimension of all datasets

in practice. For completeness, we have finally computed it for all datasets and collected the results in Table 4, where d denotes the ID being approximated. All codes have been written in Python 3.11 and are available on the GitHub page:

`https://github.com/cinziabandiziol/Topological_ID_Estimator`

4.1. **Shape Parameter Analysis.** Taking the new definition, namely that of i-dim. α PHFD, it depends on the parameter α. Now we are interested in investigating if the choice $\alpha = 1$ turns out to be equal exactly to the i-dim. PHFD, or is there a better choice of the parameter?

We have conducted a numerical analysis and, inspired by paper [31], we consider taking α, the parameter to be tested, in the range $(0,4)$. We have chosen values not so large since the analysis in [31] has taken this direction, which has been revealed to be the most meaningful. For each value of α, we computed the related i-dim. α PHFD as follows.

(1) Compute the persistent feature, usually only of dimension $0,1$.
(2) To mirror the limit $n \to \infty$, consider some number n_k closer to the maximum n available (e.g. for the benchmark datasets $n = 10000$).
(3) Consider the points $(\log(N_k), \log(E^i_\alpha(\mathcal{X}_N)))$ and compute the linear approximation (we did by using the Python function `numpy.polyfit`): the slope is then equal to $\frac{D-1}{D}$.
(4) Make the inverse and obtain the approximated value of D.

We apply this workflow to all datasets with known IDs, and we treat approximation errors as the differences between the real ID and the approximated one. There exists, say, an α^*, such that this error is closer to or equal to zero. We call this the "optimal" one.

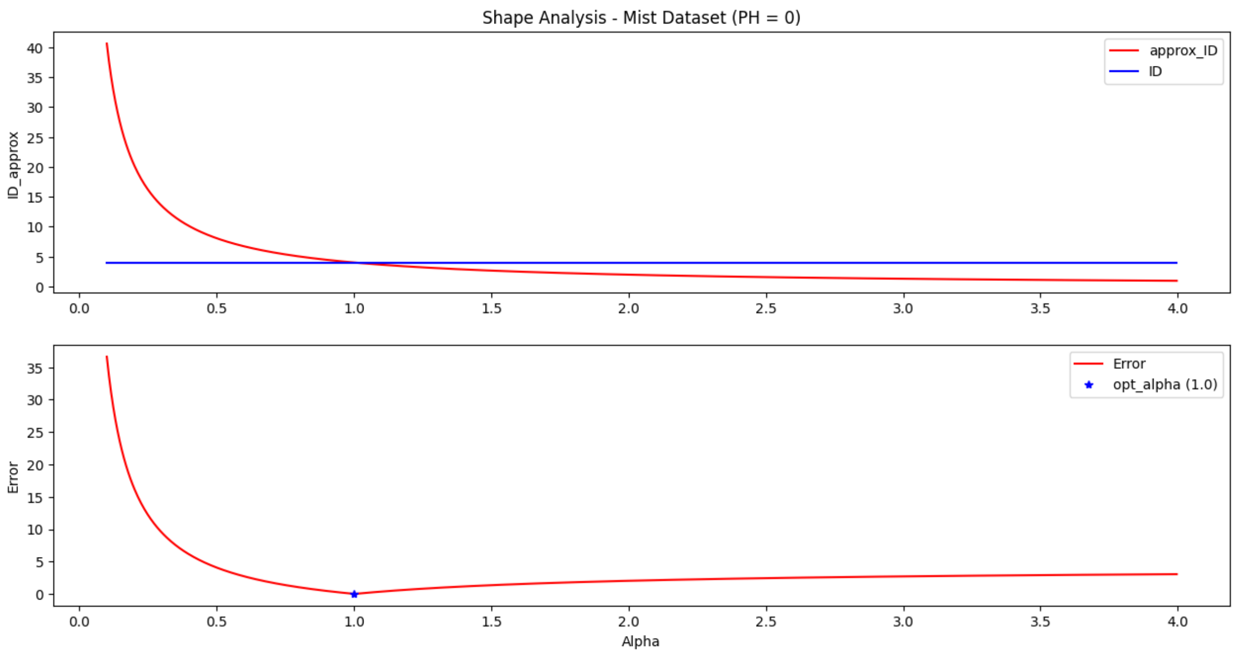

FIGURE 4. Shape Analysis for the dataset Mist with PH_0

Interestingly, experiments reveal without any doubt that, considering PH_0 and PH_1 both, the optimal value is exactly 1, revealing how the i-dim. PHFD is indeed optimal in α. We report in Figure 4.1 two plots; on the top, the approximation ID (red) and the real ID (blue); on the bottom, we plot the errors, and we mark in blue the optimal point. Notably, this behavior is observed across all benchmark datasets.

4.2. ID Estimates. We report in Table 2 the approximation of ID of all datasets using the "optimal" parameter $\alpha^* = 1$.

Remark 4.1. *Some shreds of evidence.*

- *From the results, in general, considering PH_0, the estimator performs better than on estimating PH_1. Besides, if we consider the approximation of ID for the neuroscience dataset obtained using Correlation Dimension, again, the results are not so promising. This is a point that needs more investigation and analysis.*

Dataset	d	PH_0	PH_1
Helix	2	2.01	2.38
Swiss	2	1.93	2.16
Sphere	3	2.90	3.14
NonLinear	4	3.98	6.45
Affine3d5d	3	2.84	2.91
Mist	4	4.01	6.11
CurvedManifold	12	12.73	-
NonLinear6d36d	6	5.96	9.80
Serpinski	1.58	1.61	1.87
Ikeda Attractor	1.71	2.12	2.13
Reactgo	unk	2.47	2.54
Fdgo	unk	2.14	2.17
Contextdm1	unk	3.07	3.03

TABLE 2. Computations of $0, 1$-dim. PHFD for all datasets

- *Finally, concerning the estimator $comp_{PH}$, we have tried to replicate the results obtained in* [31]. *Unfortunately, the results seem to be far from the desiderata. We have some doubts about the global definition of the estimator, since, as shown in the references, the corresponding function F should exhibit clear linear behavior over a well-defined interval. Concretely, in our application, we are not able to see it. These conclusions, of course, have sparked our curiosity to investigate the definition in depth, which represents a promising direction for future research.*

5. CONCLUSION

In this paper, we have considered the persistent homology approach as an effective tool for estimating the Intrinsic Dimension of a cloud of points. The estimation of Intrinsic Dimension is essential, for example, for quantifying the complexity of the data in terms of the minimal number of dimensions required to capture its variance. At the same time, in applications, methods that can transform the original high-dimensional data into a lower-dimensional representation are crucial. Especially in Machine Learning and Data Analysis, it can be helpful to improve model performance, reduce computation time, and mitigate the curse of dimensionality.

As claimed in Section 4, using Persistent Homology (PH) and inspired by the promising results of UMAP, we compared various estimators for the ID. In particular, we focused on i-Dimensional Persistent Homology Fractal Dimension [1], Persistent Homology Dimension and Persistent Homology Complexity [31].

We then decided to combine both definitions, to hopefully obtain another good and interesting estimator that we called i-dimensional α Persistent Homology Fractal Dimension. The choice of $\alpha = 1$ proved optimal for estimating the corresponding ID across almost all benchmark datasets and the features PH_0, PH_1. For some datasets, like Ikeda attractor, the ID estimated is far from the expected one. Also, for some datasets and using other estimators, like the CD or the $compr_{PH}$, the results are not promising. These issues should be investigated more deeply in future work.

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CINZIA BANDIZIOL
UNIVERSITY OF PADOVA
DEPARTMENT OF MATHEMATICS "TULLIO LEVI-CIVITA"
35122, PADUA, ITALY
Email address: cinzia.bandiziol@phd.unipd.it

STEFANO DE MARCHI
UNIVERSITY OF PADOVA
DEPARTMENT OF MEDICINE
35128, PADUA, ITALY
PADOVA NEUROSCIENCE CENTER
35122, PADUA, ITALY
SELÇUK UNIVERSITY
DEPARTMENT OF MATHEMATICS
SELÇUKLU, KONYA, TÜRKIYE
Email address: stefano.demarchi@unipd.it

MICHELE ALLEGRA
UNIVERSITY OF PADOVA
DEPARTMENT OF PHYSICS AND ASTRONOMY
PADOVA NEUROSCIENCE CENTER
35122, PADUA, ITALY
Email address: michele.allegra@unipd.it

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Research Article

Viscosity approximation involving generalized cocoercive mapping in Hadamard space

BASHIR ALI* AND ABDULNASIR BALA NUHU

ABSTRACT. In this paper, we introduce a new type of mapping which we term generalized cocoercive mapping in a CAT(0) space, we prove some properties of the new mapping, we also construct implicit viscosity type algorithm for approximating common solution of fixed point of (f, g)-generalized κ-strictly pseudononspreading mapping, quasi-nonexpansive mapping, family of θ_i-generalized demimetric mapping, mixed equilibrium problem and variational inequality problem involving the new mapping. Strong convergence is obtained under some mild conditions and without considering cases as in many results in the literature. Our results improved and generalized many results in the literature and our technique of proof is new and of independent interest.

Keywords: Viscosity approximation, generalized demimetric, generalized cocoercive.

2020 Mathematics Subject Classification: 47H09, 47J25.

1. INTRODUCTION

Let X be a nonempty set and $T : X \to X$ be a nonlinear map. Many physical problems can be formulated as the problem of finding

$$x \in X \text{ such that } Tx = x. \tag{1.1}$$

Fixed point of T is a point satisfying (1.1), the set of all fixed points of T is denoted by $Fix(T)$. The fixed point of the map T plays the role of an equilibrium of a system (usually of differential equations) defined in terms of the map T. The concept of equilibrium system is essential in biology, economics, noncooperative game theory, ergodic theory, physics, chemistry and so on. Therefore, fixed point theorems are related to these fields. Fixed point theory is an essential tool for establishing existence of solutions of differential equations, integro-differential equations, minimization, variational inequalities, mixed equilibrium, split feasibility problem and so on (see for example [6, 8, 12, 14, 17, 18, 19, 21, 22] and [25] for more details on fixed point theory).

Let (X, ρ) be a metric space and $x, y \in X$, a geodesic from x to y is a function $\gamma : [a, b] \subseteq \mathbb{R} \to X$ satisfying the following axioms:

(G_1) : $\gamma(a) = x, \gamma(b) = y$,
(G_2) : $\rho(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for each $t_1, t_2 \in [a, b]$.

Geodesic segment from x to y is the image of the function γ, ($i.e$ $\gamma[a, b]$). A metric space (X, d) is called geodesic if every for every $x, y \in X$, there exists $\gamma : [a, b] \to X$ satisfying (G_1) and (G_2). We say that a metric space X is uniquely geodesic space if for every $x, y \in X$, there exists a unique function $\gamma : [a, b] \to X$ satisfying (G_1) and (G_2), in this case $\gamma[a, b]$ is denoted by $[x, y]$.

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*Corresponding author: Bashir Ali; bashiralik@yahoo.com
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Let X be a geodesic space and $x_1, x_2, x_3 \in X$, a geodesic triangle in X denoted by $\Delta(x_1, x_2, x_3)$ is a set consisting of x_1, x_2, x_3 as its vertices and three geodesic segments joining each pair of the vertices as its sides. For every geodesic triangle in X, there exists a plane triangle called a comparison triangle denoted by $\bar{\Delta}(x_1, x_2, x_3)$ and define as $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ satisfying $\rho(x_i, x_j) = \rho_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j})$ for every $i, j \in \{1, 2, 3\}$.

A geodesic space (X, ρ) is called a CAT(0) space if each geodesic triangle Δ in X satisfies the following inequality

$$\rho(x, y) \leq \rho_{\mathbb{R}^2}(\bar{x}, \bar{y}), \quad \forall (x, y) \in \Delta^2, (\bar{x}, \bar{y}) \in \bar{\Delta}^2,$$

where $\bar{\Delta}$ is a comparison triangle for Δ, it is known that CAT(0) space is uniquely geodesic space. Examples of CAT(0) spaces are Hilbert spaces and R-Trees (see [10, 11, 26]).

Let X be a CAT(0) space, $(1-t)x \oplus ty$ denote the unique point z in the geodesic segment joining x to y for every $x, y \in X$ such that $\rho(z, x) = t\rho(x, y)$ and $\rho(z, y) = (1-t)\rho(x, y)$, where $t \in [0, 1]$, let $[x, y] = \{(1-t)x \oplus ty : t \in [0, 1]\}$, then $K \subseteq X$ is convex if $[x, y] \subseteq K \ \forall x, y \in X$. A complete CAT(0) space is called a Hadamard space.

Berg and Nikolaev [4] introduced the notion of quasilinearization mapping in CAT(0) spaces, which they defined as a mapping $\langle ., . \rangle : (X \times X) \times (X \times X) \longrightarrow \mathbb{R}$ by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\Big(\rho^2(a, d) + \rho^2(b, c) - \rho^2(a, c) - \rho^2(b, d)\Big), \quad \forall (a, b, c, d) \in X^4, \tag{1.2}$$

called the quasilinearization mapping (see [4] for more information). It follows from the definition above that for $(a, b, c, d, e) \in X^5$:

(1) $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{dc} \rangle$,
(2) $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = \rho^2(a, d)$,
(3) $\langle \overrightarrow{ab}, \overrightarrow{dc} \rangle = \langle \overrightarrow{ab}, \overrightarrow{de} \rangle + \langle \overrightarrow{ab}, \overrightarrow{ec} \rangle$.

A metric space X is said to satisfy the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq \rho(a, b)\rho(c, d), \quad \forall (a, b, c, d) \in X^4. \tag{1.3}$$

It is known that a geodesic space is CAT(0) space if and only if (1.3) holds, (see [5]). Kakavandi and Amini [15] introduced the concept of dual space of a CAT(0) space X (see [15] for more details). Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in a Hadamard space X and $x \in X$, let

$$r(x, \{x_n\}_{n=1}^{\infty}) := \limsup_{n\to\infty} \rho(x, x_n)$$

the asymptotic radius of $\{x_n\}_{n=1}^{\infty}$ is given by

$$r(\{x_n\}_{n=1}^{\infty}) := \inf_{x \in X} r(x, \{x_n\}_{n=1}^{\infty})$$

and the asymptotic center of $\{x_n\}_{n=1}^{\infty}$ is the set

$$A(\{x_n\}_{n=1}^{\infty}) := \{x \in X : r(x, \{x_n\}_{n=1}^{\infty}) = r(\{x_n\}_{n=1}^{\infty})\}.$$

If X is a Hadamard space then the asymptotic center of $\{x_n\}$ is a singleton set (see [9]).

Definition 1.1. *Let X be a Hadamard space, a sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to Δ-converges to a point $x^* \in X$, if $\limsup_{n\to\infty} \rho(x, x_{n_k}) = x^*$, for every subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$.*

Many authors studied the notion of Δ-convergence (see [2, 6, 9, 19] for more details). Ugwunnadi et al. study a new class of nonspreading-type mappings in the setting of Hadamard space which they called generalized κ-strictly pseudononspreading mapping defined as follows: Let X be a metric space and $K \subseteq X$, a mapping $G : K \to X$ is generalized κ-strictly pseudononspreading, if there exists two functions $f, g : K \to [0, \tau]$, $\tau < 1$ and $0 \leq \kappa < 1$ satisfying two conditions:

$$(1-\kappa)\rho^2(Gx, Gy) \leq \kappa\rho^2(x,y) + (f(x)-\kappa)\rho^2(Gx, y) + (g(x)-\kappa)\rho^2(x, Gy) \\ + \kappa(\rho^2(x, Gx) + \rho^2(y, Gy)), \ \ \forall x, y \in K$$

and $(f(x) + g(x)) \in (0, 1]$, $\forall x \in K$.

They showed that generalized κ-strictly nonspreading mapping is a generalization of the class of strictly pseudononspreading and the class of generalized nonspreading mappings and studied some properties of the mapping with some examples and they also constructed a Halpern-type scheme which they showed that it converges strongly to fixed point of the map G. For more details see [34]. Takahashi [30] introduced a new class of nonlinear mappings in a real Hilbert space which is define as follows:

Let H be a real Hilbert space and K be a nonempty, closed and convex subset of H, a mapping $U : K \longrightarrow H$ is called α-demimetric, if $Fix(U) \neq \emptyset$ and there exists $\alpha \in (-\infty, 1)$ such that

$$\langle x - y, x - Ux\rangle \geq \frac{1-\alpha}{2}\|x - Ux\|^2.$$

The class of demimetric mappings is important in optimization theory since it contains many common types of operators. For example, the class of α-demicontractive mapping with $\alpha \in [0, 1)$, the metric projections, generalized hybrid mappings, the resolvents of maximal monotone operators (which are well-known useful tools for solving optimization problems) in Hilbert spaces are subclasses of the class of α-demimetric mappings (see [3, 30]). Many authors studied this class of mappings in Banach spaces (see [20, 30, 31, 32]). Aremu et al. [3] extended the above mapping in the setting of Hadamard spaces as follows: Let H be a Hadamard space and K be a nonempty, closed and convex subset of X, a mapping $U : K \longrightarrow X$ is α-demimetric, if $Fix(U) \neq \emptyset$ and there exists $\alpha \in (-\infty, 1)$ such that

$$\langle \overrightarrow{xy}, \overrightarrow{xUx}\rangle \geq \frac{1-k}{2}\rho^2(x, Ux), \quad \forall x \in K, \ y \in Fix(U).$$

They gave an example of a demimetric mapping and established some fixed point theorems for this class of mappings and proved a strong convergence theorem for approximating a common solution of finite family of minimization problems and fixed point problems for this class of mappings in Hadamard spaces. Kawasaki and Takahashi [16] generalized the class of demimetric mappings as follows:

Let E be a smooth real Banach space and K be a nonempty, closed and convex subset of a E, let $\alpha \neq 0$, a map $U : K \longrightarrow E$ is α-generalized demimetric if $Fix(U) \neq \emptyset$ and

$$\alpha\langle x - y, J(x - Ux)\rangle \geq \|x - Ux\|^2, \quad \forall x \in K, \ y \in Fix(U),$$

where J is a duality mapping on E. Takahashi [33] studied this class of mappings in Banach spaces. Recently, Ogwo et al. [23] introduced generalized demimetric in the setting of Hadamard space as follows:

Let X be a Hadamard space, let $\alpha \neq 0$, a mapping $U : X \longrightarrow X$ is α-generalized demimetric, if $Fix(T) \neq \emptyset$ and

$$\alpha\langle \overrightarrow{xy}, \overrightarrow{xUx}\rangle \geq \rho^2(x, Ux), \quad \forall x \in X, \ y \in Fix(U).$$

They gave some examples and properties of generalized demimetric mappings in Hadamard spaces, they also proved a strong convergence theorem involving the map. Let K be a nonempty, closed and convex subset of a Hadamard space X, a mapping $U : K \longrightarrow K$ is Δ-demiclosed, if for any bounded sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\Delta - \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \rho(x_n, Ux_n) = 0$, then $x = Ux$. A mapping $U : K \longrightarrow X$ is

(i) nonexpansive [29], if

$$\rho(Ux, Uy) \leq \rho(x, y), \ \ \forall x, y \in K,$$

(ii) quasi-nonexpansive, if $Fix(U) \neq \emptyset$ and

$$\rho(q, Ux) \leq \rho(q, x) \ \ \ \forall (x, q) \in K \times Fix(U),$$

(iii) α-inverse strongly monotone [2] if there exists $\alpha > 0$ such that

$$\alpha\phi_U(x, y) \leq \rho^2(x, y) - \langle \overrightarrow{UxUy}, \overrightarrow{xy} \rangle, \ \ \ \forall x, y \in K,$$

where,

$$\phi_U(x, y) = \rho^2(x, y) - 2\langle \overrightarrow{UxUy}, \overrightarrow{xy} \rangle + \rho^2(Ux, Uy). \tag{1.4}$$

It is known that (see [2]), $\phi_U(x, y) \geq 0, \ \ \forall x, y \in K$.

(iv) firmly nonexpansive if

$$\langle \overrightarrow{UxUy}, \overrightarrow{xy} \rangle \geq \rho^2(Ux, Uy), \ \ \ \forall x, y \in K.$$

The metric projection $P_K : X \longrightarrow K$ assigns to every $x \in X$, a unique point $P_K(x)$ in K such that

$$\rho(x, y) \geq \rho(x, P_K x), \ \ \forall y \in K,$$

the map P_K is firmly nonexpansive [7]. The Variational Inequality Problem $(V.I.P.)$ in Hilbert space is formulated as follows

$$\text{find } x \in K \text{ such that } \langle Bx, x - y \rangle \leq 0, \ \ \ \forall y \in K, \tag{1.5}$$

where K is a nonempty closed and convex subset of H and B is a nonlinear mapping defined on K. Stampacchia [28] introduced $V.I.P.$ for modeling problems arising in mechanics and the regularity problem for partial differential equations, Stampacchia [28] studied a generalization of the Lax-Milgram theorem and called all problems of this kind $V.I.P.$s. The theory of $V.I.P.$ has numerous applications in diverse fields such as physics, engineering, economics, mathematical programming and others (see [27, 28] and references therein). Alizadeh et al. [2] extended (1.5) to the setting of Hadamard space H as follows:

$$\text{find } x \in K \text{ such that } \langle \overrightarrow{Bxx}, \overrightarrow{yx} \rangle \leq 0, \ \ \forall y \in K, \tag{1.6}$$

when B is inverse strongly monotone, they established the existence of $V.I.P$ (1.6) in a Hadamard space. Furthermore, they constructed the following algorithm:

$$\begin{cases} x_1 \in K, \\ y_n = P_K(\beta_n x_n \oplus (1 - \beta_n)Bx_n), \\ x_{n+1} = P_K(\alpha_n x_n \oplus (1 - \alpha_n)Ty_n), \quad n \geq 1, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subseteq (0, 1)$, T and B are nonexpansive and inverse strongly monotone mappings, respectively. They also obtained Δ-convergence of algorithm (1.7) to a common solution of the $V.I.P$ (1.6) and fixed point of T. Recently G. C. Ugwunnadi et al. [35] constructed a viscosity type algorithm in a setting of Hadamard space which comprises of a demimetric mapping, a finite family of inverse strongly monotone mappings and an equilibrium problem for a bifunction, they succeded in obtaining a strong convergence of their proposed algorithm to a common solution of a variational inequality problem, fixed point problem and equilibrium problem in Hadamard space. Furthermore, they gave applications and numerical examples.

Let K be a nonempty subset of a Hadamard space X and $\varphi : K \longrightarrow \mathbb{R}$ and $F : K \times K \longrightarrow \mathbb{R}$ be a function and bifunction respectively, a minimization problem $(M.P)$ is a problem of searching for $x^* \in K$ such that

$$\varphi(x^*) \leq \varphi(x), \quad \forall x \in K, \tag{1.8}$$

the point x^* satisfying inequality (1.8) is called a minimizer of $\varphi(.)$ on K, we denote the solution set of (1.8) by $M.P(\varphi, K)$. An Equilibrium Problem $(E.P)$ is to find $x^* \in K$ satisfying

$$F(x^*, y) \geq 0, \quad \forall y \in K, \tag{1.9}$$

the point x^* satisfying inequality (1.9) is called an equilibrium point of $F(.,.)$ on K, we denote the solution set of (1.9) by $E.P(F, K)$. C. Izuchukwu et al. [13] introduced mixed equilibrium problem $(M.E.P)$ in a Hadamard space X as the problem of finding $x^* \in K$ such that

$$F(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \ \ \forall y \in K \tag{1.10}$$

the solution set of (1.10) is denoted by $MEP(F, \varphi, K)$. They obtained the following result for existence of solution:

Theorem 1.1 ([13]). *Let K be a nonempty closed and convex subset of an Hadamard space X. Let $\psi : K \longrightarrow \mathbb{R}$ be a real-valued function and $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction such that the following assumptions hold:*

- A_1: $F(x, x) \geq 0, \forall x \in K.$
- A_2: *For every $x \in K$, the set $\{y \in K : F(x, y) + \varphi(y) - \varphi(x) < 0\}$ is convex.*
- A_3: *There exists a compact subset $D \subset K$ containing a point $y_0 \in D$ such that $x \in K/D$ implies $F(x, y_0) + \varphi(y_0) - \varphi(x) < 0,$*

then, the $M.E.P(F, \varphi, K)$ in (1.10) has a solution.

For uniqueness of solution, they obtained the following result.

Theorem 1.2 ([13]). *Let K be a nonempty, closed and convex subset of a Hadamard space X, let $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction, let $\varphi : K \longrightarrow \mathbb{R}$ be a convex function such that the following conditions hold:*

- A_1: $F(x, x) = 0, \quad \forall x \in K,$
- A_2: *F is monotone, i.e , $F(x, y) + F(y, x) \leq 0, \ \ \forall x, y \in K,$*
- A_3: *$F(x, .) : K \longrightarrow \mathbb{R}$ is convex for each $x \in K$.*
- A_4: *Given any $q \in X$ and $\lambda > 0$, there exists a compact subset $D_q \subseteq K$ containing a point $y_q \in D_q$ such that $x \in K/D_q$ implies*

$$F(x, y_q) + \varphi(y_q) - \varphi(x) + \frac{1}{\lambda}\langle \overrightarrow{xy_q}, \overrightarrow{qx} \rangle < 0,$$

then (1.10) has a unique solution.

Based on the information provided above, our main goal in this paper is to introduce and study some properties of a new mapping called generalized cocoercive in Hadamard space, construct viscosity type algorithm for approximating common solution of fixed point of (f, g)-generalized κ-strictly pseudononspreading mapping, quasi-nonexpansive mapping, a finite family of θ_i-generalized demimetric mapping and mixed equilibrium problem for a bifunction and convex lower semi continuous function and variational inequality problem involving the generalized cocoercive mapping in Hadamard space. Our result generalizes and compliments some results in the literature. Furthermore our technique of proof is new and is of independent interest, as it does not involve the use of cases as done in [23], [34] and [35].

2. Preliminaries

In this section, we state some known and useful results which we are going to use in the proof of our main result.

Lemma 2.1. *Let X be a $CAT(0)$ space and $x, y, z \in X$ and $t \in [0,1]$ then the following inequalities hold*

(i) $\rho(tx \oplus (1-t)y, z) \leq t\rho(x,z) + (1-t)\rho(y,z)$ ([9]),
(ii) $\rho^2(tx \oplus (1-t)y, z) \leq t\rho^2(x,z) + (1-t)\rho^2(y,z) - t(1-t)\rho(x,y)$ ([9]),
(iii) $\rho^2(tx \oplus (1-t)y, z) \leq t^2\rho^2(x,z) + (1-t)^2\rho^2(y,z) + 2t(1-t)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$ ([7]).

Lemma 2.2 ([19]). *Every bounded sequence in a Hadamard space has a Δ-convergent subsequence.*

Lemma 2.3 ([15]). *Let H be a Hadamard space and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X then $\limsup_{n\to\infty} \langle \overrightarrow{x_n x}, \overrightarrow{yx} \rangle \geq 0,\ \forall y \in X$ if and only if $\{x_n\}_{n=1}^{\infty}$ Δ-converges to x.*

Lemma 2.4 ([2]). *Let K be a nonempty closed and convex subset of a Hadamard space H and $A : K \longrightarrow H$ be α-inverse strongly monotone . Suppose $\mu \in [0,1]$ and a map $A_\mu : K \longrightarrow X$ define by $A_\mu(x) = (1-\mu)x + \mu Ax$ for each $x \in K$. If $\mu \in (0, 2\mu)$, then A_μ is nonexpansive and $Fix(A) = Fix(A_\mu)$.*

Lemma 2.5 ([2]). *Let K be a nonempty convex subset of a Hadamard space H and $A : K \longrightarrow H$ be a mapping and for $\mu \in (0,1]$, a map $A_\mu : K \longrightarrow X$ defined as $A_\mu(x) = (1-\mu)x + \mu A(x)$, for all $x \in K$ then,*

$$VI(K, A) = VI(K, A_\mu).$$

Remark 2.1 ([24]). *As a consequence of the Lemma 2.5 above, we have*

$$Fix(P_K A) = VI(K, A) = VI(K, A_\mu) = Fix(P_K A_\mu).$$

Lemma 2.6 ([29]). *Let X be a $CAT(0)$ space, $\{x_i\}_{i=1}^{N} \subseteq X$ and $\alpha_i \in [0,1],\ \forall i \in \{1, 2, ..., N\}$ satisfying $\sum_{i=1}^{N} \alpha_i = 1$, then,*

$$\rho\Big(\bigoplus_{i=1}^{N} \alpha_i x_i, z \Big) \leq \sum_{i=1}^{N} \alpha_i \rho(x_i, z),\ \forall z \in X. \tag{2.11}$$

Lemma 2.7 ([6]). *Let X be a $CAT(0)$ space and $q \in X$, let $\{x_i\}_{i=1}^{N} \subseteq X$ and $\{\alpha_i\}_{i=1}^{N} \subseteq [0,1]$ such that $\sum_{i=1}^{N} \alpha_i = 1$, then*

$$\rho^2\left(q, \bigoplus_{i=1}^{N} \alpha_i x_i \right) \leq \sum_{i=1}^{N} \alpha_i \rho^2(q, x_i) - \sum_{i,j=1, i\neq j} \rho^2(x_i, x_j).$$

Lemma 2.8 ([8]). *Let H be a Hadamard space and $T : H \longrightarrow H$ be a nonexpansive mapping, then T is Δ-demiclosed.*

Theorem 2.3 ([34]). *Let K be a nonempty closed and convex subset of a Hadamard space H and $G : K \to K$ be (f,g)-generalized κ-strictly pseudononspreading mapping with $\kappa \in [0,1)$ and $f, g : K \to [0, \tau],\ \tau < 1$ and $(f(x) + g(x)) \in (0,1]$ for every $x \in K$. Assume $Fix(G) \neq \emptyset$ and $f(q) \neq 0$ with $\beta \in \left[\frac{\kappa}{f(q)}, 1 \right),\ \ \forall q \in Fix(G)$, then $Fix(G)$ is closed and convex.*

Remark 2.2 ([34]). *Observe that if G is $(f,g)-$generalized κ-strictly pseudononspreading with $Fix(G) \neq \emptyset$ and $f(q) \neq 0$, for each $q \in Fix(G)$, then for every $q \in Fix(G)$ and $x \in dom(G)$, we obtain*

$$\rho^2(q, Gx) \leq \rho^2(q,x) + \frac{\kappa}{f(q)}\rho^2(x, Gx). \tag{2.12}$$

Lemma 2.9 ([23]). *Let X be a CAT(0) space and $U : X \longrightarrow X$ be β generalized demimetric mapping with $\beta \neq 0$. Suppose that $U_\alpha x = \alpha x \oplus (1-\alpha)Ux$ with $\beta \leq \frac{2}{1-\alpha}$ and $\alpha \in (0,1)$, then U_α is quasi-nonexpansive and $Fix(U_\alpha) = Fix(U)$.*

Lemma 2.10 ([36]). *Let $\{\gamma_n\}_{n=1}^\infty \subset (0,1)$ and $\{\delta_n\}_{n=1}^\infty \subset \mathbb{R}$ satisfying*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\longrightarrow\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$,

if $\{a_n\}_{n=1}^\infty \subset [0,\infty)$ such that

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \ \forall n \geq 0$$

then

$$\lim_{n\longrightarrow\infty} a_n = 0.$$

Theorem 2.4 ([13]). *Let K be a nonempty, closed and convex subset of a Hadamard space X, $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction, let $\varphi : K \longrightarrow \mathbb{R}$ be a convex function such that conditions $A_1 - A_4$ of Theorem 1.2 holds. For $\lambda > 0$, we have that $T_\lambda^{\varphi,F}$ is single valued. Moreover, if $K \subset Dom(T_\lambda^{\varphi,F})$, then*

(i) *$T_\lambda^{\varphi,F}$ is firmly nonexpansive on K.*

(ii) *If $Fix(T_\lambda^{\varphi,F}) \neq \emptyset$, then*

$$\rho^2(x, T_\lambda^{\varphi,F}x) + \rho^2(T_\lambda^{\varphi,F}x, q) \leq \rho^2(x,q), \ \forall x \in K, q \in Fix(T_\lambda^{\varphi,F}). \tag{2.13}$$

(iii) $Fix(T_\lambda^{\varphi,F}) = MEP(K,F,\varphi)$.

Lemma 2.11 ([34]). *Let K be a nonempty closed and convex subset of a Hadamard space H and $G : K \to K$ be (f,g)-generalized κ-strictly pseudononspreading mapping with $0 \leq \kappa < 1$ and $f, g : K \to [0,\tau]$, $\tau < 1$ and $(f(x)+g(x)) \in (0,1]$, $\forall x \in K$. Suppose $(2\kappa + f(x)) < 1$, $\forall x \in K$ and $\{x_n\}$ is a bounded sequence in K such that $\Delta - \lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} \rho(x_n, Gx_n) = 0$ then $x^* \in Fix(G)$.*

3. Main Results

In this section, we present the main results of this paper.

Theorem 3.5. *Let H be a CAT(0) space, K be a nonempty subset of H, $A : K \to H$ be a mapping and $\phi_A : H \times H \to \mathbb{R}$ be a mapping defined as in (1.4) then the following holds:*

(i) $\phi_A(x,y) = \phi_A(y,x), \ \ \forall x,y \in K$.

(ii) $\Big(\rho(x,y) - \rho(Ax,Ay)\Big)^2 \leq \phi_A(x,y) \leq \Big(\rho(x,y) + \rho(Ax,Ay)\Big)^2, \ \ \forall x,y \in K.$

Proof. (i) Let $x, y \in K$, then from the definition of $\phi_A(.,.)$ and property of $\rho(.,.)$, we have

$$\begin{aligned}\phi_A(x,y) &= \rho^2(x,y) - 2\langle\overrightarrow{AxAy}, \overrightarrow{xy}\rangle + \rho^2(Ax,Ay) \\ &= \rho^2(y,x) - 2\langle\overrightarrow{AyAx}, \overrightarrow{yx}\rangle + \rho^2(Ay,Ax) \\ &= \phi_A(y,x).\end{aligned}$$

(ii) Applying Cauchy-Schwarz inequality twice, we obtain

$$\begin{aligned}\Big(\rho(x,y)-\rho(Ax,Ay)\Big)^2 &= \rho^2(x,y)-2\rho(x,y)\rho(Ax,Ay)+\rho^2(Ax,Ay)\\ &\le \phi_A(x,y)\\ &= \rho^2(x,y)-2\langle\overrightarrow{xy},\overrightarrow{AxAy}\rangle+\rho^2(Ax,Ay)\\ &= \rho^2(x,y)+2\langle\overrightarrow{yx},\overrightarrow{AxAy}\rangle+\rho^2(Ax,Ay)\\ &\le \rho^2(x,y)+2\rho(x,y)\rho(Ax,Ay)+\rho^2(Ax,Ay)\\ &= \Big(\rho(x,y)+\rho(Ax,Ay)\Big)^2.\end{aligned}$$

□

Remark 3.3. *We observe that from Theorem 3.5 (ii), we can easily obtain that $\phi_A(x,y)\ge 0, \forall x,y\in K$.*

4. Generalized Cocoercive Mapping in CAT(0) Spaces

The definition of generalized cocoercive is motivated from the newly introduced map called generalized inverse strongly monotone map (see [1]). Let H be a real Hilbert space with an inner product $\langle .,.\rangle$ and a norm $\|.\|$ and K be a nonempty subset of H. For $\alpha>0$, a mapping $B:K\to H$ is called generalized α-inverse strongly monotone (see [1]), if

$$\langle Bx-By,x-y\rangle \ge \alpha\|Bx-By\|^2-\|Bx\|\|By\|,\ \ \forall x,y\in K. \tag{4.14}$$

By setting $A=I-B$, where I is identity map, we see that $(I-A)$ is generalized α-cocoercive.

Definition 4.2. *Let H be a CAT(0) space, K be a nonempty subset of H, for $\alpha>0$, a mapping $B:K\to H$ is called generalized α-cocoercive, there exists a map $A:K\to H$ such that for all $x,y\in K$,*

$$\rho^2(x,y)+\rho(x,Ax)\rho(y,Ay)-\langle\overrightarrow{AxAy},\overrightarrow{xy}\rangle \ge \alpha\phi_A(x,y), \tag{4.15}$$

where ϕ_A is defined as in (1.4).

Remark 4.4. *It is not difficult to see that every α-inverse strongly monotone is generalized α-cocoercive mapping.*

Proof. For $\alpha>0$, let $B:K\to H$ be α-inverse strongly monotone mapping then there exists $A:K\to H$ such that

$$\begin{aligned}\rho^2(x,y)-\langle\overrightarrow{AxAy},\overrightarrow{xy}\rangle &\ge \alpha\phi_A(x,y)\\ &\ge \alpha\phi_A(x,y)-\rho(x,Ax)\rho(y,Ay),\end{aligned}$$

which implies B is generalized α-cocoercive. □

The converse is not true in general:

Example 4.1. *Consider $[-1,1]$ and for $d>0$, let $A:[-1,1]\to\mathbb{R}$ be defined by*

$$Ax=\begin{cases}x+d, & x\le 0\\ x-d, & x>0\end{cases} \tag{4.16}$$

then A satisfy the following inequality:

$$|x-y|^2-(Ax-Ay)(x-y)\ge \frac{1}{4}\phi_A(x,y)-|x-Ax||y-Ay|,\ \ \forall x,y\in[-1,1].$$

Solution 1. *Define a map $B : [-1,1] \to \mathbb{R}$ by*

$$Bx = \begin{cases} -d, & x \le 0 \\ d, & x > 0 \end{cases}$$

then for $x \in [-1,1]$, we have

$$\begin{aligned} x - Bx &= x - \begin{cases} -d, & x \le 0 \\ d, & x > 0 \end{cases} \\ &= \begin{cases} x+d, & x \le 0 \\ x-d, & x > 0 \end{cases} \\ &= Ax. \end{aligned}$$

Thus, $A = I - B$ and also, B satisfy the following inequality:

$$(Bx - By)(x-y) \ge \frac{1}{4}|Bx - By|^2 - |Bx||By|, \ \ \forall x, y \in [-1,1],$$

and for each $\alpha > 0$,

$$(Bx - By)(x-y) \ngeq \frac{1}{4}|Bx - By|^2 \ \ \forall x, y \in [-1,1]$$

which implies B is generalized $\frac{1}{4}$-inverse strongly monotone mapping which is not α-inverse strongly monotone mapping for any $\alpha > 0$ (see [1] for more details).

Lemma 4.12. *Let K be a nonempty closed and convex subset of a CAT(0) space H, for $\alpha > \frac{1}{2}$, let $A : K \to H$ be a generalized cocoercive mapping. Let $\mu \in [0,1]$ and define $A_\mu : K \to H$ by*

$$A_\mu x = \mu Ax \oplus (1-\mu)x, \quad \forall x \in K, \tag{4.17}$$

if $\mu \in (0, 2\alpha - 1)$, then A_μ is nonexpansive mapping and $Fix(A_\mu) = Fix(T)$.

Proof. Let $x, y \in K$, then

$$\begin{aligned} \rho^2(A_\mu x, A_\mu y) &\le (1-\mu)^2\rho^2(x,y) + \mu(1-\mu)\rho^2(x,Ay) - \mu(1-\mu)\rho^2(y,Ay) \\ &+ \mu(1-\mu)\rho^2(Ax,y) + \mu^2\rho^2(Ax,Ay) - \mu(1-\mu)\rho^2(x,Ax) \\ &\le (1-\mu)^2\rho^2(x,y) + \mu^2\rho^2(Ax,Ay) \\ &+ 2\mu(1-\mu)\Big[\rho^2(x,y) - \alpha\phi_A(x,y) + \rho(x,Ax)\rho(y,Ay)\Big] \\ &= (1-\mu)^2\rho^2(x,y) + \mu^2\rho^2(Ax,Ay) + 2\mu(1-\mu)\rho(x,Ax)\rho(y,Ay) \\ &- 2\alpha\mu(1-\mu)\phi_A(x,y) \\ &\le (1-\mu)^2\rho^2(x,y) - 2\alpha\mu(1-\mu)\phi_A(x,y) + 2\mu(1-\mu)\rho(x,Ax)\rho(y,Ay) \\ &+ \mu^2\phi_A(x,y) + 2\mu^2\langle\overrightarrow{AxAy}, \overrightarrow{xy}\rangle - \mu^2\rho^2(x,y) \\ &= \rho^2(x,y) + (-2\alpha\mu(1-\mu) + \mu^2)\phi_A(x,y) \\ &+ 2\mu^2(\langle\overrightarrow{AxAy}, \overrightarrow{xy}\rangle - \rho^2(x,y) - \rho(x,Ax)\rho(y,Ay)) + 2\mu\rho(x,Ax)\rho(y,Ay) \\ &\le \rho^2(x,y) + \mu\Big((\mu - 2\alpha)\phi_A(x,y) + 2\rho(x,Ax)\rho(y,Ay)\Big) \\ &\le \rho^2(x,y) + \mu\Big((\mu - 2\alpha + 1)\max\{\phi_A(x,y), 2\rho(x,Ax)\rho(y,Ay)\}\Big) \\ &\le \rho^2(x,y). \end{aligned}$$

Consequently, we get A_μ is nonexpansive. □

Lemma 4.13. *Let K be a nonempty closed and convex subset of a Hadamard space H. For $\alpha_i > \frac{1}{2}$, let $A_i : K \longrightarrow H$, $i \in 1, 2, ..., N$ be a family of α_i-generalized cocoercive mappings, define $A_{\mu_i} = (1-\mu_i)I \oplus \mu_i A_i$, for $\mu_i \in (0, 2\alpha_i - 1)$ $\gamma_i \in (0,1)$, $i \in \{1, 2, ..., N\}$ such that $\sum_{i=1}^{N} \gamma_i = 1$ then the map $\bigoplus_{i=1}^{N} \gamma_i P_K A_{\mu_i}$ is a nonexpansive mapping. If in addition, $\bigcap_{i=1}^{N} Fix(P_K A_{\mu_i}) \neq \emptyset$, then $Fix\Big(\bigoplus_{i=1}^{N} \gamma_i P_K A_{\mu_i}\Big) = \bigcap_{i=1}^{N} Fix(P_K A_{\mu_i})$.*

Proof. It follows from Lemma 2.5. □

Theorem 4.6. *Let H be a Hadamard space, K be a nonempty closed and convex subset of H, $\alpha > \frac{1}{2}$ and A be generalized α-cocoercive mapping of K into H, then $VI(K, A) \neq \emptyset$.*

Proof. Let $\mu \in (0, 2\alpha - 1]$, we define a mapping $B : K \to K$ by

$$Bx = P_K(\mu Ax \oplus (1-\mu)x), \ \forall x \in K. \tag{4.18}$$

Since P_K and A_μ are nonexpansive, then

$$\begin{aligned} \rho(Bx, By) &\leq \rho(A_\mu x, A_\mu y), \\ &\leq \rho(x, y) \end{aligned}$$

which implies, B is nonexpansive, there exist $\hat{x} \in K$ satisfying

$$\begin{aligned} \hat{x} = B\hat{x} &\Leftrightarrow \hat{x} = P_K(\mu A\hat{x} \oplus (1-\mu)\hat{x}) \\ &\Leftrightarrow \hat{x} \in VI(K, A). \end{aligned}$$

Therefore, $VI(K, A) \neq \emptyset$. □

Theorem 4.7. *Let H be a Hadamard space and H^* be its dual space, let K be a nonempty closed and convex subset of H. Let $A_i : K \longrightarrow H$ be family of generalized α_i-cocoercive mapping, where $\alpha_i > \frac{1}{2}$, let $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction satisfying condition A_1 to A_4 of Theorem 1.2, $\varphi : K \longrightarrow \mathbb{R}$ be a convex and lower semi-continuous function, let $G : K \to K$ be (f, g)-generalized κ-strictly pseudononspreading mapping with $\kappa \in [0, 1)$, where $f, g : K \to [0, \tau]$, $\tau < 1$, such that f, g, κ, τ satisfies assumptions in Theorem 2.3 [34], Remark 2.2 [34] and Lemma 2.11 [34]. Let $S : K \to K$ be quasi-nonexpansive mapping, let $T_i : K \longrightarrow K$ be family of θ_i-generalized demimetric mapping with $\theta_i \neq 0$. Assume that $\Omega = Fix(G) \cap Fix(S) \cap M.E.P(F, \varphi, K) \bigcap_{i=1}^{N} \Big(VI(K, A_i) \cap Fix(T_i)\Big) \neq \emptyset$ and $\{z_n\}_{n=1}^{\infty}$ is a sequence generated by*

$$\begin{cases} z_0, z_1 \in X, \\ v_n = \alpha_n z_n \oplus (1-\alpha_n) z_{n+1}, \\ w_n = T_{r_n}^{F,\varphi} v_n, \\ x_n = \delta_n w_n \oplus \beta_{n,0} G w_n \oplus \bigoplus_{i=1}^{N} \beta_{n,i} P_K A_{\mu_i} w_n, \\ y_n = a_n w_n \oplus b_{n,0} S w_n \oplus \bigoplus_{i=1}^{N} b_{n,i} T_i^{\alpha} x_n, \\ z_{n+1} = \lambda_n h(v_n) \oplus (1-\lambda_n) y_n, \ n \in \mathbb{N}, \end{cases} \tag{4.19}$$

where $h : H \to H$ is γ-contraction for some $\gamma \in [0,1)$ and for each $i \in \{1,2,...,m\}$

$$A_{\mu_i}x = (1-\mu_i)x \oplus \mu_i A_i x, \text{ with } \mu_i \in (0, 2\alpha_i - 1),$$
$$T_i^{\alpha}x = \alpha x \oplus (1-\alpha)T_i x$$

with assumption that T_i^{α} and S are Δ-demiclosed, with $\theta_i \leq \frac{2}{1-\alpha}, \alpha \in (0,1)$, $\{r_n\} \subseteq (0,1)$ and $\{\alpha_n\}, \{\beta_{n,0}\}, \{\beta_{n,i}\}, \{\delta_n\}, \{b_{n,i}\}, \{b_{n,0}\}, \{a_n\}, \{\lambda_n\}$ are sequences in $\mathbb{R}$ satisfying the following conditions:

C_1: *$\{\alpha_n\}, \{\lambda_n\}$ are in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \lambda_n = 0$ and*
$$\sum_{n=1}^{\infty} \alpha_n = +\infty = \sum_{n=1}^{\infty} \lambda_n,$$

C_2: *$\{\delta_n\}, \{b_{n,0}\}, \{b_{n,i}\} \subseteq (f_1, f_2) \subseteq (0,1)$ and $\delta_n + \beta_{n,0} + \sum_{i=1}^{N} \beta_{n,i} = 1$,*

C_3: *$\{a_n\}, \{b_{n,0}\}, \{b_{n,i}\} \subseteq (f_1, f_2) \subseteq (0,1)$ with $a_n + b_{n,0} + \sum_{i=1}^{m} b_{n,i} = 1$,*

C_4: *$\delta_n f(q) < \kappa$, for every $q \in Fix(G)$*

then $\{z_n\}$ converges strongly to $z^* \in \Omega$ satisfying $z^* = P_\Omega \circ h(z^*)$.

Proof. The proof is divided into steps:

Step 1: We show that $\{z_n\}$ is bounded. Let $q \in \Omega$, then utilizing Lemma 2.1 (i), we have

$$\rho(q, v_n) \leq \alpha_n \rho(q, z_n) + (1-\alpha_n)\rho(q, z_{n+1}). \tag{4.20}$$

Using (2.13), we get

$$\rho(q, w_n) \leq \rho(q, v_n). \tag{4.21}$$

Using Lemma 4.13 and Remark 2.2, we get

$$\begin{aligned}
\rho^2(q, x_n) &\leq \delta_n \rho^2(q, w_n) + \beta_{n,0}\rho^2(q, Gw_n) + \sum_{i=1}^{N} \beta_{n,i}\rho^2(q, P_K A_{\mu_i} w_n) \\
&\quad - \delta_n \beta_{n,0}\rho^2(w_n, Gw_n) \\
&\leq \rho^2(q, w_n) - \Big(\delta_n - \frac{\kappa}{f(q)}\Big)\beta_{n,0}\rho^2(w_n, Gw_n).
\end{aligned} \tag{4.22}$$

By applying Condition C_4, we obtain

$$\rho(q, x_n) \leq \rho(q, w_n). \tag{4.23}$$

Using Lemma 2.9, (4.23) and Condition C_3, we get

$$\begin{aligned}
\rho(q, y_n) &\leq a_n \rho(q, w_n) + b_{n,0}\rho(q, Sw_n) + \sum_{i=1}^{N} b_{n,i}\rho(q, T_i^{\alpha} x_n) \\
&\leq \rho(q, w_n).
\end{aligned} \tag{4.24}$$

From Condition C_1, we get

$$\lambda_n + k_0 < 1,$$

therefore, we let

$$\nu_n = \frac{1-(\lambda_n + k_0)}{1-k_0}, \; \forall n \in \mathbb{N}, \text{ for some } k_0 \in (0,1),$$

by utilizing (4.20), (4.21) and (4.24), we get

$$\begin{aligned}\rho(q,z_{n+1}) &\leq \lambda_n\rho(q,h(v_n))+(1-\lambda_n)\rho(q,y_n)\\ &\leq \lambda_n\rho(q,h(q))+(1-\lambda_n(1-\gamma))\rho(q,v_n)\\ &\leq \lambda_n\rho(q,h(q))+(1-\lambda_n(1-\gamma))\Big(\alpha_n\rho(q,z_n)+(1-\alpha_n)\rho(q,z_{n+1})\Big).\end{aligned}$$

Since $(1-\lambda_n(1-\gamma))(1-\alpha)<1$, then there exists $k_0\in(0,1)$ such that

$$(1-\lambda_n(1-\gamma))(1-\alpha_n)\leq k_0<1,$$

this implies

$$1-(1-\lambda_n(1-\gamma))(1-\alpha_n)\geq(1-k_0).$$

By using Condition C_1, we can easily get

$$1\geq\alpha_n-\lambda_n(1-\alpha_n(1-\gamma))+k_0,$$

therefore we have,

$$\begin{aligned}\rho(q,z_{n+1}) &\leq \frac{\lambda_n}{1-[1-\lambda_n(1-\gamma)](1-\alpha_n)}\rho(q,h(q))\\ &\quad+\frac{(1-\lambda_n(1-\gamma))\alpha_n}{1-[1-\lambda_n(1-\gamma)](1-\alpha_n)}\rho(q,z_n)\\ &\leq \frac{\lambda_n}{1-k_0}\rho(q,h(q))+\frac{(1-\lambda_n(1-\gamma))\alpha_n}{1-k_0}\rho(q,z_n)\\ &\leq (1-\nu_n)\rho(q,h(q))+\nu_n\rho(q,z_n)\\ &\leq \max\{\rho(q,h(q)),\rho(q,z_n)\},\end{aligned}$$

which we can show by induction that,

$$\rho(q,z_{n+1})\leq\max\{\rho(q,h(q)),\rho(q,z_0)\},\quad\forall n\in\mathbb{N}.$$

Hence, $\{\rho(q,z_n)\}$ is bounded, which implies $\{z_n\}$ is bounded. Utilizing (4.20), (4.21), (4.23), (4.24), Lemma 4.13, definition of S and Lemma 2.9, we get $\{v_n\}$, $\{w_n\}$, $\{x_n\}$, $\{y_n\}$, $\{P_KA_{\mu_i}w_n\}$, $\{Sw_n\}$ and $\{T_i^\alpha x_n\}$ are bounded respectively, for each $i\in\{1,2,...,N\}$. Also, we observe that

$$\begin{aligned}\rho^2(q,h(v_n)) &\leq \Big(\rho(q,h(q))+\rho(h(q),h(v_n))\Big)^2\\ &\leq \Big(\rho(q,h(q))+\gamma\rho(q,v_n)\Big)^2,\end{aligned}$$

which implies $\{\rho^2(q,h(v_n))\}$ is bounded as $\{\rho(q,v_n)\}$ is bounded.

Step 2: Next, we show that

$$\lim_{n\to\infty}\rho(z_{n+1},v_n)=\lim_{n\to\infty}\rho(z_{n+1},w_n)=\lim_{n\to\infty}\rho(z_{n+1},x_n)=0.$$

Indeed, utilizing Lemma 2.9 and (4.22) in algorithm (4.19), we get

$$\begin{aligned} \rho^2(q,y_n) &\le a_n\rho^2(q,w_n) + b_{n,0}\rho^2(q,Sw_n) + \sum_{i=1}^{N} b_{n,i}\rho^2(q,T_i^{\alpha}x_n) \\ &\le a_n\rho^2(q,w_n) + b_{n,0}\rho^2(q,Sw_n) \\ &+ \sum_{i=1}^{N} b_{n,i}\Big[\rho^2(q,w_n) - \Big(\delta_n - \frac{\kappa}{f(q)}\Big)\beta_{n,0}\rho^2(w_n,Gw_n)\Big] \\ &\le \rho^2(q,w_n) - b_{n,i}\beta_{n,0}\Big(\delta_n - \frac{\kappa}{f(q)}\Big)\rho^2(w_n,Gw_n). \end{aligned} \tag{4.25}$$

Substituting (4.21), (4.25) and (4.20) in algorithm (4.19), we get

$$\begin{aligned} \rho^2(q,z_{n+1}) \le \lambda_n\rho^2(q,h(v_n)) + (1-\lambda_n)\Big(\rho^2(q,v_n) \\ - b_{n,i}\beta_{n,0}\Big(\delta_n - \frac{\kappa}{f(q)}\Big)\rho^2(w_n,Gw_n)\Big), \end{aligned}$$

which implies for each $i \in \{1,2,...,N\}$,

$$\begin{aligned} b_{n,i}\beta_{n,0}\Big(\delta_n - \frac{\kappa}{f(q)}\Big)\rho^2(w_n,Gw_n) \le \frac{\lambda_n}{1-\lambda_n}\rho^2\Big(q,h(v_n) - \rho^2(q,z_{n+1})\Big) \\ + \Big(\rho^2(q,v_n) - \rho^2(q,z_{n+1})\Big). \end{aligned} \tag{4.26}$$

Using (4.20) in (4.26), we get

$$\begin{aligned} b_{n,i}\beta_{n,0}\Big(\delta_n - \frac{\kappa}{f(q)}\Big)\rho^2(w_n,Gw_n) \le \frac{\lambda_n}{1-\lambda_n}\rho^2\Big(q,h(v_n) - \rho^2(q,z_{n+1})\Big) \\ + \alpha_n\Big(\rho^2(q,z_n) - \rho^2(q,z_{n+1})\Big). \end{aligned}$$

Using Conditions C_1, C_2 and C_3, we get

$$\lim_{n\to\infty} \rho(w_n,Gw_n) = 0. \tag{4.27}$$

By using Lemma 2.7, Lemma 4.13 and Condition C_3 in algorithm (4.19), we get for each $i \in \{1,2,...,N\}$,

$$\rho^2(q,x_n) \le \rho^2(q,w_n) - \delta_n\beta_{n,i}\rho^2(w_n,P_K A_{\mu_i}w_n) + \frac{\kappa}{f(q)}\rho^2(w_n,Gw_n), \tag{4.28}$$

which implies, by using Lemma 2.9, (4.21), Conditions C_3, C_4 and (4.28), we have

$$\begin{aligned}
\rho^2(q, y_n) &\leq a_n\rho^2(q, w_n) + b_{n,0}\rho^2(q, Sw_n) + \sum_{i=1}^{N} b_{n,i}\rho^2(q, T_i^{\alpha}x_n) \\
&\leq a_n\rho^2(q, w_n) + b_{n,0}\rho^2(q, Sw_n) + \sum_{i=1}^{N} b_{n,i}\rho^2(q, x_n) \\
&\leq a_n\rho^2(q, w_n) + b_{n,0}\rho^2(q, Sw_n) \\
&+ \sum_{i=1}^{N} b_{n,i}\Big(\rho^2(q, w_n) - \delta_n\beta_{n,j}\rho^2(w_n, P_K A_{\mu_i} w_n) + \frac{\kappa}{f(q)}\rho^2(w_n, Gw_n)\Big) \\
&\leq \rho^2(q, v_n) - b_{n,i}\delta_n\beta_{n,j}\rho^2(w_n, P_K A_{\mu_i} w_n) + \sum_{i=1}^{N} b_{n,i}\frac{\kappa}{f(q)}\rho^2(w_n, Gw_n). \qquad (4.29)
\end{aligned}$$

By applying (4.20) and (4.29) in algorithm (4.19), we get for each $i, j \in \{1, 2, ..., N\}$

$$\begin{aligned}
b_{n,i}\delta_n\beta_{n,j}\rho^2(w_n, P_K A_{\mu_i} w_n) &\leq \frac{\lambda_n}{1-\lambda_n}\Big(\rho^2(q, h(v_n)) - \rho^2(q, z_{n+1})\Big) \\
&+ \rho^2(q, v_n) - \rho^2(q, z_{n+1}) \\
&+ \sum_{i=1}^{N} b_{n,i}\frac{\kappa}{f(q)(1-\lambda_n)}\rho^2(w_n, Gw_n) \\
&\leq \frac{\lambda_n}{1-\lambda_n}\Big(\rho^2(q, h(v_n)) - \rho^2(q, z_{n+1})\Big) \\
&+ \alpha_n\Big(\rho^2(q, z_n) - \rho^2(q, z_{n+1})\Big) \\
&+ \sum_{i=1}^{N} b_{n,i}\frac{\kappa}{f(q)(1-\lambda_n)}\rho^2(w_n, Gw_n). \qquad (4.30)
\end{aligned}$$

By applying Conditions C_1, C_2, C_3 and (4.27) in (4.30), we obtain

$$\lim_{n\to\infty} \rho(w_n, P_K A_{\mu_i} w_n) = 0, \quad \forall i \in \{1, 2, ..., N\}. \qquad (4.31)$$

Similarly, by applying Lemma 2.9, Condition C_3 and (4.21) in algorithm (4.19), we get

$$\begin{aligned}
\rho^2(q, y_n) &\leq a_n\rho^2(q, w_n) + b_{n,0}\rho^2(q, Sw_n) \\
&+ \sum_{i=1}^{N} b_{n,i}\rho^2(q, T_i^{\alpha}x_n) - a_n b_{n,0}\rho^2(w_n, Sw_n) \\
&\leq \rho^2(q, v_n) - a_n b_{n,0}\rho^2(w_n, Sw_n). \qquad (4.32)
\end{aligned}$$

By applying (4.32) and algorithm (4.20) in (4.19), we get

$$\begin{aligned}
\rho^2(q, z_{n+1}) &\leq \lambda_n\rho^2(q, h(v_n)) + (1-\lambda_n)\Big(\rho^2(q, v_n) - a_n b_{n,0}\rho^2(w_n, Sw_n)\Big) \\
&\leq \lambda_n\rho^2(q, h(v_n)) + (1-\lambda_n)\Big(\Big[\alpha_n\rho^2(q, z_n) \\
&+ (1-\alpha_n)\rho^2(q, z_{n+1})\Big] - a_n b_{n,0}\rho^2(w_n, Sw_n)\Big),
\end{aligned}$$

which implies

$$a_n b_{n,0}\rho^2(w_n, Sw_n) \leq \frac{\lambda_n}{1-\lambda_n}\Big(\rho^2(q, h(v_n)) - \rho^2(q, z_{n+1})\Big) + \alpha_n\Big(\rho^2(q, z_n) - \rho^2(q, z_{n+1})\Big). \tag{4.33}$$

Utilizing Conditions C_1, C_2 and C_4 in (4.33), we obtain

$$\lim_{n\to\infty} \rho(w_n, Sw_n) = 0. \tag{4.34}$$

Similarly, by applying Lemma 2.9, (4.20), (4.21), (4.23) and (4.24) in algorithm (4.19), we get

$$\begin{aligned} a_n b_{n,i}\rho^2(w_n, T_i^{\alpha} x_n) &\leq \frac{\lambda_n}{1-\lambda_n}\Big(\rho^2(q, h(v_n)) - \rho^2(q, z_{n+1})\Big) \\ &\quad + \rho^2(q, v_n) - \rho^2(q, z_{n+1}) \\ &\leq \frac{\lambda_n}{1-\lambda_n}\Big(\rho^2(q, h(v_n)) - \rho^2(q, z_{n+1})\Big) \\ &\quad + \alpha_n\Big(\rho^2(q, z_n) - \rho^2(q, z_{n+1})\Big). \end{aligned} \tag{4.35}$$

Using Conditions C_1 and C_3 in (4.35), we have

$$\lim_{n\to\infty} \rho(w_n, T_i^{\alpha} x_n) = 0, \ \forall i \in \{1, 2, ..., N\}. \tag{4.36}$$

Using Lemma 2.6, we have

$$\rho(w_n, x_n) \leq \beta_{n,0}\rho(w_n, Gw_n) + \sum_{i=1}^{N} \beta_{n,i}\rho(w_n, P_K A_{\mu_i} w_n).$$

Using (4.27) and (4.31), we get

$$\lim_{n\to\infty} \rho(w_n, x_n) = 0. \tag{4.37}$$

Utilizing Lemma 2.1 (ii), (4.20) and (4.24), we obtain

$$\begin{aligned} \rho^2(v_n, w_n) &\leq \frac{\lambda_n}{1-\lambda_n}\Big(\rho^2(q, h(v_n)) - \rho^2(q, z_{n+1})\Big) \\ &\quad + \rho^2(q, v_n) - \rho^2(q, z_{n+1}) \\ &\leq \frac{\lambda_n}{1-\lambda_n}\Big(\rho^2(q, h(v_n)) - \rho^2(q, z_{n+1})\Big) \\ &\quad + \alpha_n\Big(\rho^2(q, z_n) - \rho^2(q, z_{n+1})\Big). \end{aligned} \tag{4.38}$$

Applying Condition C_1 in (4.38), we get

$$\lim_{n\to\infty} \rho(v_n, w_n) = 0. \tag{4.39}$$

Using Lemma 2.6 in algorithm (4.19),

$$\rho(w_n, y_n) \leq b_{n,0}\rho(w_n, Sw_n) + \sum_{i=1}^{N} b_{n,i}\rho(w_n, T_i^{\alpha} x_n).$$

By applying (4.34), (4.36) and Conditions C_3 and C_4, we get

$$\lim_{n\to\infty} \rho(w_n, y_n) = 0. \tag{4.40}$$

Combining (4.40) and (4.37), we have

$$\lim_{n\to\infty}\rho(y_n,x_n)=0. \tag{4.41}$$

Using Lemma 2.1 (i), we get

$$\begin{aligned}\rho(v_n,z_{n+1}) &\le \lambda_n\rho(v_n,h(v_n))+(1-\lambda_n)\rho(v_n,y_n)\\ &\le \lambda_n\rho(v_n,h(v_n))+(1-\lambda_n)\Big(\rho(v_n,w_n)+\rho(w_n,y_n)\Big).\end{aligned}$$

Utilizing (4.39) and (4.40), we get

$$\lim_{n\to\infty}\rho(v_n,z_{n+1})=0. \tag{4.42}$$

Using (4.39) and (4.42), we have

$$\lim_{n\to\infty}\rho(w_n,z_{n+1})=0. \tag{4.43}$$

Utilizing (4.37) and (4.43), we get

$$\lim_{n\to\infty}\rho(x_n,z_{n+1})=0. \tag{4.44}$$

Applying (4.40) and (4.43), we obtain

$$\lim_{n\to\infty}\rho(y_n,z_{n+1})=0. \tag{4.45}$$

By utilizing (4.36) and (4.37), we get

$$\lim_{n\to\infty}\rho(x_n,T_i^{\alpha}x_n)=0. \tag{4.46}$$

Let $\{z_{n_j}\}$ be a subsequence of $\{z_n\}$ such that $\Delta-\lim_{j\to\infty}z_{n_j}=\bar{z}$, for some $\bar{z}\in H$ and

$$\limsup_{n\to\infty}\langle\overrightarrow{h(z^*)z^*},\overrightarrow{z_{n+1}z^*}\rangle=\lim_{j\to\infty}\langle\overrightarrow{h(z^*)z^*},\overrightarrow{z_{n_j+1}z^*}\rangle$$

where $z^*=P_{\Omega}h(z^*)$.

Step 3: Next, we show that $\bar{z}\in\Omega$. Since $\Delta-\lim_{j\to\infty}z_{n_j}=\bar{z}$, then using (4.43) and (4.44) respectively, we also have $\Delta-\lim_{j\to\infty}w_{n_j}=\bar{z}$ and $\Delta-\lim_{j\to\infty}x_{n_j}=\bar{z}$. Utilizing Lemma 2.11 and (4.27), we get

$$\bar{z}\in Fix(G).$$

Using Lemma 4.13, Lemma 2.8, (4.31) and Remark 2.1, we get

$$\bar{z}\in VI(K,A).$$

By applying (4.39) and Lemma 2.8, we have

$$\bar{z}\in M.E.P(F,\varphi).$$

Utilizing Δ-demiclosed assumption on S and (4.34), we get

$$\bar{z}\in Fix(S).$$

Using (4.46), Lemma 2.9 and Δ-demiclosed assumption on T_i^{α}, for each $i\in\{1,2,...,N\}$, we obtain

$$\bar{z}\in Fix(T).$$

Hence, $\bar{z}\in\Omega$.

Step 4: Next, we show that $\limsup_{n\to\infty}\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_n z^*}\rangle \leq 0$.

Using (4.45), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_n z^*}\rangle &= \limsup_{n\to\infty}\Big(\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_n z_{n+1}}\rangle + \langle\overrightarrow{h(z^*)z^*},\overrightarrow{z_{n+1}z^*}\rangle\Big)\\ &\leq \rho(z^*,h(z^*))\lim_{n\to\infty}\rho(y_n,z_{n+1}) + \limsup_{n\to\infty}\langle\overrightarrow{h(z^*)z^*},\overrightarrow{z_{n+1}z^*}\rangle\\ &= \langle\overrightarrow{h(z^*)z^*},\overrightarrow{\bar{z}z^*}\rangle \leq 0. \end{aligned} \tag{4.47}$$

Step 5: Finally, we show that the sequence $\{z_n\}$ converges strongly to z^*. By utilizing Lemma 2.1 (iii), (4.20), (4.21) and (4.24), we have

$$\begin{aligned}\rho^2(z^*,z_{n+1}) &\leq \lambda_n^2\rho^2(z^*,h(v_n)) + (1-\lambda_n)^2\rho^2(z^*,y_n)\\ &\quad + 2\lambda_n(1-\lambda_n)\langle\overrightarrow{h(v_n)z^*},\overrightarrow{y_n z^*}\rangle\\ &\leq \lambda_n^2 M + (1-\lambda_n)^2\rho^2(z^*,v_n) + 2\lambda_n(1-\lambda_n)\Big(\langle\overrightarrow{h(v_n)h(z^*)},\overrightarrow{y_n z^*}\rangle + \langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_n z^*}\rangle\Big)\\ &\leq \lambda_n^2 M + (1-\lambda_n)^2\rho^2(z^*,v_n) + 2\lambda_n(1-\lambda_n)\Big(\gamma\rho(z^*,v_n)\rho(z^*,y_n) + \langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_n z^*}\rangle\Big)\\ &\leq \lambda_n^2 M + (1-\lambda_n)^2\rho^2(z^*,v_n) + 2\lambda_n(1-\lambda_n)\Big(\gamma\rho^2(z^*,v_n) + \langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_n z^*}\rangle\Big)\\ &= \lambda_n^2 M + \Big[1-\lambda_n\Big(2(1-\gamma)-\lambda_n(1-2\gamma)\Big)\Big]\rho^2(z^*,v_n) + 2\lambda_n(1-\lambda_n)\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_n z^*}\rangle\\ &\leq \lambda_n^2 M + \Big[1-\lambda_n\Big(2(1-\gamma)-\lambda_n(1-2\gamma)\Big)\Big]\Big(\alpha_n\rho^2(z^*,z_n) + (1-\alpha_n)\rho^2(z^*,z_{n+1})\Big)\\ &\quad + 2\lambda_n(1-\lambda_n)\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_n z^*}\rangle.\end{aligned}$$

By utilizing Condition C_1, there exists $M_0 < 1$ and $\bar{N}\in\mathbb{N}$ such that

$$\Big[1-\lambda_n\Big(2(1-\gamma)-\lambda_n(1-2\gamma)\Big)\Big] \leq M_0, \quad \forall n\geq \bar{N}.$$

Now,

$$\begin{aligned}\rho^2(z^*,z_{n+1}) &\leq \lambda_n^2 M + M_0\Big(\alpha_n\rho^2(z^*,z_n) + (1-\alpha_n)\rho^2(z^*,z_{n+1})\Big)\\ &\quad + 2\lambda_n(1-\lambda_n)\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_n z^*}\rangle\\ &\leq \lambda_n^2 M + M_0\Big(\alpha_n\rho^2(z^*,z_n) + \rho^2(z^*,z_{n+1})\Big)\\ &\quad + 2\lambda_n(1-\lambda_n)\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_n z^*}\rangle,\end{aligned}$$

which implies

$$(1-M_0)\rho^2(z^*,z_{n+1}) \leq \lambda_n^2 M + M_0\alpha_n\rho^2(z^*,z_n) + 2\lambda_n(1-\lambda_n)\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_n z^*}\rangle,$$

which implies

$$\begin{aligned}\rho^2(z^*,z_{n+1}) &\leq \frac{\lambda_n^2 M}{1-M_0} + \frac{M_0\alpha_n}{1-M_0}\rho^2(z^*,z_n) + \frac{2\lambda_n(1-\lambda_n)}{1-M_0}\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_n z^*}\rangle\\ &\leq \frac{\lambda_n^2 M}{1-M_0} + (1-\lambda_n)\rho^2(z^*,z_n) + \frac{2\lambda_n(1-\lambda_n)}{1-M_0}\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_n z^*}\rangle\\ &= (1-\lambda_n)\rho^2(z^*,z_n) + \lambda_n\Big[\frac{\lambda_n M}{1-M_0} + \frac{2(1-\lambda_n)}{1-M_0}\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_n z^*}\rangle\Big].\end{aligned}$$

Using Condition C_1, Lemma 2.10 and (4.47), we obtain $\lim_{n\to\infty} z_n = z^*$. □

Corollary 4.1. *Let H be a Hadamard space and H^* be its dual space, let K be a nonempty closed and convex subset of H. Let $A_i : K \to H$ be family of α_i-inverse strongly monotone mapping, let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying condition A_1 to A_4 of Theorem 1.2, $\varphi : K \to \mathbb{R}$ be a convex and lower semi-continuous function, let $G : K \to K$ be (f, g)-generalized κ-strictly pseudononspreading mapping with $\kappa \in [0, 1)$, where $f, g : K \to [0, \tau]$, $\tau < 1$ such that f, g, κ, τ satisfies assumptions in Theorem 2.3 [34], Remark 2.2 [34] and Lemma 2.11 [34]. Let $S : K \to K$ be nonexpansive mapping, let $T_i : H \to H$ be family of θ_i-generalized demimetric mapping with $\theta_i \neq 0$. Assume that*

$$\Omega = Fix(G) \cap Fix(S) \cap M.E.P(F, \varphi, K) \bigcap_{i=1}^{N} \Big(VI(K, A_i) \cap Fix(T_i)\Big) \neq \emptyset \text{ and } \{z_n\}_{n=1}^{\infty} \text{ is a sequence}$$

generated by

$$\begin{cases} z_0, z_1 \in X, \\ v_n = \alpha_n z_n \oplus (1 - \alpha_n) z_{n+1}, \\ w_n = T_{r_n}^{F,\varphi} v_n, \\ x_n = \delta_n w_n \oplus \beta_{n,0} G w_n \oplus \bigoplus_{i=1}^{N} \beta_{n,i} P_K A_{\mu_i} w_n, \\ y_n = a_n w_n \oplus b_{n,0} S w_n \oplus \bigoplus_{i=1}^{N} b_{n,i} T_i^{\alpha} x_n, \\ z_{n+1} = \lambda_n h(v_n) \oplus (1 - \lambda_n) y_n, \quad n \in \mathbb{N}, \end{cases} \tag{4.48}$$

where $h : H \to H$ is γ-contraction, for some $\gamma \in [0, 1)$ and for each $i \in \{1, 2, ..., m\}$,

$$A_{\mu_i} x = (1 - \mu_i) x \oplus \mu_i A_i x, \text{ with } \mu_i \in (0, 2\alpha_i - 1),$$
$$T_i^{\alpha} x = \alpha x \oplus (1 - \alpha) T_i x$$

with assumption that T_i^{α} and S are Δ-demiclosed, with $\theta_i \leq \frac{2}{1-\alpha}, \alpha \in (0, 1)$, $\{r_n\} \subseteq (0, 1)$ and $\{\alpha_n\}, \{\beta_{n,0}\}, \{\beta_{n,i}\}, \{\delta_n\}, \{b_{n,i}\}, \{b_{n,0}\}, \{a_n\}, \{\lambda_n\}$ are sequences in $\mathbb{R}$ satisfying the following conditions:

- C_1: *$\{\alpha_n\}, \{\lambda_n\}$ are in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \lambda_n = 0$ and* $\sum_{n=1}^{\infty} \alpha_n = +\infty = \sum_{n=1}^{\infty} \lambda_n$,
- C_2: *$\{a_{n,0}\}, \{\beta_{n,0}\}, \{\beta_{n,i}\}, \{a_{n,i}\} \subseteq (f_1, f_2) \subseteq (0, 1)$ and* $\beta_{n,0} + \sum_{i=1}^{N} \beta_{n,i} = a_{n,0} + \sum_{i=1}^{N} a_{n,i} = 1$,
- C_3: *$\{\delta_n\}, \{b_{n,0}\}, \{b_{n,i}\} \subseteq (f_1, f_2) \subseteq (0, 1)$ and $\delta_n + \beta_{n,0} + \sum_{i=1}^{N} \beta_{n,i} = 1$,*
- C_4: *$\{a_n\}, \{b_{n,0}\}, \{b_{n,i}\} \subseteq (f_1, f_2) \subseteq (0, 1)$ with $a_n + b_{n,0} + \sum_{i=1}^{m} b_{n,i} = 1$,*

then $\{z_n\}$ converges strongly to $z^* \in \Omega$ satisfying $z^* = P_\Omega \circ h(z^*)$.

Corollary 4.2. *Let H be a real Hilbert space, let K be a nonempty closed and convex subset of H. Let $A_i : K \to H$ be family of generalized α_i-cocoercive mapping, where $\alpha_i > \frac{1}{2}$, let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying condition A_1 to A_4 of Theorem 1.2, $\varphi : K \to \mathbb{R}$ be a convex and lower semi-continuous function, let $G : K \to K$ be (f, g)-generalized κ-strictly pseudononspreading mapping with $\kappa \in [0, 1)$, where $f, g : K \to [0, \tau]$, $\tau < 1$ such that f, g, κ, τ satisfies assumptions in Theorem*

2.3 [34], *Remark* 2.2 [34] *and Lemma* 2.11 [34]. *Let* $S : K \to K$ *be quasi-nonexpansive mapping, let* $T_i : K \to K$ *be family of* θ_i*-generalized demimetric mapping with* $\theta_i \neq 0$. *Assume that*

$\Omega = Fix(G) \cap Fix(S) \cap M.E.P(F, \varphi, K) \bigcap_{i=1}^{N} \Big(VI(K, A_i) \cap Fix(T_i)\Big) \neq \emptyset$ *and* $\{z_n\}_{n=1}^{\infty}$ *is a sequence generated by*

$$\begin{cases} z_0, z_1 \in X, \\ v_n = \alpha_n z_n + (1 - \alpha_n) z_{n+1}, \\ w_n = T_{r_n}^{F,\varphi} v_n, \\ x_n = \delta_n w_n + \beta_{n,0} G w_n + \sum_{i=1}^{N} \beta_{n,i} P_K A_{\mu_i} w_n, \\ y_n = a_n w_n + b_{n,0} S w_n + \sum_{i=1}^{N} b_{n,i} T_i^{\alpha} x_n, \\ z_{n+1} = \lambda_n h(v_n) + (1 - \lambda_n) y_n, \quad n \in \mathbb{N}, \end{cases} \tag{4.49}$$

where $h : H \to H$ *is* γ*-contraction for some* $\gamma \in [0, 1)$ *and for each* $i \in \{1, 2, ..., m\}$,

$$A_{\mu_i} x = (1 - \mu_i) x + \mu_i A_i x, \text{ with } \mu_i \in (0, 2\alpha_i - 1),$$
$$T_i^{\alpha} x = \alpha x + (1 - \alpha) T_i x$$

with assumption that T_i^{α} *and* S *are* Δ*-demiclosed, with* $\theta_i \leq \frac{2}{1-\alpha}, \alpha \in (0, 1)$, $\{r_n\} \subseteq (0, 1)$ *and* $\{\alpha_n\}, \{\beta_{n,0}\}, \{\beta_{n,i}\}, \{\delta_n\}, \{b_{n,i}\}, \{b_{n,0}\}, \{a_n\}, \{\lambda_n\}$ *are sequences in* $\mathbb{R}$ *satisfying the following conditions:*

C_1: $\{\alpha_n\}, \{\lambda_n\}$ *are in* $(0, 1)$ *such that* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \lambda_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = +\infty = \sum_{n=1}^{\infty} \lambda_n$,

C_2: $\{a_{n,0}\}, \{\beta_{n,0}\}, \{\beta_{n,i}\}, \{a_{n,i}\} \subseteq (f_1, f_2) \subseteq (0, 1)$ *and* $\beta_{n,0} + \sum_{i=1}^{N} \beta_{n,i} = a_{n,0} + \sum_{i=1}^{N} a_{n,i} = 1$,

C_3: $\{\delta_n\}, \{b_{n,0}\}, \{b_{n,i}\} \subseteq (f_1, f_2) \subseteq (0, 1)$ *and* $\delta_n + \beta_{n,0} + \sum_{i=1}^{N} \beta_{n,i} = 1$,

C_4: $\{a_n\}, \{b_{n,0}\}, \{b_{n,i}\} \subseteq (f_1, f_2) \subseteq (0, 1)$ *with* $a_n + b_{n,0} + \sum_{i=1}^{m} b_{n,i} = 1$,

then $\{z_n\}$ *converges strongly to* $z^* \in \Omega$ *satisfying* $z^* = P_\Omega \circ h(z^*)$.

5. Conclusion and Future Work

In this paper, we introduce and study some properties of a new mapping called generalized cocoercive in Hadamard space. We propose implicit viscosity-type algorithm for obtaining an element in the set of solutions of some mixed equilibrium problem, set of fixed point of (f, g)-generalized κ-strictly pseudononspreading mapping, set of fixed point of quasi-nonexpansive mapping, set of common solutions of family of variational inequality problems involving generalized cocoercive mappings and the set of common fixed point of family generalized demimetric mappings. The method of proof adapted in this paper is new, of independent interest and simpler than the standard technique used in many recent results using cases. For possible future work, one may be interested

- to get the result in more general metrically convex spaces, e.g hyperbolic spaces.
- to study an explicit version of the scheme studied in this paper.

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BASHIR ALI
BAYERO UNIVERSITY
DEPARTMENT OF MATHEMATICAL SCIENCES
KANO, NIGERIA
Email address: bashiralik@yahoo.com

ABDULNASIR BALA NUHU
BAYERO UNIVERSITY
DEPARTMENT OF MATHEMATICAL SCIENCES
KANO, NIGERIA
ALIKO DANGOTE UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS
KANO, NIGERIA
Email address: abdulnasirnuhu@gmail.com