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Research Article

On the limit operators of q-King type sequences of operators

Dedicated to Professor Ioan Raşa, on the occasion of his 75th birthday

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ABSTRACT. We determine the limit operators for a sequence of generalized q-King operators and the sequence of q-Aldaz-Kounchev-Render operators. For each sequence of operators we prove the uniform convergence to the corresponding limit operators, for which we provide quantitative estimates.

Keywords: Bernstein operators, King operators, q-Bernstein operators, q-King type operators, modulus of continuity.

2020 Mathematics Subject Classification: 41A10, 41A25, 41A36.

1. INTRODUCTION

The connection between regular summability matrices and convergent positive linear operators leads to the following operators introduced by King [13, p. 204, (2.1)]:

$$(V_n f)(x) \equiv V_n(f;x) = \sum_{k=0}^{n} p_{n,k}(r_n(x)) f\left(\frac{k}{n}\right),$$

where $f \in C[0,1]$, $\{r_n\}$ is a sequence of continuous functions defined on $[0,1]$ with $0 \le r_n(x) \le 1$ and $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0,1]$. For the special case $r_n(x) = x$, $x \in [0,1]$, we receive the Bernstein operators

$$(B_n f)(x) \equiv B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right). \tag{1.1}$$

We consider the monomial functions $e_i(x) = x^i$, where $x \in [0,1]$ and $i = 0, 1, 2, \ldots$ The King operators which preserve e_0 and e_2 are given by

$$(V_n^* f)(x) \equiv V_n^*(f;x) = \sum_{k=0}^{n} p_{n,k}(r_n^*(x)) f\left(\frac{k}{n}\right), \tag{1.2}$$

where

$$r_n^*(x) = \begin{cases} x^2, & \text{if } \quad n = 1 \\ \frac{1}{2(n-1)}\left(-1 + \sqrt{4n(n-1)x^2 + 1}\right), & \text{if } \quad n = 2, 3, \ldots \end{cases}$$

(see [13, p. 205]). Thus $(V_n^* e_0)(x) = 1$, $(V_n^* e_1)(x) = r_n^*(x)$ and $(V_n^* e_2)(x) = x^2$, $x \in [0,1]$, in contrast with Bernstein operators: $(B_n e_0)(x) = 1$, $(B_n e_1)(x) = x$ and $(B_n e_2)(x) = x^2 + \frac{x(1-x)}{n}$,

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$x \in [0,1]$. The following quantitative estimation is obtained in [13, p. 206, Theorem 3.1]:

$$|(V_n^* f)(x) - f(x)| \leq 2\,\omega(f; \sqrt{2x(x - r_n^*(x))}\,), \quad x \in [0,1],$$

where the usual modulus of continuity of $f \in C[0,1]$ is defined by

$$\omega(f;\delta) = \sup\{|f(x) - f(y)| : x, y \in [0,1], |x - y| < \delta\}, \quad \delta > 0. \tag{1.3}$$

A systematic study of the operators V_n^* $(n = 1, 2, 3, \ldots)$ is due to Gonska and Pițul [11], who determine new estimates for the rate of convergence in terms of the first and second modulus of continuity. The existence of a sequence of linear positive bounded *polynomial* operators on $C[0,1]$, possessing e_0 and e_2 as fixed points, was proved in [6]. Furthermore, in [10], we proved the unique existence of the functions r_n $(n = 1, 2, 3, \ldots)$ on $[0,1]$ such that the corresponding sequence of King operators approximate each continuous function on $[0,1]$ and preserve e_0 and e_j, where $j \in \{2, 3, \ldots\}$ is fixed. Main results concerning certain King type modifications of the Bernstein operators and the Szász-Mirakyan operators were presented in the survey paper [1].

Replacing $f\left(\frac{k}{n}\right)$ in (1.1) with $f\left(\sqrt[j]{\frac{k(k-1)\ldots(k-j+1)}{n(n-1)\ldots(n-j+1)}}\right)$, $n \geq j \geq 2$, Aldaz, Kounchev and Render [4, p. 12, Proposition 11] defined a new King type operator

$$(U_n f)(x) \equiv U_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\sqrt[j]{\frac{k(k-1)\ldots(k-j+1)}{n(n-1)\ldots(n-j+1)}}\right), \tag{1.4}$$

where $f \in C[0,1]$ and $x \in [0,1]$. These operators reproduce e_0 and e_j, where $j \in \{2, 3, \ldots\}$ is fixed, and $U_n f \to f$ uniformly for all $f \in C[0,1]$. In [7], we proved that there exist infinitely many sequences of Bernstein type operators, which have similar properties to (1.4). Further properties of the Bernstein type operators of Aldaz, Kounchev and Render were obtained in the papers [2] and [5].

The q-Bernstein operators introduced by Phillips [16] are a new generalization of (1.1). Let $0 < q \leq 1$ and denote the q-integers by $[n]_q = 1 + q + \ldots + q^{n-1}$ for $n = 1, 2, \ldots$ and $[0]_q = 0$. Besides, let $[n]_q! = [1]_q[2]_q \ldots [n]_q$ for $n = 1, 2, \ldots$ and $[0]_q! = 1$. Then the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad 0 \leq k \leq n.$$

The Bernstein operators based on q-integers are given by

$$(B_{n,q} f)(x) \equiv B_{n,q}(f;x) = \sum_{k=0}^{n} p_{n,k}(q;x) f\left(\frac{[k]_q}{[n]_q}\right), \tag{1.5}$$

where $f \in C[0,1]$ and

$$p_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)(1-xq)\ldots(1-xq^{n-k-1}), \quad x \in [0,1]$$

(an empty product in (1.5) is taken to equal 1). For $q = 1$, (1.5) reduces to (1.1). Il'inskii and Ostrovska [12] proved the existence of the limit $\lim\limits_{n\to\infty} B_{n,q}$ for $q \in (0,1)$ given, obtaining the so-called limit q-Bernstein operator defined by

$$(B_{\infty,q} f)(x) \equiv B_{\infty,q}(f;x) = \begin{cases} \sum\limits_{k=0}^{\infty} p_{\infty,k}(q;x) f(1-q^k), & \text{if} \quad x \in [0,1) \\ f(1), & \text{if} \quad x = 1, \end{cases} \tag{1.6}$$

where $p_{\infty,k}(q;x) = \frac{x^k}{(1-q)^k [k]_q!} \prod_{s=0}^{\infty}(1-xq^s)$. They proved for $q \in (0,1)$ fixed that $B_{n,q}f \to B_{\infty,q}f$ uniformly for each $f \in C[0,1]$, and $B_{\infty,q}f \to f$ uniformly for any $f \in C[0,1]$ and $q \uparrow 1$ (see [12, p. 104, Theorem 2 and Theorem 1]). Moreover, Wang and Meng showed in [20, p. 153, Theorem 1] that if $q \in (0,1)$ and $f \in C[0,1]$, then

$$|(B_{n,q}f)(x) - (B_{\infty,q}f)(x)| \leq \left(2 + \frac{4\ln\frac{1}{1-q}}{q(1-q)}\right)\omega(f;q^n), \quad x \in [0,1]. \tag{1.7}$$

Videnskii obtained in [19, p. 221, Theorem 8.1] that if $f \in C[0,1]$, then

$$|(B_{\infty,q}f)(x) - f(x)| \leq 2\,\omega\left(f; \frac{1}{2}\sqrt{1-q}\right), \quad x \in [0,1]. \tag{1.8}$$

The King type operators based on q-integers were introduced in [3, p. 7] as follows (see (1.2) and (1.5)):

$$(V^*_{n,q}f)(x) \equiv V^*_{n,q}(f;x) = \sum_{k=0}^{n} p_{n,k}(q; r^*_{n,q}(x)) f\left(\frac{[k]_q}{[n]_q}\right), \tag{1.9}$$

where $0 < q \leq 1$, $f \in C[0,1]$, $n \geq 2$ and

$$r^*_{n,q}(x) = \frac{1}{2([n]_q - 1)}\left(-1 + \sqrt{4[n]_q[n-1]_q x^2 + 1}\right), \quad x \in [0,1].$$

We mention that $V^*_{n,q}$ preserves the functions e_0 and e_2. The generalized q-King operators were given in [8, p. 346] as follows:

$$(V_{n,q}f)(x) \equiv V_{n,q}(f;x) = \sum_{k=0}^{n} p_{n,k}(q; r_{n,q}(x)) f\left(\frac{[k]_q}{[n]_q}\right), \tag{1.10}$$

where $0 < q \leq 1$, the continuous functions $r_{n,q}$ ($n = 1, 2, 3, \ldots$) are defined on $[0,1]$ with $0 \leq r_{n,q}(x) \leq 1$, $f \in C[0,1]$ and $p_{n,k}(q;x)$ are the polynomials considered in (1.5). We proved [8, p. 349, Theorem 1] that there exists a unique sequence of continuous functions $\{r_{n,q}\}$ such that $V_{n,q}$ reproduces e_0 and e_j, where $j \in \{2, 3, \ldots\}$ is fixed, and $V_{n,q_n}f \to f$ uniformly for each $f \in C[0,1]$ and $q_n \in (0,1)$, $q_n \to 1$; the rate of convergence is estimated with the aid of the modulus of continuity (1.3). For $n \geq j$ and $q \in (0,1)$, the function $r_{n,q}$ will be the unique solution of the equation

$$(B_{n,q}e_j)(y) - x^j \equiv \sum_{k=0}^{n} p_{n,k}(q;y)\left(\frac{[k]_q}{[n]_q}\right)^j - x^j = 0, \quad x \in [0,1] \tag{1.11}$$

(see [8, p. 349, Theorem 1] and Lemma 2.2 below). For $j = 2$, $V_{n,q}$ is identical with $V^*_{n,q}$.

The q-Aldaz-Kounchev-Render operators were defined in [9, p. 756, (3.1)]: for $0 < q \leq 1$, $n \geq j$, $f \in C[0,1]$ and $x \in [0,1]$, we set

$$(U_{n,q}f)(x) \equiv U_{n,q}(f;x) = \sum_{k=0}^{n} p_{n,k}(q;x) f\left(\sqrt[j]{\frac{[k]_q[k-1]_q \ldots [k-j+1]_q}{[n]_q[n-1]_q \ldots [n-j+1]_q}}\right), \tag{1.12}$$

where $j \in \{2, 3, \ldots\}$ is fixed and $p_{n,k}(q;x)$ is given in (1.5). We mention that $[k]_q[k-1]_q \ldots [k-j+1]_q = [0]_q = 0$ for $k = 0, 1, \ldots, j-1$, and the operator $U_{n,1}$ is identical with U_n defined by (1.4). Furthermore, we have $U_{n,q}e_0 = e_0$ and $U_{n,q}e_j = e_j$. Among others, a new Korovkin type theorem and its converse theorem are established in [9]. As applications we obtained quantitative estimates for q-Bernstein type operators which preserve e_0 and e_j.

The main goal of the paper is to determine the limit operators of the sequences of operators given by (1.10) and (1.12). We establish the essential properties of the functions $r_{n,q}$ defined by (1.11) that guarantee the uniform convergence of $V_{n,q}f$ to its limit operator for each $f \in C[0,1]$ and $q \in (0,1)$ given. We also give an estimate similar to (1.8) for the limit operator of the sequence of operators $\{V_{n,q}\}$. Furthermore, we establish quantitative estimates similar to (1.7) and (1.8) for the limit operator of the sequence of operators defined by (1.12).

2. The Case of Generalized q-King Operators

In what follows we need some lemmas.

Lemma 2.1. *Let $j \in \{2,3,\ldots\}$ be given, $n \geq j$ and $0 < q < 1$. Then*

$$(B_{n,q}e_j)(y) = \sum_{k=0}^{n} p_{n,k}(q;y)\left(\frac{[k]_q}{[n]_q}\right)^j, \quad y \in [0,1]$$

is a polynomial in y of degree $\leq j$. Moreover, $(B_{n,q}e_j)(y) = a_0 y^j + a_1 y^{j-1} + \ldots + a_{j-1}y$, $y \in [0,1]$, where $a_0, a_1, \ldots, a_{j-1}$ depend on n and q, and $a_0 = \left(1 - \frac{[1]_q}{[n]_q}\right)\left(1 - \frac{[2]_q}{[n]_q}\right)\ldots\left(1 - \frac{[j-1]_q}{[n]_q}\right)$, a_0, a_1, …, $a_{j-1} > 0$, $a_0 + a_1 + \ldots + a_{j-1} = 1$.

For the proof see [8, p. 346, Lemma 1].

Lemma 2.2. *Let $j \in \{2,3,\ldots\}$ be given, $n \geq j$ and $0 < q < 1$. Then, the equation*

$$(B_{n,q}e_j)(y) - x^j = 0, \quad x \in [0,1]$$

has a unique solution $r_{n,q}(x)$, $x \in [0,1]$ such that $r_{n,q}(0) = 0$, $r_{n,q}(1) = 1$ and $0 < r_{n,q}(x) < 1$ for $x \in (0,1)$.

Proof. By [15, p. 236, (6)-(7)], we have

$$(B_{n,q}f)(y) = \sum_{k=0}^{n} \lambda_{k,n}\, f\left[0, \frac{1}{[n]_q}, \ldots, \frac{[k]_q}{[n]_q}\right] y^k,$$

where $\lambda_{0,n} = \lambda_{1,n} = 1$, $\lambda_{k,n} = \prod_{i=1}^{k-1}\left(1 - \frac{[i]_q}{[n]_q}\right)$, $k = 2,3,\ldots,n$ and $f[y_0,y_1,\ldots,y_k]$ is the k-th order divided difference of f with distinct nodes $y_0, y_1, \ldots, y_k$:

$$f[y_0] = f(y_0) \quad f[y_0,y_1,\ldots,y_k] = \frac{f[y_1,\ldots,y_k] - f[y_0,\ldots,y_{k-1}]}{y_k - y_0}.$$

Hence

$$(B_{n,q}e_j)(y) = \sum_{k=0}^{n} \lambda_{k,n}\, e_j\left[0, \frac{1}{[n]_q}, \ldots, \frac{[k]_q}{[n]_q}\right] y^k. \tag{2.13}$$

In view of [17, p. 10, (1.33)], there exists $\xi_k \in \left(0, \frac{[k]_q}{[n]_q}\right)$ such that

$$\begin{aligned} e_j\left[0, \frac{1}{[n]_q}, \ldots, \frac{[k]_q}{[n]_q}\right] &= \frac{1}{k!}e_j^{(k)}(\xi_k) \\ &= \begin{cases} 0, & \text{if} \quad k = 0 \\ \frac{1}{k!}j(j-1)\ldots(j-k+1)(\xi_k)^{j-k}, & \text{if} \quad 1 \leq k \leq j \\ 0, & \text{if} \quad j+1 \leq k \leq n. \end{cases} \end{aligned} \tag{2.14}$$

By (2.13) and (2.14), we get

$$(B_{n,q}e_j)'(y) = \sum_{k=1}^{n} \lambda_{k,n}\, e_j \left[0, \frac{1}{[n]_q}, \ldots, \frac{[k]_q}{[n]_q}\right] ky^{k-1} > 0 \tag{2.15}$$

for all $y \in [0,1]$, thus the function $B_{n,q}e_j$ is strictly increasing on $[0,1]$. Because $(B_{n,q}e_j)(0) = 0$, $(B_{n,q}e_j)(1) = 1$ and $B_{n,q}e_j \in C[0,1]$, therefore the equation $(B_{n,q}e_j)(y) - x^j = 0$, $x \in [0,1]$ has a unique solution noted by $y = r_{n,q}(x)$, $x \in [0,1]$ such that $r_{n,q}(0) = 0$, $r_{n,q}(1) = 1$ and $0 < r_{n,q}(x) < 1$ for $x \in (0,1)$. □

The essential properties of $r_{n,q}$ $(n = j, j+1, j+2, \ldots)$ from our point of view are summarized by the following lemma.

Lemma 2.3. *Let $j \in \{2, 3, \ldots\}$ be given, $n \geq j$ and $0 < q < 1$. The functions $r_{n,q}$ $(n \geq j)$ determined in Lemma 2.2 satisfy the following properties:*

a) $r_{n,q}$ is differentiable on $[0,1]$;
b) $r_{n,q}(x) \leq r_{n+1,q}(x) \leq x$ for all $x \in [0,1]$;
c) there exists a function $r_q : [0,1] \to \mathbb{R}$ continuous on the $[0,1]$ such that $\lim_{n\to\infty} r_{n,q}(x) = r_q(x)$ uniformly for $x \in [0,1]$;
d) $r_{n,q}(x) \leq r_q(x) \leq x$ for all $x \in [0,1]$.

Proof. a) Because the function $B_{n,q}e_j$ is continuous and strictly increasing on $[0,1]$, there exists the continuous and strictly increasing inverse function $(B_{n,q}e_j)^{-1}$ on $[0,1]$, due to the following continuous inverse theorem: If $\varphi : [a,b] \to \mathbb{R}$ is a strictly increasing and continuous function, then the inverse mapping $\varphi^{-1} : [\varphi(a), \varphi(b)] \to [a,b]$ exists and is strictly increasing and continuous function on $[\varphi(a), \varphi(b)]$.

On the other hand, by Lemma 2.2, we have $(B_{n,q}e_j)(r_{n,q}(x)) - x^j = 0$, $x \in [0,1]$, which implies that $r_{n,q}(x) = ((B_{n,q}e_j)^{-1} \circ e_j)(x)$, $x \in [0,1]$. Thus $r_{n,q}$ is differentiable on $[0,1]$ and

$$r'_{n,q}(x) = ((B_{n,q}e_j)^{-1})'(x^j) \cdot e'_j(x) = \frac{jx^{j-1}}{(B_{n,q}e_j)'(r_{n,q}(x))}, \quad x \in [0,1], \tag{2.16}$$

because of (2.15) and $(B_{n,q}e_j)(r_{n,q}(x)) = x^j$, $x \in [0,1]$.
b) By [17, p. 268, (7.56)], we have $B_{n,q}(e_1; x) = x$, $x \in [0,1]$. Applying Lemma 2.2 and Jensen's inequality to the convex function e_j on $[0,1]$, we get

$$\begin{aligned} x^j &= B_{n,q}(e_j; r_{n,q}(x)) = \sum_{k=0}^{n} p_{n,k}(q; r_{n,q}(x)) \left(\frac{[k]_q}{[n]_q}\right)^j \geq \left(\sum_{k=0}^{n} p_{n,k}(q; r_{n,q}(x)) \left(\frac{[k]_q}{[n]_q}\right)\right)^j \\ &= (B_{n,q}(e_1; r_{n,q}(x)))^j = (r_{n,q}(x))^j. \end{aligned}$$

Hence $r_{n,q}(x) \leq x$, $x \in [0,1]$. Furthermore, by [17, p. 270, Theorem 7.3.4], $(B_{n,q}e_j)(x) \geq (B_{n+1,q}e_j)(x)$, $x \in [0,1]$. Because $(B_{n,q}e_j)(0) = 0 = (B_{n+1,q}e_j)(0)$ and $(B_{n,q}e_j)(1) = 1 = (B_{n+1,q}e_j)(1)$, by continuous inverse theorem (see also [10, p. 91, Lemma 2.1]), we find that $(B_{n,q}e_j)^{-1}(y) \leq (B_{n+1,q}e_j)^{-1}(y)$, $y \in [0,1]$. Then $r_{n,q}(x) = (B_{n,q}e_j)^{-1}(x^j) \leq (B_{n+1,q}e_j)^{-1}(x^j) = r_{n+1,q}(x)$, $x \in [0,1]$.
c) Due to (2.16) and (2.15), we have $r'_{n,q}(x) \geq 0$ for each $x \in [0,1]$. Thus the functions $r_{n,q}$ $(n = j, j+1, j+2, \ldots)$ are increasing on $[0,1]$.

In view of b), the sequence $\{r_{n,q}(x)\}_{n\geq j}$ is increasing and bounded above for all $x \in [0,1]$, therefore it is convergent. We set $r_q(x) = \lim_{n\to\infty} r_{n,q}(x)$, $x \in [0,1]$.

Let $x_0, x \in [0,1]$. By Lemma 2.1, we have

$$\begin{aligned}
&(B_{n,q}e_j)(r_{n,q}(x)) - (B_{n,q}e_j)(r_{n,q}(x_0))\\
&= a_0\left\{(r_{n,q}(x))^j - (r_{n,q}(x_0))^j\right\} + a_1\left\{(r_{n,q}(x))^{j-1} - (r_{n,q}(x_0))^{j-1}\right\} + \dots\\
&+ a_{j-1}\left\{r_{n,q}(x) - r_{n,q}(x_0)\right\}\\
&= \left\{r_{n,q}(x) - r_{n,q}(x_0)\right\}\left\{a_0\left((r_{n,q}(x))^{j-1} + \dots + (r_{n,q}(x_0))^{j-1}\right)\right.\\
&\left. + a_1\left((r_{n,q}(x))^{j-2} + \dots + (r_{n,q}(x_0))^{j-2}\right) + \dots + a_{j-1}\right\}.
\end{aligned}$$

Hence, by Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}
&|(B_{n,q}e_j)(r_{n,q}(x)) - (B_{n,q}e_j)(r_{n,q}(x_0))|\\
&\geq |r_{n,q}(x) - r_{n,q}(x_0)|\, a_0\left\{(r_{n,q}(x))^{j-1} + \dots + (r_{n,q}(x_0))^{j-1}\right\}\\
&\geq |r_{n,q}(x) - r_{n,q}(x_0)|\left(1 - \frac{[1]_q}{[n]_q}\right)\left(1 - \frac{[2]_q}{[n]_q}\right)\dots\left(1 - \frac{[j-1]_q}{[n]_q}\right)(r_{n,q}(x_0))^{j-1}.
\end{aligned}$$

Due to Lemma 2.2,

$$\begin{aligned}
&|r_{n,q}(x) - r_{n,q}(x_0)|\left(1 - \frac{[1]_q}{[n]_q}\right)\left(1 - \frac{[2]_q}{[n]_q}\right)\dots\left(1 - \frac{[j-1]_q}{[n]_q}\right)(r_{n,q}(x_0))^{j-1}\\
&\leq |x^j - x_0^j| = |x - x_0||x^{j-1} + \dots + x_0^{j-1}| \leq j|x - x_0|.
\end{aligned}$$

By passing to the limit in terms of n, we get

$$|r_q(x) - r_q(x_0)|\, q\, q^2 \dots q^{j-1}\,(r_q(x_0))^{j-1} \leq j|x - x_0|.$$

If $r_q(x_0) \neq 0$, then we obtain the continuity of r_q at x_0. Furthermore, in view of Lemma 2.2, we have $r_{n,q}(0) = 0$ and $0 < r_{n,q}(x) < 1$ for $x \in (0,1)$. At the same time, the sequence $\{r_{n,q}(x)\}_{n\geq j}$ is increasing for all $x \in (0,1)$, therefore if $r_q(x_0) = 0$, then $x_0 = 0$. Because $0 \leq r_{n,q}(x) \leq x$, $x \in [0,1]$ (see b)), we have $|r_{n,q}(x) - r_q(x_0)| \leq x$, $x \in [0,1]$. By passing to the limit in terms of n, we find that $|r_q(x) - r_q(x_0)| \leq x$, $x \in [0,1]$, therefore r_q is continuous at $x_0 = 0$.

Based on the properties established for the functions r_q and $r_{n,q}$ $(n = 1,2,3,\dots)$, we obtain the uniform convergence $\lim\limits_{n\to\infty} r_{n,q}(x) = r_q(x)$ on $[0,1]$ by Pólya's theorem of uniform convergence (see [18, p. 81, Problem 127]).

d) Due to b), we have $r_{n,q}(x) \leq r_{m,q}(x) \leq x$ for each $x \in [0,1]$ and $n \leq m$. By passing to the limit in terms of m, in view of c), we get $r_{n,q}(x) \leq r_q(x) \leq x$, $x \in [0,1]$, which completes the proof of the lemma. □

Taking into account (1.10), (1.5) and (1.6), we have $V_{n,q}(f;x) = B_{n,q}(f;r_{n,q}(x))$, therefore the corresponding limit operator is defined by

$$(V_{\infty,q}f)(x) \equiv V_{\infty,q}(f;x) = B_{\infty,q}(f;r_q(x)) = \begin{cases} \sum\limits_{k=0}^{\infty} p_{\infty,k}(q;r_q(x))f(1-q^k), & \text{if} \quad x \in [0,1)\\ f(1), & \text{if} \quad x = 1. \end{cases}$$

We have the following result.

Theorem 2.1. *Let $q \in (0,1)$ be given. Then for each $f \in C[0,1]$ we have*

$$\lim_{n\to\infty}(V_{n,q}f)(x) = (V_{\infty,q}f)(x)$$

uniformly for $x \in [0,1]$.

Proof. For all $x \in [0,1]$, we have

$$\begin{aligned}
&|(V_{n,q}f)(x) - (V_{\infty,q}f)(x)| \\
&\le |B_{n,q}(f; r_{n,q}(x)) - B_{\infty,q}(f; r_{n,q}(x))| + |B_{\infty,q}(f; r_{n,q}(x)) - B_{\infty,q}(f; r_q(x))|. \qquad (2.17)
\end{aligned}$$

Because $\omega(f; q^n) \to 0$ as $n \to \infty$, by (1.7), for each $\varepsilon > 0$ there exists $n'_\varepsilon \in \mathbb{N}$ such that for all $n > n'_\varepsilon$ and $x \in [0,1]$, we have

$$|B_{n,q}(f; r_{n,q}(x)) - B_{\infty,q}(f; r_{n,q}(x))| < \frac{\varepsilon}{2}. \qquad (2.18)$$

On the other hand, [12, p. 104, Theorem 2] implies that $\lim\limits_{n\to\infty} (B_{n,q}f)(x) = (B_{\infty,q}f)(x)$ uniformly on $[0,1]$. Thus $B_{\infty,q}f \in C[0,1]$, therefore $B_{\infty,q}f$ is uniformly continuous on $[0,1]$ for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $u, v \in [0,1]$, $|u - v| < \delta$, we have

$$|B_{\infty,q}(f; u) - B_{\infty,q}(f; v)| < \frac{\varepsilon}{2}. \qquad (2.19)$$

In view of Lemma 2.3, c), we have $\lim\limits_{n\to\infty} r_{n,q}(x) = r_q(x)$ uniformly on $[0,1]$. Then there exists $n''_\varepsilon \in \mathbb{N}$ such that for each $n > n''_\varepsilon$ and $x \in [0,1]$, we have

$$|r_{n,q}(x) - r_q(x)| < \delta. \qquad (2.20)$$

Combining (2.19) and (2.20), we obtain for all $n > n''_\varepsilon$ and $x \in [0,1]$ that

$$|B_{\infty,q}(f; r_{n,q}(x)) - B_{\infty,q}(f; r_q(x))| < \frac{\varepsilon}{2}. \qquad (2.21)$$

In conclusion, by (2.17), (2.18) and (2.21), we get that for each $\varepsilon > 0$ there exists $n_\varepsilon = \max(n'_\varepsilon, n''_\varepsilon)$ such that for all $n > n_\varepsilon$ and $x \in [0,1]$, we have $|(V_{n,q}f)(x) - (V_{\infty,q}f)(x)| < \varepsilon$, which means that $\lim\limits_{n\to\infty} (V_{n,q}f)(x) = (V_{\infty,q}f)(x)$ uniformly for $x \in [0,1]$. □

Theorem 2.2. *Let $j \in \{2,3,\ldots\}$ be given and $q \in (0,1)$. If $f \in C[0,1]$ and $x \in [0,1]$, then*

$$|(V_{\infty,q}f)(x) - f(x)| \le \left(1 + \sqrt{\frac{1}{2} + 2\sqrt[j]{\frac{1}{4} j^2 (j-1)^2}}\right) \omega(f; \sqrt[j]{1-q}). \qquad (2.22)$$

Proof. Because $\sum\limits_{k=0}^{\infty} p_{\infty,k}(q; x) = 1$, $x \in [0,1)$ (see [12, p. 103, (13)]), and applying the proof of general Popoviciu's theorem [14, p. 20, Theorem 1.6.1], we obtain

$$|(V_{\infty,q}f)(x) - f(x)| \le \omega(f; \delta) \left\{ 1 + \delta^{-1} \left(\sum_{k=0}^{\infty} p_{\infty,k}(q; r_q(x))(1 - q^k - x)^2 \right)^{1/2} \right\}, \qquad (2.23)$$

where $\delta > 0$ and $x \in [0,1)$. Because of $(a+b)^2 \le 2(a^2 + b^2)$, $a, b \in \mathbb{R}$ and $B_{\infty,q}((e_1 - xe_0)^2; x) = (1-q)x(1-x)$ (see [19, p. 221, (7.12)]), we get

$$\begin{aligned}
&\sum_{k=0}^{\infty} p_{\infty,k}(q; r_q(x))(1 - q^k - x)^2 \\
&\le 2 \sum_{k=0}^{\infty} p_{\infty,k}(q; r_q(x))(1 - q^k - r_q(x))^2 + 2 \sum_{k=0}^{\infty} p_{\infty,k}(q; r_q x)(x - r_q(x))^2 \\
&= 2B_{\infty,q}((e_1 - r_q(x)e_0)^2; r_q(x)) + 2(x - r_q(x))^2 \\
&= 2(1-q) r_q(x)(1 - r_q(x)) + 2(x - r_q(x))^2. \qquad (2.24)
\end{aligned}$$

Due to the estimation $|x - r_{n,q}(x)| \leq ([1]_q + [2]_q + \ldots + [j-1]_q)^{1/j} [n]_q^{-1/j}$ given in [8, p. 350, (10)], we find

$$(x - r_{n,q}(x))^2 \leq ([1]_q + [2]_q + \ldots + [j-1]_q)^{2/j} [n]_q^{-2/j} \leq (1 + 2 + \ldots + j - 1)^{2/j} [n]_q^{-2/j}.$$

By passing to the limit in terms of n and using Lemma 2.3, c), we obtain

$$(x - r_q(x))^2 \leq \left(\frac{1}{2} j(j-1)\right)^{2/j} (1-q)^{2/j}, \quad x \in [0,1).$$

Hence, by (2.24),

$$\sum_{k=0}^{\infty} p_{\infty,k}(q; r_q(x))(1 - q^k - r_q(x))^2$$
$$\leq \frac{1}{2}(1-q) + 2\left(\frac{1}{2} j(j-1)\right)^{2/j} (1-q)^{2/j} \leq \left(\frac{1}{2} + 2\sqrt[j]{\frac{1}{4} j^2 (j-1)^2}\right)(1-q)^{2/j}.$$

Now, (2.23) implies that

$$|(V_{\infty,q} f)(x) - f(x)| \leq \omega(f; \delta)\left\{1 + \delta^{-1}\sqrt{\frac{1}{2} + 2\sqrt[j]{\frac{1}{4} j^2 (j-1)^2}}\,(1-q)^{1/j}\right\}.$$

Taking $\delta = (1-q)^{1/j}$, we obtain (2.22) for $x \in [0,1)$. Finally, if $x = 1$, then $(V_{\infty,q} f)(1) = (B_{\infty,q} f)(r_q(1)) = (B_{\infty,q} f)(1) = f(1)$, because $1 = r_{n,q}(1) = \lim_{n \to \infty} r_{n,q}(1) = r_q(1)$ in view of Lemma 2.2 and Lemma 2.3, c). Thus (2.22) is true for all $x \in [0,1]$. $\square$

Remark 2.1. *The limit operator of the sequence of operators given by (1.9) is the following:*

$$(V^*_{\infty,q} f)(x) = (B_{\infty,q} f)(r^*_q(x)) = \begin{cases} \sum_{k=0}^{\infty} p_{\infty,k}(q; r^*_q(x)) f(1-q^k), & \text{if} \quad x \in [0,1) \\ f(1), & \text{if} \quad x = 1, \end{cases}$$

where $r^*_q(x) = \frac{1}{2q}\left(-1 + q + \sqrt{x^2 + (1-q)^2}\right)$, $x \in [0,1]$.

3. The Case of q-Aldaz-Kounchev-Render Operators

The q-Aldaz-Kounchev-Render operators (see (1.12)) depend on the nodes

$$a_{n,k} \equiv a_{n,k}(q,j)$$
$$= \sqrt[j]{\frac{[k]_q [k-1]_q \ldots [k-j+1]_q}{[n]_q [n-1]_q \ldots [n-j+1]_q}} = \sqrt[j]{\frac{(1-q^k)(1-q^{k-1}) \ldots (1-q^{k-j+1})}{(1-q^n)(1-q^{n-1}) \ldots (1-q^{n-j+1})}}, \quad k = 0, 1, \ldots, n.$$

With the notation

$$a_{\infty,k} \equiv a_{\infty,k}(q,j) = \sqrt[j]{(1-q^k)(1-q^{k-1}) \ldots (1-q^{k-j+1})}, \quad k = 0, 1, 2, \ldots,$$

the corresponding limit operator of the sequence of operators given by (1.12) is the following:

$$(U_{\infty,q} f)(x) = U_{\infty,q}(f; x) = \begin{cases} \sum_{k=0}^{\infty} p_{\infty,k}(q; x) f(a_{\infty,q}), & \text{if} \quad x \in [0,1) \\ f(1), & \text{if} \quad x = 1. \end{cases} \tag{3.25}$$

The next result contains the properties of the nodes $a_{n,k}$ and $a_{\infty,k}$.

Lemma 3.4. *Let $j \in \{2, 3, \ldots\}$ be given and $n \geq j$. Then for the numbers $a_{n,k}$ $(k = 0, 1, \ldots, n)$ and $a_{\infty,k}$ $(k = 0, 1, 2, \ldots)$, we have*

a) $a_{n,0} = a_{n,1} = \ldots = a_{n,j-1} = 0$, $a_{n,n} = 1$ *and* $a_{\infty,0} = a_{\infty,1} = \ldots = a_{\infty,j-1} = 0$;
b) $0 < a_{n,k} - a_{\infty,k} < q^{n-j+1}$ *for* $k = j, j+1, \ldots, n$;
c) $\frac{[j]_q}{j}\, q^{k-j+1} < 1 - a_{\infty,k} < q^{k-j+1}$ *for* $k = j, j+1, j+2, \ldots$

Proof. *a)* It follows immediately from the definitions of $a_{n,k}$ and $a_{\infty,k}$. For $k = 0, 1, \ldots, j-1$, the value of $a_{n,k}$ can be calculated using its second form.
b) Because $0 < q^{k-i} < 1$ and $0 < q^{n-i} < 1$ for $k \in \{j, j+1, \ldots, n\}$ and $i \in \{0, 1, \ldots, j-1\}$, we get

$$\begin{aligned} &a_{n,k} - a_{\infty,k} \\ &= \sqrt[j]{\frac{(1-q^k)(1-q^{k-1})\ldots(1-q^{k-j+1})}{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-j+1})}} \left(1 - \sqrt[j]{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-j+1})}\right) > 0. \end{aligned}$$

On the other hand, by the inequality of geometric and harmonic means and the inequalities $1 - q^n > 1 - q^{n-1} > \ldots > 1 - q^{n-j+1}$, $n \geq j$, we obtain

$$\begin{aligned} &\sqrt[j]{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-j+1})} \\ &> \frac{j}{\frac{1}{1-q^n} + \frac{1}{1-q^{n-1}} + \ldots + \frac{1}{1-q^{n-j+1}}} \geq \frac{1}{\frac{1}{1-q^{n-j+1}}} = 1 - q^{n-j+1}. \end{aligned} \tag{3.26}$$

But $\frac{1-q^{k-i}}{1-q^{n-i}} \leq 1$ for $i \in \{0, 1, \ldots, j-1\}$, since $q^{n-i} \leq q^{k-i}$, where $n \geq j$, $k \in \{j, j+1, \ldots, n\}$ and $i \in \{0, 1, \ldots, j-1\}$. In conclusion $a_{n,k} - a_{\infty,k} < q^{n-j+1}$ for $k = j, j+1, \ldots, n$.
c) By the inequality of arithmetic and geometric means, for $k = j, j+1, \ldots, n$, we have

$$\begin{aligned} &\sqrt[j]{(1-q^k)(1-q^{k-1})\ldots(1-q^{k-j+1})} \\ &< \frac{1-q^k + 1 - q^{k-1} + \ldots + 1 - q^{k-j+1}}{j} = \frac{j - q^{k-j+1}(1 + q + \ldots + q^{j-1})}{j} = 1 - \frac{[j]_q}{j} q^{k-j+1} \end{aligned}$$

or

$$\frac{[j]_q}{j} q^{k-j+1} < 1 - \sqrt[j]{(1-q^k)(1-q^{k-1})\ldots(1-q^{k-j+1})} = 1 - a_{\infty,k}. \tag{3.27}$$

Analogously to (3.26), we have for $k = j, j+1, \ldots$ that

$$\sqrt[j]{(1-q^k)(1-q^{k-1})\ldots(1-q^{k-j+1})} > 1 - q^{k-j+1} \quad \text{or} \quad 1 - a_{\infty,k} < q^{k-j+1}. \tag{3.28}$$

Then, (3.27) and (3.28) imply the desired inequalities. □

We have the following result.

Theorem 3.3. *Let $j \in \{2, 3, \ldots\}$ be given, $n \geq j$ and $0 < q < 1$. Then*

$$\begin{aligned} &|(U_{n,q}f)(x) - (U_{\infty,q}f)(x)| \\ &\leq 2\left(1 + \frac{2}{q(1-q)} \ln\left(\frac{1}{1-q}\right)\right) \omega\left(f; q^{n-j+1}\right) + |f(0) - f(1)| \frac{2}{q(1-q)} \ln\left(\frac{1}{1-q}\right) q^{n-j+1} \end{aligned} \tag{3.29}$$

for each $x \in [0, 1]$ and $f \in C[0, 1]$.

Proof. By (1.12) and (3.25), we have $(U_{n,q}f)(1) = f(1) = (U_{\infty,q}f)(1)$, which implies the validity of (3.29) for $x = 1$.

Now, let $x \in [0,1)$. In view of $\sum_{k=0}^{n} p_{n,k}(q;x) = 1$ (see [17, p. 268, (7.55)]), $\sum_{k=0}^{\infty} p_{\infty,k}(q;x) = 1$ (see [12, p. 103, (13)]) and Lemma 3.4, a), we have

$$
\begin{aligned}
&|(U_{n,q}f)(x) - (U_{\infty,q}f)(x)| \\
&= \Big| \sum_{k=0}^{n} p_{n,k}(q;x) f(a_{n,k}) - \sum_{k=0}^{\infty} p_{\infty,k}(q;x) f(a_{\infty,k}) \Big| \\
&= \Big| \sum_{k=0}^{n} p_{n,k}(q;x) f(a_{n,k}) - \sum_{k=0}^{\infty} p_{\infty,k}(q;x) f(a_{\infty,k}) - f(1)\{\sum_{k=0}^{n} p_{n,k}(q;x) - \sum_{k=0}^{\infty} p_{\infty,k}(q;x)\} \Big| \\
&= \Big| \sum_{k=0}^{j-1} \{p_{n,k}(q;x) - p_{\infty,k}(q;x)\}\{f(0) - f(1)\} + \sum_{k=j}^{n} p_{n,k}(q;x) f(a_{n,k}) \\
&- \sum_{k=j}^{\infty} p_{\infty,k}(q;x) f(a_{\infty,k}) - f(1)\{\sum_{k=j}^{n} p_{n,k}(q;x) - \sum_{k=j}^{\infty} p_{\infty,k}(q;x)\} \Big| \\
&= \Big| \sum_{k=0}^{j-1} \{p_{n,k}(q;x) - p_{\infty,k}(q;x)\}\{f(0) - f(1)\} + \sum_{k=j}^{n} p_{n,k}(q;x)\{f(a_{n,k}) - f(a_{\infty,k})\} \\
&+ \sum_{k=j}^{n} \{p_{n,k}(q;x) - p_{\infty,k}(q;x)\}\{f(a_{\infty,k}) - f(1)\} - \sum_{k=n+1}^{\infty} p_{\infty,k}(q;x)\{f(a_{\infty,k}) - f(1)\} \Big| \\
&\le |f(0) - f(1)| \sum_{k=0}^{j-1} |p_{n,k}(q;x) - p_{\infty,k}(q;x)| + \sum_{k=j}^{n} p_{n,k}(q;x)|f(a_{n,k}) - f(a_{\infty,k})| \\
&+ \sum_{k=j}^{n} |p_{n,k}(q;x) - p_{\infty,k}(q;x)||f(a_{\infty,k}) - f(1)| + \sum_{k=n+1}^{\infty} p_{\infty,k}(q;x)|f(a_{\infty,k}) - f(1)| \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}
\tag{3.30}
$$

First, we estimate I_1. In view of [20, p. 156, (2.9)], we have

$$
\sum_{k=0}^{n} q^k |p_{n,k}(q;x) - p_{\infty,k}(q;x)| \le \frac{2q^n}{q(1-q)} \ln\left(\frac{1}{1-q}\right). \tag{3.31}
$$

Therefore

$$
\begin{aligned}
I_1 &= |f(0) - f(1)| q^{-j+1} \sum_{k=0}^{j-1} q^{j-1} |p_{n,k}(q;x) - p_{\infty,k}(q;x)| \\
&\le |f(0) - f(1)| q^{-j+1} \sum_{k=0}^{j-1} q^{k} |p_{n,k}(q;x) - p_{\infty,k}(q;x)| \\
&\le |f(0) - f(1)| q^{-j+1} \sum_{k=0}^{n} q^{k} |p_{n,k}(q;x) - p_{\infty,k}(q;x)|
\end{aligned}
$$

$$\leq |f(0)-f(1)|\frac{2}{q(1-q)}\ln\left(\frac{1}{1-q}\right)q^{n-j+1}. \tag{3.32}$$

For the estimation of I_2, we apply Lemma 3.4, b):

$$I_2 \leq \sum_{k=j}^{n} p_{n,k}(q;x)\omega\left(f;a_{n,k}-a_{\infty,k}\right) \leq \sum_{k=j}^{n} p_{n,k}(q;x)\omega\left(f;q^{n-j+1}\right)$$

$$\leq \omega\left(f;q^{n-j+1}\right)\sum_{k=0}^{n} p_{n,k}(q;x) = \omega\left(f;q^{n-j+1}\right). \tag{3.33}$$

At the estimation of I_3 we use the property $\omega(f;\lambda\delta) \leq (1+\lambda)\omega(f;\delta)$, where $\lambda > 0$, $\delta > 0$. Then, by Lemma 3.4, c), we have for $k \in \{j, j+1, \ldots, n\}$ that

$$\begin{aligned}
|f(a_{\infty,k})-f(1)| &\leq \omega\left(f;1-a_{\infty,k}\right) \leq \left(1+q^{-n+j-1}(1-a_{\infty,k})\right)\omega\left(f;q^{n-j+1}\right)\\
&= q^{-n+j-1}\left(q^{n-j+1}+(1-a_{\infty,k})\right)\omega\left(f;q^{n-j+1}\right)\\
&\leq q^{-n+j-1}\left(q^{n-j+1}+q^{k-j+1}\right)\omega\left(f;q^{n-j+1}\right)\\
&\leq q^{n-j+1}\,2q^{k-j+1}\omega\left(f;q^{n-j+1}\right) = 2q^{k-n}\omega\left(f;q^{n-j+1}\right).
\end{aligned}$$

Then, by (3.31),

$$I_3 \leq 2q^{-n}\omega\left(f;q^{n-j+1}\right)\sum_{k=j}^{n} q^k|p_{n,k}(q;x)-p_{\infty,k}(q;x)|$$

$$\leq 2q^{-n}\omega\left(f;q^{n-j+1}\right)\sum_{k=0}^{n} q^k|p_{n,k}(q;x)-p_{\infty,k}(q;x)|$$

$$\leq \frac{4}{q(1-q)}\ln\left(\frac{1}{1-q}\right)\omega\left(f;q^{n-j+1}\right). \tag{3.34}$$

If $k \geq n+1$, then, in view Lemma 3.4, c), we get

$$|f(a_{\infty,k})-f(1)| \leq \omega\left(f;1-a_{\infty,k}\right) \leq \omega\left(f;q^{k-j+1}\right) \leq \omega\left(f;q^{n-j+1}\right).$$

Hence, we obtain the estimation of I_4:

$$I_4 \leq \sum_{k=n+1}^{\infty} p_{\infty,k}(q;x)\omega\left(f;q^{n-j+1}\right) \leq \omega\left(f;q^{n-j+1}\right)\sum_{k=0}^{\infty} p_{\infty,k}(q;x) = \omega\left(f;q^{n-j+1}\right). \tag{3.35}$$

Combining (3.30), (3.32)-(3.35), we obtain (3.29). □

Theorem 3.4. *Let $j \in \{2,3,\ldots\}$ be given and $q \in (0,1)$. If $f \in C[0,1]$ and $x \in [0,1]$, then*

$$|(U_{\infty,q}f)(x)-f(x)| \leq \left(1+\sqrt{\frac{1}{2}+2(j-1)^2}\right)\omega\left(f;\sqrt{1-q}\right). \tag{3.36}$$

Proof. Analogously to the proof of Theorem 2.2, we have

$$|(U_{\infty,q}f)(x)-f(x)| \leq \omega(f;\delta)\left\{1+\delta^{-1}\left(\sum_{k=0}^{\infty} p_{\infty,k}(q;x)(a_{\infty,k}-x)^2\right)^{1/2}\right\}$$

$$= \omega(f;\delta)\left\{1+\delta^{-1}\left(U_{\infty,q}((e_1-xe_0)^2;x)\right)^{1/2}\right\}, \tag{3.37}$$

where $\delta > 0$ and $x \in [0,1)$.

On the other hand, taking into account (1.12), the inequality $(a+b)^2 \leq 2(a^2+b^2)$, $a,b \in \mathbb{R}$, $B_{n,q}((e_1 - xe_0)^2; x) = \frac{1}{[n]_q} x(1-x)$ [17, pp. 268-269, (7.55)-(7.57)] and Lemma 3.4, a), we get

$$\begin{aligned}
U_{n,q}((e_1 - xe_0)^2; x) &= \sum_{k=0}^{n} p_{n,k}(q;x)(a_{n,k} - x)^2 \\
&\leq 2\sum_{k=0}^{n} p_{n,k}(q;x)\left(a_{n,k} - \frac{[k]_q}{[n]_q}\right)^2 + 2\sum_{k=0}^{n} p_{n,k}(q;x)\left(\frac{[k]_q}{[n]_q} - x\right)^2 \\
&= 2\sum_{k=0}^{n} p_{n,k}(q;x)\left(a_{n,k} - \frac{[k]_q}{[n]_q}\right)^2 + \frac{2}{[n]_q} x(1-x) \\
&= 2\sum_{k=0}^{j-1} p_{n,k}(q;x)\left(\frac{[k]_q}{[n]_q}\right)^2 + 2\sum_{k=j}^{n} p_{n,k}(q;x)\left(a_{n,k} - \frac{[k]_q}{[n]_q}\right)^2 + \frac{2}{[n]_q} x(1-x) \\
&\leq 2\sum_{k=0}^{j-1} p_{n,k}(q;x)\left(\frac{[j-1]_q}{[n]_q}\right)^2 + 2\sum_{k=j}^{n} p_{n,k}(q;x)\left(a_{n,k} - \frac{[k]_q}{[n]_q}\right)^2 \\
&\quad + \frac{2}{[n]_q} x(1-x). \qquad (3.38)
\end{aligned}$$

Because

$$\frac{[k-j+1]_q}{[n-j+1]_q} \leq \ldots \leq \frac{[k-1]_q}{[n-1]_q} \leq \frac{[k]_q}{[n]_q}, \quad k \in \{j, j+1, \ldots n\},$$

we obtain that

$$\frac{[k-j+1]_q}{[n-j+1]_q} \leq a_{n,k} = \sqrt[j]{\frac{[k]_q[k-1]_q \ldots [k-j+1]_q}{[n]_q[n-1]_q \ldots [n-j+1]_q}} \leq \frac{[k]_q}{[n]_q},$$

where $k \in \{j, j+1, \ldots n\}$. Hence

$$\begin{aligned}
0 &\leq \frac{[k]_q}{[n]_q} - a_{n,k} \leq \frac{[k]_q}{[n]_q} - \frac{[k-j+1]_q}{[n-j+1]_q} \\
&= \frac{[k]_q[n-j+1]_q - [n]_q[k-j+1]_q}{[n]_q[n-j+1]_q} = q^{k-j+1}\frac{[n-k]_q}{[n-j+1]_q}\frac{[j-1]_q}{[n]_q} \leq \frac{[j-1]_q}{[n]_q}
\end{aligned}$$

for $k \in \{j, j+1, \ldots n\}$. Then, by (3.38) and $(B_{n,q}e_0)(x) = 1$ [17, p. 268, (7.55)], we find

$$\begin{aligned}
&U_{n,q}((e_1 - xe_0)^2; x) \\
&\leq 2\sum_{k=0}^{j-1} p_{n,k}(q;x)\left(\frac{[j-1]_q}{[n]_q}\right)^2 + 2\sum_{k=j}^{n} p_{n,k}(q;x)\left(\frac{[j-1]_q}{[n]_q}\right)^2 + \frac{2}{[n]_q} x(1-x) \\
&\leq 2\left(\frac{[j-1]_q}{[n]_q}\right)^2 + \frac{1}{2[n]_q} \leq \left(2[j-1]_q^2 + \frac{1}{2}\right)\frac{1}{[n]_q} \leq \left(\frac{1}{2} + 2(j-1)^2\right)\frac{1}{[n]_q}.
\end{aligned}$$

By passing to the limit $n \to \infty$, in view of Theorem 3.3 and $\lim_{n\to\infty} [n]_q = (1-q)^{-1}$, we get

$$U_{\infty,q}((e_1 - xe_0)^2; x) \leq \left(\frac{1}{2} + 2(j-1)^2\right)(1-q).$$

Hence, by (3.37), we have

$$|(U_{\infty,q}f)(x) - f(x)| \le \omega(f;\delta)\left\{1 + \delta^{-1}\sqrt{\frac{1}{2} + 2(j-1)^2}\sqrt{1-q}\right\}.$$

Choosing $\delta = \sqrt{1-q}$, we obtain (3.36) for $x \in [0,1)$.

Finally, if $x = 1$, then (3.25) implies that $(U_{\infty,q}f)(1) = f(1)$, which means that (3.36) is true for $x = 1$. Thus the theorem is proved. □

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Survey Article

A survey on recent notions of singularity for kernels of nonlinear integral operators

Dedicated to Professor Ioan Raşa, on the occasion of his 75th birthday

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ABSTRACT. In this paper, we review several notions of singularity for kernels of abstract nonlinear integral operators acting on functions defined over locally compact Hausdorff topological spaces or groups. A recent general definition of abstract nonlinear operators is also discussed, which includes some discrete operators, as the sampling series in a nonlinear form.

Keywords: Singularity, generalized Lipschitz conditions, nonlinear integral operators, locally compact topological spaces, modular spaces.

2020 Mathematics Subject Classification: 41A25, 41A35, 47A58.

1. INTRODUCTION

This article deals with issues related to approximation with nonlinear operators in function spaces. Although these topics are not strictly related to those successfully developed by Prof. Ioan Rasa over many years, they have found their foundation and inspiration in the classical theory of approximation with linear operators, to which he made numerous and fundamental contributions. It is impossible to select the main articles from his vast scientific production here, as they are all not only of a high standard but also very varied.

We begin by considering the classical linear framework, so as to highlight the foundations that inspired the theory of approximation with nonlinear operators, first introduced by Prof. Julian Musielak.

In classical Fourier analysis, one of the main tasks is the problem of reconstruction of a periodic function f by means of its Fourier coefficients (see e.g. [29]). The classical methods are based on the approximation properties of sequences of convolution integral operators of type

$$(T_n f)(x) := \int_{-\pi}^{\pi} K_n(t) f(x-t)\,dt \qquad (x \in \mathbb{R}), \tag{1.1}$$

where K_n is a (measurable) 2π-periodic function and the sequence $(K_n)_{n\in\mathbb{N}}$, satisfies some assumptions that characterize it as an "approximate identity". Denoted by $L^1_{2\pi}$ the space comprising all 2π-periodic functions which are (locally) integrable on $\mathbb{R}$, endowed with the usual norm $\|g\|_{L^1_{2\pi}} := \int_{-\pi}^{\pi} |g(x)|dx$, these assumptions can be summarized in the following:

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(p.1) there exists a constant $M > 0$ such that $\|K_n\|_{L^1_{2\pi}} \leq M$ for every $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} K_n(t)dt = 1,$$

(p.2) for every $\delta \in]0, \pi[$, it holds

$$\lim_{n\to\infty} \int_{\delta\leq|t|\leq\pi} |K_n(t)|dt = 0.$$

The above assumptions define what in the language of approximation theory is called "singularity" of the sequence $(K_n)_n$.

An analogous problem in Fourier analysis is the reconstruction of a (measurable) function $f : \mathbb{R} \to \mathbb{R}$ by means of its Fourier transform. In this case, the reconstruction is based on the approximation properties of convolution integral operators of type

$$(T_n f)(x) := \int_{-\infty}^{\infty} K_n(t) f(x-t)dt \qquad (x \in \mathbb{R}). \tag{1.2}$$

Here, K_n is a measurable function over $\mathbb{R}$. Denoting by $L^1(\mathbb{R})$ the usual Lebesgue space of all the integrable functions on $\mathbb{R}$, endowed with the norm $\|g\|_1 = \int_{\mathbb{R}} |g(t)|dt$, the notion of singularity for the sequence $(K_n)_{n\in\mathbb{N}}$ is characterized by the following assumptions:

(r.1) there exists a constant $M > 0$ such that $\|K_n\|_1 \leq M$, for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} K_n(t)dt = 1,$$

(r.2) for every $\delta > 0$,

$$\lim_{n\to\infty} \int_{|t|\geq\delta} |K_n(t)|dt = 0.$$

In the study of the approximation properties of the operators (1.1) and (1.2) a relevant role is played by the translation operator $(\tau_h f)(x) := f(x-h)$, $x, h \in \mathbb{R}$, that is invariant with respect to the Lebesgue measure.

More recently, Paul Butzer and Stephan Jansche (see [24]-[28]), introduced a modern approximation theory in the framework of Mellin analysis, which is based on sequences of convolution integral operators of type

$$(T_n f)(x) := \int_0^{\infty} K_n(xt^{-1}) f(t) \frac{dt}{t} \qquad (x \in \mathbb{R}^+), \tag{1.3}$$

for functions $f \in L^1_\mu(\mathbb{R}^+)$ where μ is the measure defined on the Borel $\sigma-$algebra of $\mathbb{R}^+$, denoted by $\mathcal{B}(\mathbb{R}^+)$, defined by

$$\mu(A) := \int_A \frac{dt}{t} \qquad (A \in \mathcal{B}(\mathbb{R}^+)).$$

For a complete treatment of the approximation by the operators (1.3) in the framework of Mellin analysis see also the new monograph [6]. The space $L^1_\mu(\mathbb{R}^+)$ is endowed with the norm

$$\|g\|_{L^1_\mu} := \int_0^{\infty} |g(t)| \frac{dt}{t} \qquad (g \in L^1_\mu(\mathbb{R}^+)).$$

Here, $K_n : \mathbb{R}^+ \to \mathbb{R}$ is a measurable function and the singularity assumptions of the sequence $(K_n)_n$ are now expressed by the following:

(m.1) there is a constant $M > 0$ such that $\|K_n\|_{L^1_\mu} \leq M$ for every $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \int_0^\infty K_n(t)\frac{dt}{t} = 1,$$

(m.2) for any $\delta > 1$, setting $U_\delta :=]1/\delta, \delta[$, it holds

$$\lim_{n\to\infty} \int_{\mathbb{R}^+ \setminus U_\delta} |K_n(t)|\frac{dt}{t} = 0.$$

Here, the translation operator is indeed a "dilation" operator, defined by $(\widetilde{\tau}_h f)(x) := f(xh^{-1})$, for $x, h \in \mathbb{R}^+$, that is invariant with respect to the measure μ. All operators (1.1), (1.2) and (1.3), admit a unique approach, using the language of locally compact topological groups and its associated Haar measure μ defined over the Borel sets (see [35, Chapter 10] and [36]). For example, in the periodic case, the underlying measure space can be identified with the group

$$S^1 := \{z \in \mathbb{C} : z = e^{it},\ t \in [0, 2\pi]\}.$$

For operators (1.2) the underlying space is the additive group $(\mathbb{R}, +)$ while for operators (1.3) it is the multiplicative group $(\mathbb{R}^+, \cdot)$.

The general approach consists in the study of convolution integral operators of type

$$(T_n f)(x) := \int_G K_n(t) f(x-t) d\mu(t) \qquad (x \in G), \tag{1.4}$$

where G is a locally compact topological group, endowed with its operation "+" (we adopt here the additive notation) and with its Haar measure μ, that is invariant with respect to the group operation, and $f : G \to \mathbb{R}$ is a μ-measurable function.

In order to define the notion of singularity for the sequence $(K_n)_n$, we assume here that the locally compact group $(G, +)$ is abelian. We denote by $\mathcal{U}$ a base of compact neighborhood of the neutral element θ of G. Let us denote by $L^1(G, \mu)$ the usual Lebesgue space comprising all the integrable functions g with respect the measure μ, endowed with the norm $\|g\|_{L^1(G,\mu)} := \int_G |g(t)| d\mu(t)$. The notion of singularity reads now as follows:

(g.1) there exists a constant $M > 0$ such that $\|K_n\|_{L^1(G,\mu)} \leq M$, for every $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \int_G K_n(t) d\mu(t) = 1,$$

(g.2) for every $U \in \mathcal{U}$ it holds

$$\lim_{n\to\infty} \int_{G\setminus U} |K_n(t)| d\mu(t) = 0.$$

This formulation is the starting point for a general treatment of the notion of singularity for kernels of nonlinear integral operators of convolution type, which was developed starting from the pioneering work by Julian Musielak (see [40], [42]) and then developed in various directions in the framework of approximation theory, see e.g. [3], [7]-[14], [16, 17, 18, 21, 39, 44, 46], see also the monograph [19].

The main problems to be solved in order to obtain a suitable approximation theory by means of nonlinear operators are essentially the following two: a) How to replace the linearity? b) How we can extend the notion of singularity for the kernels in a consistent way? The first problem can be adequately solved employing certain generalized Lipschitz conditions on the kernel functions K_n. This is quite usual in the theory of nonlinear integral operators, for example in solving nonlinear integral equations and in fixed point theory. The second problem represents the main objective of this survey article: we report the original definitions of singularity introduced by Musielak. Then we describe some useful generalizations that enable us to obtain more easily examples of singular sequences. In particular, we consider singularity for kernels of nonlinear integral operators of Urysohn type, for which it is not necessary to employ any algebraic structure, so that our framework will be an abstract Hausdorff and locally compact topological space. Finally, we discuss the final form of singularity for a general class of nonlinear operators, which includes also nonlinear discrete operators of sampling type, (see [1, 13, 14, 22, 23, 44]). Since the theory developed in the last years is devoted to the study of convergence properties in abstract spaces, we premise a section in which we introduce the notion of modular space, that includes large classes of function spaces (L^p-spaces, Orlicz spaces, Musielak-Orlicz spaces, etc).

2. Basic Notions

Let $(G,+)$ be a locally compact and σ-compact Hausdorff topological group, provided with its Haar measure μ defined on the σ-algebra Σ of all Haar measurable sets (see e.g [35, 36]). For any measurable set $A \subset G$, we denote by χ_A the characteristic function of the set A. For a sake of simplicity, we will assume that $(G,+)$ is abelian. Let $\mathcal{U}$ the family of (measurable) neighborhoods of the neutral element θ.

Let $L^0(G,\mu) \equiv L^0(G)$ be the space of all the μ-measurable functions defined on G, finite $\mu-$a.e. and with equality a.e.

Let $\varrho : L^0(G) \to \widetilde{\mathbb{R}}^+ = [0,+\infty]$ be a measurable modular functional, that is ϱ satisfies the following assumptions:

1. $\varrho(f) = 0$ if and only if $f = 0$, a.e.
2. $\varrho(-f) = \varrho(f)$,
3. $\varrho(\alpha f + \beta g) \leq \varrho(f) + \varrho(g)$, for every $f, g \in L^0(G)$, $\alpha, \beta \geq 0,\ \alpha + \beta = 1$.
4. $\varrho(F(t,\cdot))$ is a $\Sigma-$measurable function of $t \in G$ for any $\mu \otimes \mu-$measurable function $F : G \times G \to \mathbb{R}^+$.

In what follows a μ-measurable (or $\mu \otimes \mu$-measurable) set or function will be called simply "measurable", when the context is clear. By means of the functional ϱ we introduce the vector subspace of $L^0(G)$, denoted by $L^\varrho(G,\mu) \equiv L^\varrho(G)$, defined by

$$L^\varrho(G) = \{f \in L^0(G) : \lim_{\lambda \to 0^+} \varrho(\lambda f) = 0\}.$$

The subspace $L^\varrho(G)$ is called the modular space generated by ϱ. A general theory of modular spaces can be found in [41] (see also [19], [38]).

The following notions on measurable modulars will be used (see [19, Chapters 1, 2])

a. ϱ is quasi-convex if there is a constant $M \geq 1$ such that

$$\varrho\left[\int_G p(t)\, h(t,\cdot)\, d\mu(t)\right] \leq M \int_G p(t)\, \varrho(M h(t,\cdot))\, d\mu(t),$$

for every $p \in L^1(G,\mu)$, $p \geq 0$, with $\|p\|_{L^1(G,\mu)} = 1$ and for any (globally) measurable $h : G \times G \to \mathbb{R}_0^+$.

b. ϱ is monotone if $f, g \in L^0(G)$, $|f| \leq |g|$ implies $\varrho(f) \leq \varrho(g)$.

c. ϱ is finite if $\chi_A \in L^\varrho(G)$ whenever A is a measurable subset of G such that $\mu(A) < \infty$.

d. ϱ is absolutely continuous if there exists an $\alpha > 0$ such that for every $f \in L^0(G)$, with $\varrho(f) < +\infty$, the following two conditions are satisfied:

   i. for every $\epsilon > 0$, there is a measurable compact subset $V \subset G$ such that $\mu(V) < \infty$ and $\varrho(\alpha f \chi_{G \setminus V}) < \epsilon$

ii. for every $\epsilon > 0$ there is a $\delta > 0$ such that $\varrho(\alpha f \chi_B) < \epsilon$, for any measurable subset $B \subset G$ with $\mu(B) < \delta$.

Among the examples of modular spaces, we quote here the Lebesgue spaces $L^p(G, \mu) \equiv L^p(G)$, the Orlicz spaces $L^\varphi(G, \mu) \equiv L^\varphi(G)$ generated by a φ-function, the generalized Orlicz spaces, and many others (for details see [19], [41]). We will say that a net of functions $(f_w)_{w>0} \subset L^\varrho(G)$ is modularly convergent to a function $f \in L^\varrho(G)$, if there is $\lambda > 0$, such that

$$\lim_{w\to\infty} \varrho[\lambda(f_w - f)] = 0.$$

We denote this by $f_w \to f$ (ϱ). This notion extends the norm-convergence in L^p-spaces. Moreover it is weaker than the norm-convergence induced by the Luxemburg norm generated by ϱ and they are equivalent if and only if the modular satisfies a Δ_2-condition, (see [19, p. 9], [41]).

3. Singularity for Kernels of Nonlinear Integral Operators of Convolution Type

J. Musielak (cfr. [40]) introduced for the first time the study of approximation theory by means of nonlinear convolution integral operators, defined by

$$(T_w f)(x) := \int_a^b K_w\big(x - t, f(t)\big)dt \qquad (x \in \mathbb{R}).$$

Here, the family $(K_w)_{w>0}$ is a family of kernel functions $K_w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, with $w \in \mathbb{R}^+$, that satisfies a generalized Lipschitz condition with respect to the second variable of type

$$|K_w(t, u) - K_w(t, v)| \le L_w(t)|u - v| \qquad (t \in [a, b],\ u, v \in \mathbb{R}),$$

and the functions $K_w(\cdot, u)$ are extended periodically outside the interval $[a, b]$. These operators were subsequently generalized to obtain the form

$$(T_w f)(x) := \int_G K_w\big(x - t, f(t)\big)d\mu(t) \qquad (x \in G), \tag{3.5}$$

where G is locally compact topological abelian group, endowed with its Haar measure on the Borel sets.

Let $\mathcal{P}$ be the class comprising all the (globally) measurable functions $\psi : G \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that the following assumptions hold:

1. $\psi(t, \cdot)$ is a continuous, nondecreasing function, for every $t \in G$.
2. $\psi(t, 0) = 0$, $\psi(t, u) > 0$ for every $u > 0$ and $t \in G$.

The notion of singularity for the family $(K_w)_{w>0}$, $K_w : G\times\mathbb{R} \to \mathbb{R}$, is expressed in the following way (see [19]). First, we will assume that K_w is globally measurable with respect to the σ-algebra in the product space $G \times \mathbb{R}$, $K_w(\cdot, u) \in L^1(G, \mu)$ and $K_w(t, 0) = 0$, for every $w > 0$. Setting $\mathbb{K} = (K_w)_{w>0}$, we denote by $\mathcal{K}$ the class of all families $\mathbb{K}$ whose elements K_w satisfy the above assumptions. We say that $\mathbb{K} \in \mathcal{K}$ satisfies an $(L, 1)$-Lipschitz condition, or that $\mathbb{K}$ is $(L, 1)$-Lipschitz, if for every $w > 0$ there is a nonnegative measurable function $L_w : G \to \mathbb{R}_0^+$ such that

$$|K_w(t, u) - K_w(t, v)| \le L_w(t)|u - v| \qquad (t \in G,\ u, v \in \mathbb{R}).$$

For a fixed function $\psi \in \mathcal{P}$, we say that $\mathbb{K}$ is (L, ψ)-Lipschitz if for every $w > 0$ there exists a nonnegative measurable function $L_w : G \to \mathbb{R}_0^+$ such that

$$|K_w(t, u) - K_w(t, v)| \le L_w(t)\psi(t, |u - v|) \qquad (t \in G,\ u, v \in \mathbb{R}).$$

The corresponding notions of singularity of $\mathbb{K}$ associated with the above Lipschitz conditions, are as follows:

Definition 3.1. *We say that $\mathbb{K} \in \mathcal{K}$ is singular if*

(s.1) $L_w \in L^1(G, \mu)$ *and there exists a constant $D > 0$ such that $\|L_w\|_{L^1(G,\mu)} \leq D$ for every $w > 0$,*

(s.2) *For any $n \in \mathbb{N}$, it holds*

$$r_n(w) := \sup_{1/n \leq |u| \leq n} \left| \frac{1}{u} \int_G K_w(t, u) d\mu(t) - 1 \right| \to 0 \qquad (w \to \infty),$$

(s.3) *Setting $p_w(t) := \dfrac{L_w(t)}{\|L_w\|_{L^1(G,\mu)}}$, for $t \in G$ and $w > 0$, it holds*

$$\lim_{w \to \infty} \int_{G \setminus U} p_w(t) d\mu(t) = 0,$$

for every $U \in \mathcal{U}$.

Definition 3.2. *We say that $\mathbb{K}$ is strongly singular if the assumption $(s.2)$ is replaced by the stronger one*

$$\sup_{n \in \mathbb{N}} r_n(w) \to 0 \qquad (w \to \infty).$$

For the above definitions and their consequences see [19, Chapter 3].

Remark 3.1. *1. If we define $K_w(t, u) := K_w^*(t)u$, for $t \in G$ and $u \in \mathbb{R}$, where $(K_w^*)_{w>0}$ is a net of measurable functions, then taking $L_w(t) := |K_w^*(t)|$ we see that the notion of singularity for kernels of linear operators is a particular case of the strong singularity expressed in Definition 3.2.*

2. The notion of singularity given in Definition 3.1 is strictly connected with the Lipschitz conditions. This is quite natural, due the nature of the above definitions. In particular the assumption $(s.2)$ in Definition 3.1 may be interpreted as a statement that $K_w(t, u)$ behaves nearly u for large values of w. This would imply that a generalized Lipschitz condition with a function $\psi \in \mathcal{P}$, is nearly an usual $(L, 1)$-Lipschitz condition. As showed in [17] by an example, this is not true.

Using the above notions in [19, Chapter 3] an approximation theory with the nonlinear convolution operators (3.5) was developed for functions in a modular space $L^\rho(G)$ generated by a modular ρ satisfying certain specific conditions. For details see [19].

The weak point of the definition of singularity for kernels satisfying (L, ψ)-Lipschitz conditions, with a general $\psi \in \mathcal{P}$, lies in the difficulty of finding examples that connect the Lipschitz condition with assumption $(s.2)$ of Definition 3.1 without leading us back to a Lipschitz condition of type $(L, 1)$. In the next section, we will describe a generalization of the notion of singularity which give a contribution to solve the above difficulties.

4. A Generalization of Singularity for Nonlinear Convolution Integral Operators

In this section, we report a generalized notion of singularity for kernels $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}$ of nonlinear integral operators of convolution type (3.5). This generalization, introduced in [10], is based on the notion of "regular nets" of functions in modular spaces. As a first step, we modify slightly the generalized Lipschitz condition.

Definition 4.3. *Let $\Psi = (\psi_w)_{w>0}$ be a family of functions from the class $\mathcal{P}$. We say that $\mathbb{K}$ is (L, Ψ)-Lipschitz if for every $w > 0$, there is a nonnegative measurable function $L_w : G \to \mathbb{R}_0^+$ such that*

$$|K_w(t, u) - K_w(t, v)| \leq L_w(t)\psi_w(t, |u - v|) \qquad (t \in G,\ u, v \in \mathbb{R}).$$

4.1. **Regular Nets in Modular Spaces.** Let ϱ be a modular on $L^0(G)$ and let $f \in L^\varrho(G)$ be a fixed function.

Definition 4.4. *A net $(\xi_w)_{w>0}$ of measurable functions $\xi_w : \mathbb{R} \to \mathbb{R}$ is said to be $\varrho-$regular for f if we have $\xi_w \circ f \to f$ (ϱ), $w \to \infty$.*

In order to give some sufficient conditions in order that a net $(\xi_w)_{w>0}$ is $\varrho-$regular, we begin with the following definition.

Definition 4.5. *We say that a family of functions $(g_w)_{w>0} \subset L^\varrho(G)$ is $\varrho-$ uniformly equicontinuous if there is a constant $\beta > 0$ such that:*

1. *For every $\epsilon > 0$ there is a compact neighborhood of θ, $U \in \mathcal{U}$, such that*
$$\varrho(\beta g_w \chi_{G\setminus U}) < \epsilon,$$
for every $w > 0$.
2. *We have $\varrho[\beta g_w \chi_{B_k}] \to 0$ as $k \to \infty$, for every sequence $\{B_k\}$ with $B_k \in \Sigma$ for every $k \in \mathbb{N}$, $B_{k+1} \subset B_k$, and $\mu(B_k) \to 0$.*

We have the following (see [10]).

Theorem 4.1. *Let ϱ be a finite, monotone, absolutely continuous modular on $L^0(G)$, $f \in L^\varrho(G)$ and let $\xi_w : \mathbb{R} \to \mathbb{R}$ be a net of functions with the following properties*

1. *$\lim_{w\to\infty} \xi_w(u) = u$ uniformly on the compact sets of $\mathbb{R}$.*
2. *The family $(g_w)_{w>0}$, with $g_w(s) = \xi_w \circ f(s)$ is a $\varrho-$uniformly equicontinuous net of functions in $L^\varrho(G)$.*

Then $\xi_w \circ f \to f$ (ϱ).

The following corollary provides an interesting special case.

Corollary 4.1. *Let ϱ be a finite, monotone, absolutely continuous modular on $L^0(G)$, $f \in L^\varrho(G)$ and let $\xi_w : \mathbb{R} \to \mathbb{R}$ be a net of measurable functions with the following properties*

1. *$\lim_{w\to\infty} \xi_w(u) = u$ uniformly on the compact sets of $\mathbb{R}$.*
2. *$|\xi_w(u)| \leq \gamma(u)$ for every $u \in \mathbb{R}$, where $\gamma : \mathbb{R} \to \mathbb{R}_0^+$ is a measurable function such that $\gamma \circ f \in L^\varrho(G)$.*

Then $(\xi_w)_{w>0}$ is $\varrho-$regular for f.

The proof is an immediate consequence of Theorem 4.1; indeed, if $\gamma \circ f \in L^\varrho(G)$, then it is easy to show, from absolute continuity of ϱ, that $(\xi_w \circ f)_{w>0}$ is $\varrho-$uniformly equicontinuous. As an example, let $f \in L^\varrho(G)$ and let ξ_w be defined by $\xi_w(u) = u^{1-\frac{1}{w}}$, for $u \geq 0$ and extended in odd way to the real axis for sufficiently large w. Put $\gamma(u) = \sup_{w \geq \widetilde{w}} |\xi_w(u)|$. If $\gamma \circ f \in L^\varrho(G)$, then $\{\xi_w \circ f\}_{w>0}$ is $\varrho-$uniformly equicontinuous.

4.2. **A Generalization of Singularity via Regular Nets of Functions.** In this section, we report the notion of singularity defined employing regular nets of functions, studied in [10].

Let ϱ be a modular on $L^0(G)$. Let $\Xi = (\xi_w)_{w>0}$, $\xi_w : \mathbb{R} \to \mathbb{R}$, be a net of measurable functions such that $\xi_w(u) \neq 0$ if $u \neq 0$. We denote by $L^\varrho_\Xi(G)$ the class of all functions $f \in L^\varrho(G)$ such that $(\xi_w)_{w>0}$ is $\varrho-$regular with respect to f.

Let $\gamma : \mathbb{R} \to \mathbb{R}_0^+$ be a fixed measurable function such that $|\xi_w(u)| \leq \gamma(u)$ for every $w > 0$ and $u \in \mathbb{R}$. Setting
$$L^\varrho_\gamma(G) := \{f \in L^\varrho(G) : \gamma \circ f \in L^\varrho(G)\},$$
it is easy to see that $L^\varrho_\Xi(G) \supset L^\varrho_\gamma(G)$ whenever $(\xi_w)_{w>0}$ satisfies (i) of Corollary 4.1. For a class $\Psi = (\psi_w)_{w>0} \subset \mathcal{P}$, let $\mathbb{K} = \{K_w\}_{w>0} \in \mathcal{K}$ be a (L, Ψ)-Lipschitz kernel.

Definition 4.6. *We say that $\mathbb{K}$ is Ξ-singular if the following assumptions hold:*

(Ξ.1) *for every $w > 0$, $L_w \in L^1(G, \mu)$, and there is $D > 0$ such that $\|L_w\|_{L^1(G,\mu)} =: D_w \leq D$, for every $w > 0$.*

(Ξ.2) *Setting for every $n = 1, 2, \ldots,$*

$$r_w(n) = \sup_{(1/n) \leq |u| \leq n} \left| \frac{1}{\xi_w(u)} \int_G K_w(t, u)\, d\mu(t) - 1 \right|,$$

for every $w > 0$, it results $\lim_{w\to\infty} r_w(n) = 0$, for every $n \in \mathbb{N}$.

(Ξ.3) *Setting $p_w(t) = L_w(t)/D_w$, for every $w > 0$, it results*

$$\lim_{w\to\infty} \int_{G\setminus U} p_w(t) d\mu(t) = 0,$$

for every $U \in \mathcal{U}$.

Next, we say that $\mathbb{K}$ is $\Xi-$*strongly singular*, if (Ξ.2) is replaced with $r_w := \sup_{n\in\mathbb{N}} r_w(n) \to 0$ as $w \to \infty$. It is clear that if $\mathbb{K}$ is $\Xi-$strongly singular then it is also $\Xi-$singular.

Using this generalized notion of singularity for the kernel of the nonlinear operator (3.5) it is proved a general modular approximation thorem in modular space, under certain side conditions. For details see [10]. In order to compare this new definition of singularity, with the previous one with (L, Ψ)-Lipschitz kernels (see Definition 3.1 and Definition 4.3), we have the following proposition (see [10]).

Proposition 4.1. *Let $\Xi = (\xi_w)_{w>0}$ be a family of measurable functions $\xi_w : \mathbb{R} \to \mathbb{R}$ such that $\xi_w(u) \neq 0$, for any $u \neq 0$ and $\xi_w(u) \to u$ uniformly on every compact set in $\mathbb{R}$. Let $\mathbb{K}$ be an (L, Ψ)-Lipschitz kernel, $\Psi \subset \mathcal{P}$, such that $(L_w(\cdot)\psi_w(\cdot, n))_{w>0}$ is uniformly bounded in $L^1(G, \mu)$ for every $n \in \mathbb{N}$. Then $\Xi-$singularity and singularity of $\mathbb{K}$ are equivalent.*

As an example, let $\Xi = (\xi_w)_{w\geq 0}$ be a family of Holderian functions, with degree α_w, with $\alpha_w \to 1$, as $w \to \infty$. Put

$$K_w(t, u) = L_w(t)\xi_w(u).$$

In this case, all the assumptions of Proposition 4.1 are satisfied and so for such kernel the equivalence between the two definitions of singularity holds. In particular if $L_w \geq 0$ and $\|L_w\|_{L^1(G,\mu)} = 1$, and $\xi_w(u) = u^{1-1/w}$ for $u \geq 0$, and $\xi_w(u) = -|u|^{1-1/w}$ for $u \leq 0$, with sufficiently large $w > 0$, we obtain an example of singular and $\Xi-$singular kernel. Note that the equivalence given in Proposition 4.1 is not true for strong singularity.

5. Singularity for Kernels of Urysohn Integral Operators

In this section, we consider nonlinear integral operators which have no convolution structures, of type

$$(U_w f)(s) := \int_G K_w(s, t, f(t)) d\mu(t) \qquad (w > 0,\ s \in G).$$

Since we do not use a translation operator, we do not need an algebraic structure on G. Therefore our framework is the class of locally compact Hausdorff topological spaces. However, we will need also a uniform structure generating the topology of G. This enables us to employ some features of the topology of a topological group.

Let G be a locally compact Hausdorff topological space, provided with a regular measure μ on the $\sigma-$algebra $\mathcal{B}$ of the Borel sets of G. As before, by $L^0(G)$ we will denote the set of all measurable real functions on G (and as usual we identify functions which agree on a measurable subset H with $\mu(G \setminus H) = 0$). As before, by $L^1(G, \mu) \equiv L^1(G)$ we denote the Lebesgue space comprising all the integrable functions with respect to the measure μ.

We will assume that the topology of G is uniformizable, i.e. there is a (diagonal) uniform structure $\mathcal{V} = \{V\} \subset G \times G$, which generates the topology of G, (see, e.g. [47]). We recall that for $s \in G$, a basic neighborhood of s is given by

$$V_s = \{t \in G : (s,t) \in V\}$$

for any $V \in \mathcal{V}$. By local compactness, we will assume that for every $s \in G$, the base $\{V_s : V \in \mathcal{V}\}$ contains compact sets. Note that the uniform structure is employed only in order to define uniform continuity and to study uniform convergence.

Let $\mathcal{L}$ be the class of all $\mu \otimes \mu-$measurable functions $L : G \times G \to \mathbb{R}_0^+$ such that the sections $L(\cdot, t)$ and $L(s, \cdot)$ belong to $L^1(G, \mu)$, for every $t \in G$ and $s \in G$ respectively.

Let $\mathcal{P}^*$ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that ψ is a continuous (nondecreasing) function and $\psi(0) = 0,\ \psi(u) > 0,$ for $u > 0$. We will denote by $\mathcal{K}^*$ the class comprising the families $\mathbb{K} = (K_w)_{w>0}$, where $K_w : G \times G \times \mathbb{R} \to \mathbb{R}$, such that K_w is globally measurable and $K_w(s,t,0) = 0$ for every $(s,t) \in G \times G$.

We introduce the following slight modification of the generalized Lipschitz condition considered in [13, 14], (see also [15], in a more general case).

Definition 5.7. *Let $\Psi = (\psi_w)_{w>0} \subset \mathcal{P}^*$. We will say that $\mathbb{K} \in \mathcal{K}$ is (L, Ψ)-Lipschitz, if for every $w > 0$ there exists $L_w \in \mathcal{L}$ such that*

$$|K_w(s,t,u) - K_w(s,t,v)| \leq L_w(s,t)\psi_w(|u-v|) \qquad (s,t \in G).$$

If $\psi_w(u) = u$, for $u \geq 0$, and for every $w > 0$, we will say that $\mathbb{K}^*$ is $(L, 1)$-Lipschitz. We are now ready to introduce the following notion of singularity.

Definition 5.8. *We say that a (L, Ψ)-Lipschitz family $\mathbb{K}$ is singular if the following assumptions hold:*

(u.1) *There is $D > 0$ such that, for every $t, s \in G,\ w > 0$, we have*

$$\alpha_w(s) := \|L_w(s,\cdot)\|_{L^1(G,\mu)} \leq D, \ \ \widetilde{L}_w(t) := \|\frac{L_w(\cdot,t)}{\alpha_w(\cdot)}\|_{L^1(G,\mu)} \leq D,$$

(u.2) *For every $s \in G$ and $u \in \mathbb{R}$, we have*

$$\lim_{w\to\infty} \int_G K_w(s,t,u)d\mu(t) = u.$$

(u.3) *For every $s \in G$, and for every neighborhood V_s of s we have*

$$\lim_{w\to\infty} \int_{G\setminus V_s} L_w(s,t)d\mu(t) = 0.$$

We say that $\mathbb{K}$ is uniformly singular if conditions (u.2) and (u.3) are replaced by the following ones:

(u.2)′ *We have*

$$\lim_{w\to\infty} \int_G K_w(s,t,u)d\mu(t) = u,$$

uniformly with respect to $s \in G$ and $u \in C$, where C is any compact subset of $\mathbb{R} \setminus \{0\}$.

(u.3)′ *For every $V \in \mathcal{V}$ we have*

$$\lim_{w\to\infty} \int_{G\setminus V_s} L_w(s,t)d\mu(t) = 0$$

uniformly with respect to $s \in G$.

Remark 5.2. *1. Let us remark that in particular, in assumption (u.1) we can assume that there are constants $d, D > 0$ such that $d \leq \alpha_w(s) \leq D$ for every $w > 0, s \in G$ and $\|L_w(\cdot,t)\|_{L^1(G,\mu)} \leq D$, for every $w > 0$, $t \in G$. In this case it is easily seen that the second inequality in assumption 1 holds with the constant D/d. Moreover the above relations can be satisfied only for sufficiently large $w > 0$.*

2. *If G is a locally compact topological abelian group, setting $K_w(s,t,u) := K_w(s-t,u)$, for $s, t, \in G$ and $u \in \mathbb{R}$, in order to compare the definition of singularity given above with the one used for convolution operators in topological groups, considered in the previous sections, it should be noted that the present definition is more general and has the advantage to obtain examples of singular kernels in an easier way.*
3. *Employing the above definition of singularity, under the notion of regular families of measures, introduced in [13, 14] and some other side conditions, pointwise and uniform convergence theorems are obtained, and also a modular convergence theorem is established.*

6. Singularity for Kernels of Abstract Sampling-Type Operator

In recent years, it was introduced a general approach for the study of the convergence properties of sequences or nets of linear or nonlinear operators which include various kind of operators: from integral operators, to discrete operators of sampling type (see [1, 2, 4, 13, 14, 22, 23]). This approach allows us to develop a unitary treatment of the approximation properties of abstract operators in various functional spaces. Once again, the concept of singularity plays a fundamental role. In this section, we report the notion of singularity suitable for the study of these abstract operators.

Let G be a locally compact Hausdorff topological space, provided with its family of Borel sets $\mathcal{B}$ of G. Here we denote by μ_G be a regular and $\sigma-$finite measure defined on $\mathcal{B}$. As in the previous section, we will assume that the topology of G is uniformizable, i.e. there is a uniform structure $\mathcal{U} \subset G \times G$ which generates the topology of G. For every $U \in \mathcal{U}$, we put $U_s = \{t \in G : (s,t) \in U\}$. By local compactness, we will assume that for every $s \in G$, the base $\{U_s : U \in \mathcal{U}\}$ contains compact sets.

Let now $\{H_w\}_{w>0}$ be a net of nonempty closed subsets of G such that $G = \overline{\cup_{w>0} H_w}$.

For every $w > 0$ we will denote by μ_w a regular and σ-finite measure on H_w, defined on the Borel σ-algebra generated by the family $\{A \cap H_w$: A open subset of $G\}$.

Let $\mathcal{L}$ be the class of all the families of globally measurable functions $\mathbb{L} = (L_w)_{w>0}$, $L_w : G \times H_w \to \mathbb{R}_0^+$ such that for every $w > 0$ the sections $L_w(\cdot,t)$ and $L_w(s,\cdot)$ belong to $L^1(G,\mu)$ for every $t \in H_w$ and to $L^1(H_w,\mu_w)$ for every $s \in G$ respectively. If $\mathcal{P}^*$ is the class introduced in Section 5, let $\Psi := (\psi_w)_{w>0} \subset \mathcal{P}^*$ be a family of functions such that the following two assumptions hold:

1. $(\psi_w)_{w>0}$ is equicontinuous at $u = 0$,
2. for every $u \geq 0$ the net $(\psi_w(u))_{w>0}$ is bounded.

We denote by $\widetilde{\mathcal{K}}$ the class of all families of functions $\mathbb{K} = (K_w)_{w>0}$, where the functions $K_w : G \times H_w \times \mathbb{R} \to \mathbb{R}$, are such that for any $w > 0$, $K_w(\cdot,\cdot,u)$ is globally measurable on $G \times H_w$ for every $u \in \mathbb{R}$ and $K_w(s,t,0) = 0$, for every $(s,t) \in G \times H_w$.

Definition 6.9. *We say that $\mathbb{K} = (K_w)_{w>0}$ is (L,Ψ)-Lipschitz if there are a family $\mathbb{L} = (L_w)_{w>0} \in \mathcal{L}$ and a constant $D > 0$ such that*

$$0 < \beta_w(s) := \int_{H_w} L_w(s,t)d\mu_w(t) \leq D$$

for all $s \in G, w > 0$ *and*

$$|K_w(s,t,u) - K_w(s,t,v)| \leq L_w(s,t)\psi_w(|u-v|)$$

for every $s \in G,\ t \in H_w$ *and* $u, v \in \mathbb{R}$.

For a given (L, Ψ)-Lipschitz family $\mathbb{K} = (K_w)_{w>0} \in \widetilde{\mathcal{K}}$ we introduce the following family of nonlinear integral operators $\mathbf{T} = (T_w)_{w>0}$ given by

$$(T_w f)(s) = \int_{H_w} K_w(s,t,f(t))d\mu_w(t) \quad s \in G,\ w > 0 \tag{6.6}$$

where $f \in Dom\mathbf{T} = \cap_{w>0} DomT_w$; here $DomT_w$ is the subset of $L^0(G)$ on which $T_w f$ is well defined as a μ_G-measurable function of $s \in G$. The notion of singularity is now modified in the following way.

Definition 6.10. *We say that the* (L, Ψ)*-family* $\mathbb{K} = (K_w)_{w>0} \in \widetilde{\mathcal{K}}$ *is singular if the following assumptions hold*

(s.1) *There is a net* $(\zeta_w)_{w>0}$ *of positive real numbers such that for every* $w > 0$ *and* $t \in H_w$ *we have*

$$\int_G L_w(s,t)d\mu_G(s) \leq \zeta_w \leq D.$$

(s.2) *For every* $s \in G$ *and for every* $u \in \mathbb{R}$ *we have*

$$\lim_{w \to \infty} \int_{H_w} K_w(s,t,u)d\mu_w(t) = u.$$

(s.3) *For every* $s \in G$ *and for every* $U \in \mathcal{U}$ *we have*

$$\lim_{w \to \infty} \int_{H_w \setminus U_s} L_w(s,t)d\mu_w(t) = 0.$$

We say that $\mathbb{K}$ *is uniformly singular if conditions (s.2) and (s.3) are replaced by the following ones*

(s.2′) we have

$$\lim_{w \to \infty} \int_{H_w} K_w(s,t,u)d\mu_w(t) = u,$$

uniformly with respect to $s \in G$ *and* $u \in C$, *where* C *is any compact subset of* $\mathbb{R} \setminus \{0\}$.

(s.3′) for every $U \in \mathcal{U}$ *we have*

$$\lim_{w \to \infty} \int_{H_w \setminus U_s} L_w(s,t)d\mu_w(t) = 0$$

uniformly with respect to $s \in G$.

For the above concepts see [13, 14]. By specializing G and H_w, we obtain very interesting families of operators of discrete type. In particular, we can apply these results to nonlinear discrete operators (sampling series) of the form ([14])

$$(S_w f)(s) = \sum_{k \in \mathbb{Z}} L(ws - k) J_w\big(f\big(\frac{k}{w}\big)\big) \quad s \in \mathbb{R},\ w > 0.$$

Related results about nonlinear sampling type operators can be found e.g. in [32, 45]. Here, $G = \mathbb{R}$ endowed with the Lebesgue measure, $H_w = (1/w)\mathbb{Z}$ with counting measure, and the family of kernel is defined by

$$K_w(s,t,u) = L_w(s,t)J_w(u) \qquad (w > 0,\ s \in \mathbb{R},\ t \in H_w),$$

with $L_w(s,t) = L(w(s-t))$, where $L : \mathbb{R} \to \mathbb{R}$ is a continuous function with compact support such that

$$\sum_{k\in\mathbb{Z}} L(u-k) = 1 \qquad (\text{for every } u \in \mathbb{R})$$

and the family $(J_w)_{w>0}$ satisfies a $(L,\Psi)-$condition. Using this structure, reconstruction of the function f is obtained, in various sense (pointwise, uniform, modular). Analogously, taking $G = \mathbb{R}^+$ endowed with the logarithmic measure $d\mu(t) = dt/t$ and $H_w = \{t = e^{k/w} : k \in \mathbb{Z}\}$, endowed with the counting measure, we obtain nonlinear version of exponential sampling series (see [30, 31, 37]).

Other examples are given by the classical nonlinear operators defined over the euclidean space R^N, where we take $G = H_w = \mathbb{R}^N$, endowed with the Lebesgue measure, or for Mellin-type integral operators, for which we take $G = H_w = \mathbb{R}^+$, endowed with the logarithmic measure.

7. Other Developments

There exist other definitions of singularity which take into account certain properties of the kernels. In [7], there is considered a class of nonlinear integral operators, acting on functions defined over the positive real line, whose kernels satisfy some generalized Lipschitz conditions, in which the functions L_w satisfy certain general homogeneity assumptions with respect to a continuous function. To be precise, let $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that

$$(\eta(t))^{-1}\eta(ts) \leq C\eta(s) + D \qquad (t, s \in \mathbb{R}^+)$$

for suitable constants C and D. Then we say that the function $L_w : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}_0^+$ is $\eta-$homogeneous, if for any $\lambda > 0$, we have

$$\eta(t)L_w(\lambda s, \lambda t) = \eta(\lambda t)L_w(s,t),$$

for every $w > 0$, $s, t \in \mathbb{R}^+$. For this kind of kernels, the singularity is defined through the homogeneity function η. For example, assumption like (u.2) of Section 5, is now expressed as follows:

$$\lim_{w\to\infty} \int_0^\infty K_w(s,t,u)\frac{dt}{t} = \eta(s)u$$

for every $s \in \mathbb{R}^+$ and $u \in \mathbb{R}$. Employing this notion of singularity, approximation results in pointwise and uniform sense were obtained for the function $g := \eta f$. Another notion of singularity is linked with the study of quantitative estimates of convergence, for which the assumptions like (u.2) and (u.3) are given with a prescribed order (see [7]).

Related definitions are given in [5] for the nonlinear abstract sampling type operators considered in Section 6, when the convergence is studied with respect to a filter $\mathcal{F}$. In this case the singularity takes into account of this kind of convergence, and it is named $\mathcal{F}-$singularity. We shortly report this abstract approach. For a sake of simplicity, we consider now sequences instead of nets, that is we take $\mathbb{N}$ as the set of indices (instead of $\mathbb{R}^+$). For details and further references see [5] (see also [33]). We begin with some preliminary definitions.

Definition 7.11. *A nonempty family $\mathcal{F}$ of subsets of $\mathbb{N}$ is called a filter of $\mathbb{N}$ iff $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and $B \supset A$ we get $B \in \mathcal{F}$.*

As classical examples of filters we quote the filter $\mathcal{F}_{\text{cofin}}$ of all subsets of $\mathbb{N}$ whose complement is finite and the filter $\mathcal{F}_{\text{d}}$ associated with the statistical convergence, that is the class of all subsets of $\mathbb{N}$ whose asymptotic density is 1. We recall here this notion

Definition 7.12. *The asymptotic density of a set $A \subset \mathbb{N}$ is defined as*

$$d(A) = \lim_n \frac{\sharp(A \cap \{1, \dots, n\})}{n}$$

(provided that this limit exists). Here by $\sharp$ we denote the cardinality of the set in brackets (see e.g [34, 43]).

Definition 7.13. *1. A sequence $(x_n)_n$ in $\mathbb{R}$ is said to be $\mathcal{F}$-bounded iff there exists an $M > 0$ such that $\{n \in \mathbb{N} : |x_n| \leq M\} \in \mathcal{F}$.*
2. A sequence $(x_n)_n$ in G is $\mathcal{F}$-convergent to $x \in G$ (and we write $x = (\mathcal{F}) \lim_n x_n$) iff $\{n \in \mathbb{N} : (x_n, x) \in U\} \in \mathcal{F}$ whenever $U \in \mathcal{U}$.
3. A sequence $f_n : G \to \mathbb{R}$, $n \in \mathbb{N}$, is called $\mathcal{F}$-exhaustive at $s \in G$ iff for every $\varepsilon > 0$ there exist a neighborhood U_s of s and $A \in \mathcal{F}$ with $|f_n(z) - f_n(s)| \leq \varepsilon$, whenever $n \in A$ and $z \in U_s$.
4. A sequence $f_n : G \to \mathbb{R}$, $n \in \mathbb{N}$, is said to be $\mathcal{F}$-convergent uniformly to f on G iff

$$(\mathcal{F}) \lim_n [\sup_{t \in G} |f_n(t) - f(t)|] = 0.$$

Using the notations of the previous section, let $\mathbb{K} := (K_n)_{n \in \mathbb{N}} \in \widetilde{\mathcal{K}}$ be a (L, Ψ)-Lipschitz family.

Definition 7.14. *We say that $\mathbb{K}$ is $\mathcal{F}$-singular iff*

(f.1) there is a $D_1 > 0$ with

$$\Lambda := \Big\{ n \in \mathbb{N} : \int_{H_n} L_n(s,t)\, d\mu_n(t) \leq D_1 \text{ for all } s \in G \Big\} \in \mathcal{F},$$

(f.2) for every $s \in G$ and for each neighborhood $U_s \subset G$ we get

$$(\mathcal{F}) \lim_n \int_{H_n \setminus U_s} L_n(s,t)\, d\mu_n(t) = 0,$$

(f.3) for every $s \in G$ and $u \in \mathbb{R}$ we have

$$(\mathcal{F}) \lim_n \int_{H_n} K_n(s,t,u)\, d\mu_n(t) = u.$$

We say that $\mathbb{K}$ is strongly $\mathcal{F}$-singular iff it fulfills (f.1) and

(f.2)$'$ for any $s \in G$ and for every neighborhood $U_s \subset G$ there is a neighborhood $Z_s \subset G$, with

$$(\mathcal{F}) \lim_n \Big[\sup_{z \in Z_s} \int_{H_n \setminus U_s} L_n(z,t)\, d\mu_n(t) \Big] = 0,$$

(f.3)$'$ for any $s \in G$ and $u \in \mathbb{R}$ there are two neighborhoods U_s of s and W of u respectively, with

$$(\mathcal{F}) \lim_n \Big[\sup_{z \in U_s, v \in W} \Big(\int_{H_n} K_n(z,t,v)\, d\mu_n(t) - v \Big) \Big] = 0.$$

With the above notion of convergence, in [5] several convergence theorem are established including also the case of statistical convergence for sequence of operators T_n (6.6).

As a final issue, we recall also the particular case of singularity for kernels of nonlinear operators acting of real functions with bounded variation (in its various sense, Jordan variation, φ-variation, and other generalizations). In [3, 4] (see also [20]), some notions of singularity are given for kernels of type

$$K_w(t,u) := L_w(t) H_w(t) \quad (w > 0,\, t \in \mathbb{R}, u \in \mathbb{R}),$$

where $L_w \in L^1(\mathbb{R})$ or $L_w \in L^1_{2\pi}$, being $L^1_{2\pi}$ the space comprising all the Lebesgue 2π-periodic summable functions with respect to the Lebesgue measure, and H_w satisfies a strong Lipschitz condition of type $|H_w(u) - H_w(v)| \leq C|u - v|$, for every $u, v \in \mathbb{R}$, and an absolute constant $C > 0$. In the periodic case, the notion of singularity in BV spaces is as follows.

Definition 7.15. *We say that $\mathbb{K} := \{K_w\}_{w>0}$ is BV-singular if $\|L_w\|_1 \leq A$ for an absolute constant $A > 0$, $A_w := \int_{-\pi}^{\pi} L_w(t)dt \to 1$ as $w \to \infty$ and for any $\delta > 0$ one has*

$$\lim_{w\to\infty} \int_{\delta\leq|t|\leq\pi} |L_w(t)|dt = 0,$$

and setting $G_w(u) := H_w(u) - u$, for $u \in \mathbb{R}$, one has $V_J(G_w) \to 0$ as $w \to \infty$, where V_J denotes a notion of variation over a bounded interval $J \subset \mathbb{R}$.

With the above notion of BV-singularity, convergence theorems with respect the total variation (in various sense) are obtained ([3, 4, 20]).

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Research Article

Korovkin approximation of non-convex set-valued continuous functions

Dedicated to Professor Ioan Raşa, on the occasion of his 75th birthday

MICHELE CAMPITI*

ABSTRACT. In this paper, we continue the study of Korovkin approximation in cones of Hausdorff continuous set-valued functions. Besides some general results in the context of set-valued functions with compact and convex values, we consider for the first time the possibility of lacking the convex assumption using different methods, namely parametrizations and starshaped conditions. The interest of the new results consist also in considering local conditions which depend on directions of $\mathbb{R}^d$.

Keywords: Korovkin approximation, approximation of set-valued functions, non convex set-valued functions.

2020 Mathematics Subject Classification: 41A65, 41A36, 41A25, 41A63.

1. INTRODUCTION AND NOTATION

Korovkin-type approximation theory has been successfully extended to sequences of monotone operators between cones of Hausdorff continuous set-valued functions since the pioneering work by Keimel and Roth [10, 11].

Thereafter, many papers have been subsequently devoted to this study and to different problems arising in connection with this kind of approximation (see, e.g. [7, 8]). The most recent results in this direction are collected in [5, 6].

Recently, one of the most interesting topics in Korovkin approximation has been the study of set-valued functions with compact but not necessarily convex values. However, in this setting a Korovkin-type approximation theory involving these kind of functions has not been yet developed.

In [9], some cases of particular interest have been indicated. In particular, the authors suggest to consider the possibility of replacing the convex condition with suitable subsets which admit a parametrization of their boundary or, alternatively, starshaped convex sets.

Here, our aim is to state some Korovkin type results in both these cases. We begin with some notations. Let $\mathbf{B}^d$ denote the closed unit ball in $\mathbb{R}^d$ and let

- $\mathcal{F}(\mathbb{R}^d)$ the cone of all non empty subsets of $\mathbb{R}^d$,
- $\mathcal{K}(\mathbb{R}^d)$ the cone of all non empty compact subsets of $\mathbb{R}^d$,
- $\mathcal{K}_c(\mathbb{R}^d)$ the cone of all non empty compact convex subsets of $\mathbb{R}^d$.

All the above cones are endowed with the natural addition and multiplication by positive scalars.

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In $\mathcal{F}(\mathbb{R}^d)$ it is also defined the Hausdorff distance

$$d_H(A,B) := \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}.$$

If X is a Hausdorff topological space and $\mathcal{A}(\mathbb{R}^d)$ denotes one of the above cones, we consider the cone $C(X,\mathcal{A}(\mathbb{R}^d))$ of all continuous set-valued functions defined in X and with values in $\mathcal{A}(\mathbb{R}^d)$. $C(X,\mathcal{A}(\mathbb{R}^d))$ is endowed with the Hausdorff uniform distance and is naturally ordered by inclusion.

If $F \in C(X,\mathcal{F}(\mathbb{R}^d))$, we shall denote by $\mathrm{Sel}(F)$ the set of all continuous selections of F. If $F \in C(X,\mathcal{K}_c(\mathbb{R}^d))$ then $\mathrm{Sel}(F)$ is a convex subset of $C(X,\mathcal{K}_c(\mathbb{R}^d))$. Moreover, an operator $L : C(X,\mathcal{A}(\mathbb{R}^d)) \to C(X,\mathcal{A}(\mathbb{R}^d))$ is called linear if $L(F+G) = L(F)+L(G)$ and $L(\lambda F) = \lambda L(F)$ for every $F,G \in C(X,\mathcal{A}(\mathbb{R}^d))$ and $\lambda \geq 0$ and monotone if the condition $F \leq G$ (i.e., $F(x) \subset G(x)$ for every $x \in X$) implies $L(F) \leq L(G)$. Finally, we recall the definition of Korovkin subset of $C(X,\mathcal{A}(\mathbb{R}^d))$. For the sake of simplicity, we shall refer only to the identity operator.

A subset $\mathcal{H}$ of $C(X,\mathcal{A}(\mathbb{R}^d))$ is a Korovkin subset in $C(X,\mathcal{A}(\mathbb{R}^d))$ with respect to equicontinuous nets of monotone operators (respectively, monotone linear operators) if, for every equicontinuous net $(L_i)_{i\in I}^{\leq}$ of monotone (respectively, monotone linear) operators from $C(X,\mathcal{A}(\mathbb{R}^d))$ into itself satisfying $\lim_{i\in I^{\leq}} L_i(H) = H$ for every $H \in \mathcal{H}$, we also have $\lim_{i\in I^{\leq}} L_i(F) = F$ for every $F \in C(X,\mathcal{A}(\mathbb{R}^d))$.

One of the main results concerning the compact convex case is the following.

Theorem 1.1. *Let X be a compact Hausdorff topological space and let $\mathcal{H}$ be a subset of $C(X,\mathcal{K}_c(\mathbb{R}^d))$ such that, for every $F \in C(X,\mathcal{K}_c(\mathbb{R}^d))$, $x_0 \in X$ and $\varepsilon > 0$, there exists $H \in \mathcal{H}$ satisfying the following conditions*

$$F \leq H, \qquad H(x_0) \subset F(x_0) + \varepsilon \cdot \boldsymbol{B}. \tag{1.1}$$

Then, $\mathcal{H}$ is a Korovkin system in $C(X,\mathcal{K}_c(\mathbb{R}^d))$.

From the above result it is possible to obtain different criteria and examples of Korovkin subsets. Hence, we consider the possibility of extending this result to set-valued functions which do not take their values in $\mathcal{K}_c(\mathbb{R}^d)$. In the next sections, we consider different methods.

2. Parametrization of Compact Subsets

The first method under consideration is based on the possibility of considering a parametrization of convex compact which can be extended to more general non convex subsets. In this way we can try to obtain Korovkin approximation results for the parametrized sets.

A natural way to parametrize a compact convex subset is to consider its support function (see, e.g., [8, 13]). Namely, denote by S^{d-1} be the unit sphere surface in $\mathbb{R}^d$ ($S^{d-1} := \mathbf{B}^d \setminus \overset{\circ}{\mathbf{B}}{}^d$).

The support function $\delta^* : \mathcal{K}_c(\mathbb{R}^d) \times S^{d-1} \to \mathbb{R}$ is defined by setting, for every $A \in \mathcal{K}_c(\mathbb{R}^d)$ and every direction $d \in S^{d-1}$,

$$\delta^*(A,d) := \max_{a\in A}\langle d,a\rangle\,.$$

Many properties of the map δ^* are well-known and are listed, e.g., in [8, 9, 13]. In particular, $\delta^*(A,\cdot)$ is continuous for every $A \in \mathcal{K}_c(\mathbb{R}^d)$. Moreover, a subset $A \in \mathcal{K}_c(\mathbb{R}^d)$ may be recovered from its support function and is given by

$$A = \bigcap_{d\in S^{d-1}} \{x \in \mathbb{R}^d \mid \langle d,x\rangle \leq \delta^*(A,d)\}\,.$$

Now, if X be a compact Hausdorff topological space, we can associate to every set-valued function $F \in C(X, \mathcal{K}_c(\mathbb{R}^d))$ its parametrization $p_F : X \times S^{d-1} \to \mathbb{R}$ defined by setting, for every $(x, d) \in X \times S^{d-1}$,

$$p_F(x,d) := \delta^*(F(x), d) \quad (= \max_{y \in F(x)} \langle d, y \rangle) .$$

It is well-known that f is Hausdorff continuous if and only if p_F is (uniformly) continuous on $X \times S^{d-1}$. Thus, we obtain a continuous operator $P : C(X, \mathcal{K}_c(\mathbb{R}^d)) \to C(X \times S^{d-1}, [0, +\infty[)$ defined by

$$P(F) := p_F , \qquad F \in C(X, \mathcal{K}_c(\mathbb{R}^d)) .$$

Since convex compact subsets may be recovered from their support function, the map P is invertible. The above operator allows us to study Korovkin approximation in $C(X, \mathcal{K}_c(\mathbb{R}^d))$ using Korovkin approximation results in $C(X \times S^{d-1}, \mathbb{R})$.

As regards to the space $C(X \times S^{d-1}, [0, +\infty[)$, we know many examples of Korovkin systems with respect to positive operators both in $C(X, \mathbb{R})$ than in $C(S^{d-1}), \mathbb{R})$ (see, e.g. [1, Chapter 4] and [2]) and hence we can easily obtain Korovkin systems in $C(X \times S^{d-1}, \mathbb{R})$ (see [1] for a rather complete treatment of Korovkin systems on a product space). We also observe that many classical examples are just made up by positive functions. At this point, we can fix a Korovkin system D in $C(X \times S^{d-1}, \mathbb{R})$ consisting of positive functions and consider the following subset of $C(X, \mathcal{K}_c(\mathbb{R}^d))$

$$\mathcal{D} := \{F \in C(X, \mathcal{K}_c(\mathbb{R}^d)) \mid p_F \in D\} .$$

We have the following result.

Theorem 2.2. *The set $\mathcal{D}$ is a Korovkin system in $C(X, \mathcal{K}_c(\mathbb{R}^d))$ with respect to equicontinuous nets of monotone operators.*

Proof. Let $(L_i)_{i \in I}^{\leq}$ be an equicontinuous net of monotone operators from $C(X, \mathcal{K}_c(\mathbb{R}^d))$ into itself satisfying $\lim_{i \in I^{\leq}} L_i(H) = H$ for every $H \in \mathcal{D}$. For every $i \in I$, consider the operator $M_i : C(X \times S^{d-1}, [0, +\infty[) \to C(X \times S^{d-1}, [0, +\infty[)$ defined by $\xi \in C(X \times S^{d-1}, [0, +\infty[)$ and $(x, d) \in X \times S^{d-1}$,

$$M_i(\xi)(x, d) := \delta^*(L_i(P^{-1}(\xi))(x), d) .$$

In this way, we obtain a net $(M_i)_{i \in I}^{\leq}$ of operators from $C(X \times S^{d-1}, [0, +\infty[)$ into itself which are obviously monotone and satisfy $\lim_{i \in I^{\leq}} M_i(\eta) = \eta$ for every $\eta \in D$. Since D is a Korovkin system in $C(X \times S^{d-1}, [0, +\infty[)$, we have also $\lim_{i \in I^{\leq}} M_i(\xi) = \xi$ for every $\xi \in C(X \times S^{d-1}, [0, +\infty[)$. This yields, for every $F \in C(X, \mathcal{K}(\mathbb{R}^d))$, $\lim_{i \in I^{\leq}} P(L_i(F)) = P(F)$ and finally $\lim_{i \in I^{\leq}} L_i(F) = F$, which completes the proof. □

The above representation has been very useful in order to use well-known results for single-valued functions for the Korovkin approximation of set-valued Hausdorff continuous functions with compact and convex values.

However, we cannot use the above representation to extend our results to the Korovkin approximation of set-valued functions with non convex values. Indeed, if we consider a compact non convex subset K of $\mathbb{R}^d$, we can still consider the support function of K, but if we try to recover K from the support function we obtain the convex hull of K instead of K. Hence, this representation is useful only in the context of convex subsets of $\mathbb{R}^d$.

In order to overcome this problem, we consider a different representation which allows to recover uniquely every compact subset of $\mathbb{R}$. Namely, we consider the squared distance function $\rho^* : \mathcal{K}(\mathbb{R}^d) \times \mathbb{R}^d \to [0, +\infty[$ defined by setting, for every $K \in \mathcal{K}(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$,

$$\rho^*(K, y) := \inf_{x \in K} \|x - y\|^2 .$$

The squared distance function has many regularity properties since it is connected to the Moreau-Yosida regularization of the indicator function. Moreover, $\rho^*(K,\cdot)$ is a convex function whenever $K \in \mathcal{K}_c(\mathbb{R}^d)$.

Now, consider a set-valued Hausdorff continuous function $F : X \to \mathcal{K}(\mathbb{R})$; we can associate the function $\rho_F : X \times \mathbb{R}^d \to \mathbb{R}$ defined by setting, for every $x \in X$ and $y \in \mathbb{R}$,

$$\rho_F(x,y) := \rho^*(F(x),y) \quad (= \inf_{z\in F(x)} \|z-y\|^2)\,.$$

The function ρ_F is continuous; indeed, if we fix $(x_0,y_0) \in X \times \mathbb{R}^d$, for every $(x,y) \in X \times \mathbb{R}^d$ we have

$$\begin{aligned}|\rho_F(x,y)-\rho_F(x,y)| &\le |\rho_F(x,y)-\rho_F(x_0,y)| + |\rho_F(x_0,y)-\rho_F(x_0,y_0)|\\ &= \left|\inf_{z\in F(x)}\|z-y\|^2 - \inf_{z\in F(x_0)}\|z-y\|^2\right| + |\rho^*(F(x_0),y)-\rho^*(F(x_0),y_0)|\end{aligned}$$

and the first addend in the last sum tends to 0 as $x \to x_0$ due to the Hausdorff continuity of F, while the second addend tends to 0 as $y \to y_0$ due to the continuity of squared distance function. With the same argument it can be shown that ρ_F is uniformly continuous on every compact subset $X \times B$, where B is a closed ball containing the image $\bigcup_{x\in X} F(x)$ of F. Now, consider the set

$$\Gamma := \{\rho_F \mid F \in C(X,\mathcal{K}(\mathbb{R}^d))\}\,.$$

We have $\Gamma \subset C(X \times \mathbb{R}^d,\mathbb{R})$ and we can define on Γ a cone structure by defining

$$\rho_F \oplus \rho_G := \rho_{F+G}\,, \quad \lambda \odot \rho_F := \rho_{\lambda F}\,, \qquad F,G \in C(X,\mathcal{K}(\mathbb{R}^d))\,,\ \lambda \ge 0\,,$$

and assuming $\rho_F \preceq \rho_G$ if and only if $G \le F$ (the reversed inclusion is justified by the fact that the distance is smaller from larger sets).

Finally, it can be readily seen that a sequence $(F_n)_{n\in\mathbb{N}}$ in $C(X,\mathcal{K}(\mathbb{R}^d))$ converges to a set-valued function $F \in C(X,\mathcal{K}(\mathbb{R}^d))$ with respect to the uniform Hausdorff distance is and only if the sequence $(\rho_{F_n})_{n\in\mathbb{N}}$ converges to ρ_F uniformly on compact subsets $X \times B \subset X \times \mathbb{R}^d$. These properties allows us to state the following main result of this section.

Theorem 2.3. *Let X be a compact Hausdorff topological space and let $\mathcal{H}$ be a subset of $C(X,\mathcal{K}(\mathbb{R}^d))$ such that, for every $F \in C(X,\mathcal{K}(\mathbb{R}^d))$, $x_0 \in X$, $y_0 \in \mathbb{R}^d$ and $\varepsilon > 0$, there exist $G,H \in \mathcal{H}$ satisfying the following conditions*

$$G \le F \le H\,, \qquad d(G(x_0),y_0) < d(H(x_0),y_0) + \varepsilon\,. \tag{2.2}$$

Then, $\mathcal{H}$ is a Korovkin system in $C(X,\mathcal{K}(\mathbb{R}^d))$ with respect to equicontinuous nets of monotone operators.

Proof. Let $(L_i)_{i\in I}^{\le}$ be an equicontinuous net of monotone operators from $C(X,\mathcal{K}(\mathbb{R}^d))$ into itself satisfying $\lim_{i\in I\le} L_i(H) = H$ for every $H \in \mathcal{H}$. We observe that, for every $\rho \in \Gamma$, there exists a unique $F \in C(X,\mathcal{K}(\mathbb{R}^d))$ such that $\rho = \rho_F$. This allows us to define, for every $i \in I$, the operator $\tau_i : \Gamma \to \Gamma$ by setting, for every $F \in C(X,\mathcal{K}(\mathbb{R}^d))$,

$$\tau_i(\rho_F) := \rho_{L_i(F)}\,.$$

The operator τ_i is monotone since, for every $F,G \in C(X,\mathcal{K}(\mathbb{R}^d))$,

$$\rho_F \preceq \rho_G \;\Rightarrow\; G \le F \;\Rightarrow\; L_i(G) \le L_i(F) \;\Rightarrow\; \tau_i(\rho_F) = \rho_{L_i(F)} \preceq \rho_{L_i(G)} = \tau_i(\rho_G)$$

(in the same manner, it can be shown that τ_i is linear if L_i is linear). Hence, we obtain a net $(\tau_i)_{i\in I}^{\preceq}$ of monotone operators on Γ which satisfies $\lim_{i\in I\preceq} \tau_i(\rho_H) = \rho_H$ uniformly on compact subsets $X \times B \subset X \times \mathbb{R}^d$ for every $H \in \mathcal{H}$.

Let $F \in C(X, \mathcal{K}(\mathbb{R}^d))$ and $\varepsilon > 0$. From our assumptions, for every $(x_0, y_0) \in X \times \mathbb{R}^d$, there exist $G, H \in \mathcal{H}$ satisfying the following conditions

$$G \leq F \leq H\,, \qquad d(G(x_0), y_0) < d(H(x_0), y_0) + \min\left\{\varepsilon, \frac{\varepsilon}{2d(F(x_0), y_0) + \varepsilon}\right\}.$$

Thus, we have $\rho_H \preceq \rho_F \preceq \rho_G$; moreover, since

$$d(G(x_0), y_0) - d(F(x_0), y_0) \leq d(G(x_0), y_0) - d(H(x_0), y_0) \leq \varepsilon,$$

we have $d(G(x_0), y_0) \leq d(F(x_0), y_0) + \varepsilon$ and consequently

$$\begin{aligned} 0 \leq \rho_G(x_0, y_0) - \rho_H(x_0, y_0) &= d^2(G(x_0), y_0) - d^2(H(x_0), y_0) \\ &= (d(G(x_0), y_0) - d(H(x_0), y_0))(d(G(x_0), y_0) + d(H(x_0), y_0)) \\ &\leq \frac{\varepsilon}{2d(F(x_0), y_0) + \varepsilon}\,(d(F(x_0), y_0) + \varepsilon + d(F(x_0), y_0)) = \varepsilon\,. \end{aligned}$$

Since $\rho_H(x_0, y_0) \leq \rho_F(x_0, y_0) \leq \rho_G(x_0, y_0)$, we have also

$$\rho_G(x_0, y_0) \leq \rho_F(x_0, y_0) + \varepsilon\,, \qquad \rho_F(x_0, y_0) \leq \rho_H(x_0, y_0) + \varepsilon\,,$$

and therefore

$$\rho_G(x, y) - 2\varepsilon \leq \rho_F(x, y) \leq \rho_H(x, y) + 2\varepsilon\,,$$

for (x, y) in a suitable neighborhood of (x_0, y_0). Since, for every $i \in I$, the operator τ_i is monotone, we have also

$$\tau_i(\rho_H) \preceq \tau_i(\rho_F) \preceq \tau_i(\rho_G)\,.$$

Finally, if we fix a compact subset B of $\mathbb{R}^d$ such that (x_0, y_0) is in the interior of $X \times B$, we have $\lim_{i \in I^{\preceq}} \tau_i(\rho_H) = \rho_H$ and $\lim_{i \in I^{\preceq}} \tau_i(\rho_G) = \rho_G$ uniformly in $X \times B$ and therefore we can consider $\alpha \in I$ such that, for every $i \geq \alpha$ and $(x, y) \in X \times B$,

$$\rho_H(x, y) - \varepsilon \leq \tau_i(\rho_H)(x, y) \leq \tau_i(\rho_F)(x, y) \leq \tau_i(\rho_G)(x, y) \leq \rho_G(x, y) + \varepsilon\,.$$

Collecting the last inequalities, we obtain

$$\tau_i(\rho_F)(x, y) - 3\varepsilon \leq \rho_F(x, y) \leq \tau_i(\rho_F)(x, y) + 3\varepsilon\,,$$

in a suitable neighborhood of (x_0, y_0). At this point, a straightforward argument applied to compact subsets $X \times B \subset X \times \mathbb{R}^d$ yields the convergence of $(\tau_i(\rho_F))^{\preceq}_{i\in I}$ to ρ_F uniformly on compact subsets. This is equivalent to the convergence of $(L_i(F)))^{\leq}_{i\in I}$ to F and this completes the proof. □

Theorem 2.3 extends Korovkin-type approximation theory to set-valued Hausdorff continuous functions with non convex values. This is not the only noteworthy point. Indeed, in Theorem 1.1 and all previous results, only global assumption independent of direction have been considered. On the contrary, in Theorem 2.3, the second condition in (2.2) is a local assumption that depends on each direction. We also observe that in (2.2) there is an additional internal condition which is due to the possible failure of the convexity condition. However, in some cases of particular interest this condition can be weakened. If $f \in C(X, \mathbb{R})$, we shall denote by $\{f\}$ the set-valued function defined by setting, for every $x \in X$, $\{f\}(x) := \{f(x)\}$. Moreover, we shall need to consider this corollary.

Corollary 2.1. *Let X be a compact Hausdorff topological space and let $\mathcal{C}$ a subcone of $C(X, \mathcal{K}(\mathbb{R}^d))$ such that $\{f\} \in \mathcal{C}$ for every $f \in C(X, \mathbb{R}^d)$ and such that, for every $F \in \mathcal{C}$ and $x \in X$,*

$$F(x) = \bigcup_{f \in \mathrm{Sel}(F)} \{f(x)\}\,.$$

Moreover, let $\mathcal{H}$ be a subset of $\mathcal{C}$ such that

i) *There exists a Korovkin system $S \subset C(X, \mathbb{R}^d)$ such that, for every $s \in S$, $\{s\} \in \mathcal{H}$.*
ii) *For every $F \in \mathcal{C}$, $x_0 \in X$, $y_0 \in \mathbb{R}^d$ and $\varepsilon > 0$, there exists $H \in \mathcal{H}$ satisfying the following conditions*

$$F \le H\,, \qquad d(F(x_0), y_0) < d(H(x_0), y_0) + \varepsilon\,. \tag{2.3}$$

Then, $\mathcal{H}$ is a Korovkin system in $\mathcal{C}$ with respect to equicontinuous nets of monotone operators.

Proof. Since S is a Korovkin system, assumption i) ensures the convergence on every function $\{f\}$ with $f \in C(X, \mathbb{R}^d)$.

Now, let $F \in \mathcal{C}$, $x_0 \in X$, $y_0 \in \mathbb{R}^d$ and $\varepsilon > 0$ and consider the set-valued function $H \in \mathcal{H}$ such that $F \le H$ and $d(F(x_0), y_0) < d(H(x_0), y_0) + \varepsilon/2$. From the definition of $d(F(x_0), y_0)$, there exists $z_0 \in F(x_0)$ such that $d(z_0, y_0) < d(F(x_0), y_0) + \varepsilon/2$. Moreover, we can consider a continuous selection g of F such that $g(x_0) = z_0$. If we set $G := \{g\}$, we have $G \in \mathcal{C}$, $G \le F$ and

$$d(G(x_0), y_0) = d(z_0, y_0) < d(F(x_0), y_0) + \frac{\varepsilon}{2} < d(H(x_0), y_0) + \varepsilon\,.$$

Hence, condition (2.2) is satisfied and this completes the proof. □

Different Korovkin systems in cones of set-valued functions are obtained starting with Korovkin systems in $C(X, \mathbb{R}^d)$ and therefore condition i) in Theorem 2.3 is not too restrictive. The usage of Korovkin systems for single-valued continuous functions is deepened in the next section for particular set-valued functions with non convex values.

3. Korovkin Results for Starshaped Set-Valued Functions

Set-valued functions with compact starshaped values have a particular interest for the possibility of extending classical operators to more general cones. We recall that a subset S of $\mathbb{R}^d$ is called starshaped set if there exists $x_0 \in S$ such that the segment joining x_0 and x is contained in S whenever $x \in S$.

We shall denote by $\mathcal{S}(\mathbb{R}^d)$ the set of all starshaped compact subsets of $\mathbb{R}^d$. $\mathcal{S}(\mathbb{R}^d)$ is obviously a cone endowed with the natural addition and multiplication by positive scalars. If $A \in \mathcal{S}(\mathbb{R}^d)$, we observe that the kernel $\ker A$ of A, defined by

$$ker(A) := \{x_0 \in A \mid \forall\, x \in A\,, \forall \lambda \in [0,1]:\ x_0 + \lambda(x - x_0) \in A\}\,,$$

is compact and convex. Indeed, $\ker(A)$ is obviously bounded and if we set, for every $x_0 \in A$,

$$A(x_0) := \{x \in A \mid \forall\, \lambda \in [0,1]:\ x_0 + \lambda(x - x_0) \in A\}\,,$$

we can easily show that $A(x_0)$ is closed and $\ker(A) = \bigcap_{x_0 \in A} A(x_0)$. The convexity of $\ker(A)$ is obvious. Hence, for every $F \in C(X, \mathcal{S}(\mathbb{R}^d))$, we can consider the *kernel set-valued function* $\ker(F) : X \to \mathcal{K}_c(\mathbb{R}^d)$ defined by

$$(\ker F)(x) \ = \ker\,(F(x))\,.$$

Unfortunately, we cannot ensure in general that $\ker(F)$ is Hausdorff continuous and this does not allow us to consider a kernel selection, that is, a continuous selection of $\ker(F)$.

Now, assume that $\ker F$ admits a continuous selection φ_F, which acts as a center selection. We can associate to φ_F the radial function $\psi_F : X \times S^{d-1} \to [0, +\infty[$ defined by

$$\psi_F(x, y) := \max\{t \in [0, +\infty[\; \mid \varphi_F(x) + ty \in A\}\,.$$

Even under our restrictive assumptions, we cannot ensure the continuity of ψ_F; for example, consider the set-valued function $F : [-\pi/2, \pi/2] \to \mathcal{S}_R(\mathbb{R}^2)$ defined by setting, for every $\theta_0 \in [-\pi/2, \pi/2]$,

$$F(\theta_0) := \left\{(\rho\cos\theta, \rho\sin\theta) \in \mathbb{R}^2 \mid 0 \le \rho \le 1\,, |\theta_0| \le |\theta| \le \pi\right\}\,.$$

If we take the constant function $\varphi_F(x) := (-1/2, 0)$ as central selection, then the radial function associated to φ_F is not continuous at $\theta_0 = 0$. Therefore, we shall restrict our attention to the subcone $\mathcal{C}$ of $C(X, \mathcal{S}_R(\mathbb{R}^d))$ consisting of all $F \in C(X, \mathcal{S}_R(\mathbb{R}^d))$ which admit a continuous central selection φ_F of $\ker(F)$ and a continuous radial function ψ_F with positive values.

It is clear that we can associate a unique set-valued function $F_{\varphi,\psi} : X \to \mathcal{S}_R(\mathbb{R}^d)$ to every pair $(\varphi, \psi) \in C(X, \mathbb{R}^d) \times C(X \times S^{d-1}, [0, +\infty[)$ by setting, for every $x \in X$,

$$F_{\varphi,\psi}(x) := \{\varphi(x) + ty \in \mathbb{R}^d \mid y \in S^{d-1}\,,\ t \in [0, \psi(x, y)]\}\,. \tag{3.4}$$

In the next Proposition, we show that $F_{\varphi,\psi}$ is Hausdorff continuous.

Proposition 3.1. *Let $\varphi : X \to \mathbb{R}^d$ and $\psi : X \times S^{d-1} \to \mathbb{R}$ be continuous and assume that ψ takes strictly positive values. Then, the set-valued function $F_{\varphi,\psi} : X \to \mathcal{S}_R(\mathbb{R}^d)$ defined by (3.4) is Hausdorff continuous.*

Proof. Since the Hausdorff distance is invariant under translations, we can assume $\varphi = 0$. Moreover, for every $x_0, x \in X$,

$$d_H(F(x_0), F(x)) \le \sup_{y \in S^{d-1}} |\psi(x, y) - \psi(x_0, y)|$$

and since ψ is uniformly continuous

$$\lim_{x \to x_0} \left(\sup_{y \in S^{d-1}} |\psi(x, y) - \psi(x_0, y)| \right) = 0$$

and this completes the proof. □

Hence, we can associate to every pair $(\varphi, \psi) \in C(X, \mathbb{R}^d) \times C(X \times S^{d-1}, [0, +\infty[)$ a set-valued function in the cone $\mathcal{C}$. The converse property is not valid since we may have more than one central selection and radial function which are associated with the same set-valued function. Hence, we have only a partial Korovkin-type result in the cone $\mathcal{C}$.

First, we recall that a linear operator $T : C(X, \mathbb{R}^d) \to C(X, \mathbb{R}^d)$ is called a convexity monotone operator if for every $\varphi_1, \varphi_2 \in C(X, \mathbb{R}^d)$ having disjoint graphs

$$\varphi \in \mathrm{co}(\varphi_1, \varphi_2) \ \Rightarrow\ T(\varphi) \in \mathrm{co}(T(\varphi_1), T(\varphi_2))\,. \tag{3.5}$$

Many examples of Korovkin systems in spaces of vector-valued continuous functions with respect to equicontinuous nets of convexity monotone operators have been obtained in [3] and, more recently, also in [4, 5, 6]. We fix a Korovkin S in $C(X, \mathbb{R}^d)$ with respect to equicontinuous nets of convexity monotone operators.

Similarly, we have already observed in the preceding section that many examples of Korovkin systems with respect to positive linear operators are well-known both in $C(X, \mathbb{R})$ than in $C(S^{d-1}), \mathbb{R})$ (see, e.g. [1, Chapter 4] and [2]) and hence we can easily obtain Korovkin systems in $C(X \times S^{d-1}, \mathbb{R})$. Hence, we can also fix a Korovkin system D in $C(X \times S^{d-1}, \mathbb{R})$ consisting of positive functions. At this point, we can state the following convergence result. The proof is based on a simple property, which can be easily shown:

If $(\varphi_i)_{i \in I}^{\le}$ is a net of functions in $C(X, \mathbb{R}^d)$ converging uniformly to $\varphi \in C(X, \mathbb{R}^d)$ and if $(\psi_i)_{i \in I}^{\le}$ is a net of positive functions in $C(X \times S^{d-1}, \mathbb{R})$ converging uniformly to a positive function $\psi \in C(X \times S^{d-1,\mathbb{R}})$, we have that the net $(F_{\varphi_i,\psi_i})^{\leq}_{i\in I}$ converges to $F_{\varphi,\psi}$ for the Hausdorff distance.

Theorem 3.4. *Let X be a compact Hausdorff topological space and let S be a Korovkin system in $C(X,\mathbb{R}^d)$ with respect to equicontinuous nets of linear convexity monotone operators and D be a Korovkin system in $C(X \times S^{d-1},\mathbb{R})$ consisting of positive functions with respect to equicontinuous nets of positive linear operators. Consider a net $(S_i)^{\leq}_{i\in I}$ of linear convexity monotone operators from $C(X,\mathbb{R}^d)$ into itself and assume that, for every $\varphi \in S$, we have $\lim_{i\in I^{\leq}} S_i(\varphi) = \varphi$.*

Moreover, let $(T_i)^{\leq}_{i\in I}$ a net of linear positive operators from $C(X \times S^{d-1},\mathbb{R})$ into itself and assume that, for every $\psi \in D$, we have $\lim_{i\in I^{\leq}} T_i(\psi) = \psi$.

Then, the net $(F_{S_i(\varphi),T_i(\psi)})^{\leq}_{i\in I}$ converges to $F_{\varphi,\psi}$ for every $\varphi \in C(X,\mathbb{R}^d)$ and $\psi \in C(X \times S^{d-1},[0,+\infty[)$.

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Research Article

An asymptotic expansion for an integral variant of the Wright operators

Dedicated to Professor Ioan Raşa, on the occasion of his 75th birthday

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ABSTRACT. In this work, we investigate an integral variant of the Wright operators arising in approximation theory. Our main contribution is the derivation of a complete pointwise asymptotic expansion for these operators $W_n^{\beta,\gamma}$. To establish this expansion, we first develop several structural properties of the operators, emphasizing their intrinsic connections with special functions, including the Mittag-Leffler and confluent hypergeometric functions. The analysis requires a careful study of the moments of the associated integral operators and relies on a collection of auxiliary results, such as a localization theorem. The obtained asymptotic formula provides refined convergence estimates and yields deeper insight into the limiting behavior of the operators as $n \to \infty$.

Keywords: Mittag-Leffler functions, asymptotic expansion, hypergeometric series, pointwise estimations.

2020 Mathematics Subject Classification: 41A60, 41A30, 41A36.

1. INTRODUCTION

Positive linear operators constitute a fundamental tool in approximation theory, providing effective schemes for approximating continuous functions and for studying convergence and asymptotic behavior. In recent years, there has been growing interest in operator sequences constructed via special functions arising in fractional calculus and probability theory, owing to their rich analytical structure and enhanced flexibility. Among these, operators associated with the Wright function have attracted considerable attention due to their close connection with fractional diffusion processes and their ability to model non-local effects. The present paper is devoted to the study of an integral variant of the Wright operators, with particular emphasis on deriving a precise asymptotic expansion that yields refined convergence estimates and deeper insight into the limiting behavior of the operators.

The three-parameter Mittag-Leffler function $E_{\alpha,\beta}^{\gamma}$ (also known as the Prabhakar function), introduced by T. R. Prabhakar in 1971 [8], is defined as

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{\nu=0}^{\infty} \binom{\gamma+\nu-1}{\nu} \frac{z^{\nu}}{\Gamma(\alpha\nu+\beta)}, \qquad (z \in \mathbb{C}, \ \Re(\alpha) > 0), \tag{1.1}$$

where $\binom{\gamma+\nu-1}{\nu} = \dfrac{\Gamma(\gamma+\nu)}{\nu!\Gamma(\gamma)} = \dfrac{\gamma^{\overline{\nu}}}{\nu!}$ and $\Gamma(\cdot)$ denotes the Euler gamma function. Here, $\gamma^{\overline{0}} = 1$ and $\gamma^{\overline{\nu}} = \gamma(\gamma+1)\cdots(\gamma+\nu-1)$, for $\nu \in \mathbb{N}$, denotes the rising factorial. Although the three parameters α, β and γ are allowed to assume values in $\mathbb{C}$, we will use the function $E_{\alpha,\beta}^{\gamma}$ only for

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real values of α, β and γ. We mention the special instances $E_{1,\beta}^{\beta}(z) = \dfrac{e^z}{\Gamma(\beta)}$ and $E_{1,1}^{1}(z) = e^z$. In this paper, we focus on the special case $\alpha = 1$. The asymptotic formulae for the three parametric Mittag-Leffler functions for large values of indices were established in [4]. The positive linear operator using the terms of the Prabhakar function $E_{1,\beta}^{\gamma}(nx)$ as basis functions,

$$\left(L_n^{1,\beta,\gamma} f\right)(x) = \left(E_{1,\beta}^{\gamma}(nx)\right)^{-1} \sum_{\nu=0}^{\infty} \binom{\gamma+\nu-1}{\nu} \frac{(nx)^{\nu}}{\Gamma(\nu+\beta)} f\left(\frac{\nu}{n}\right)$$

was introduced and studied in [1].

Closely related to the Mittag-Leffler function is the Wright function [10], introduced by E. M. Wright as a natural generalization of exponential-type functions in the theory of entire functions. It is defined by the series

$$\phi_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(\alpha k+\beta)}, \qquad \alpha > -1,\ \beta \in \mathbb{C}.$$

The Wright function plays a significant role in fractional calculus, as it appears in the fundamental solutions of time-fractional and space-fractional diffusion equations. It is also closely connected with the Mittag-Leffler function, Lévy stable distributions, and probability density functions describing anomalous diffusion processes. Furthermore, in approximation theory and applied analysis, Wright-type functions are employed in the construction of positive linear operators [6] and in the study of their asymptotic behavior and convergence properties, highlighting their importance in both theoretical and applied contexts.

The Prabhakar function $E_{\alpha,\beta}^{\gamma}$ and the Wright function $\phi_{\alpha,\beta}$ play a central role in several areas of fractional calculus, probability theory, and anomalous relaxation phenomena. The Prabhakar function, as a three-parameter generalization of the classical Mittag-Leffler function, gives rise to the Prabhakar fractional integral operator, which extends the classical Riemann–Liouville and Caputo fractional integrals. On the other hand, the Wright function naturally appears in the fundamental solutions of time-fractional and space-fractional diffusion equations and is closely connected with stable probability distributions. Both functions are extensively used in the modeling of memory-dependent processes, viscoelastic materials, and non-local diffusion phenomena. In approximation theory, kernels expressed in terms of the Prabhakar and Wright functions enable the construction of positive linear operators with adjustable parameters, leading to enhanced flexibility and improved convergence behavior. Consequently, the availability of accurate asymptotic expansions for $E_{\alpha,\beta}^{\gamma}$ and $\phi_{\alpha,\beta}$ is essential for rigorous theoretical analysis and efficient numerical implementation, particularly in large-parameter regimes, stability investigations, and the derivation of precise rate of convergence results. Such expansions also facilitate closed-form representations of moments and provide deeper insight into the asymptotic behavior of the associated operators as $n \to \infty$, which is crucial in approximation theory and applied mathematics.

For a real number $A \geq 0$, let $E_A[0,\infty)$ be the class of all functions $f : [0,\infty) \to \mathbb{R}$ which satisfy, for some positive constant K, the growth condition $|f(t)| \leq K \exp(At)$, for $t \geq 0$. Let $E[0,\infty) = \bigcup_{A \geq 0} E_A[0,\infty)$ be the space of all functions of (at most) exponential growth. For $\beta > 0$ and $\gamma \geq 1$, we define the operators $W_n^{\beta,\gamma} : E[0,\infty) \to C[0,\infty)$ by

$$\left(W_n^{\beta,\gamma} f\right)(x) = \left(E_{1,\beta}^{\gamma}(nx)\right)^{-1} \sum_{\nu=0}^{\infty} \binom{\gamma+\nu-1}{\nu} \frac{(nx)^{\nu}}{\nu!\,\Gamma(\nu+\beta)} \int_0^{\infty} e^{-t} t^{\nu} f\left(\frac{t}{n}\right) dt. \tag{1.2}$$

It is obvious that, the operators $W_n^{\beta,\gamma}$ are linear and positive. We observe that, for sufficiently large n, the operators $W_n^{\beta,\gamma}$ are well defined on the space $E\,[0,\infty)$ since $f \in E_A\,[0,\infty)$, implies that, for $n > A$,

$$\left|\int_0^\infty e^{-t}t^\nu f\left(\frac{t}{n}\right)dt\right| \leq K\int_0^\infty e^{-t}t^\nu \exp\left(At/n\right)dt$$
$$= K\int_0^\infty t^\nu \exp\left(-t\left(1-A/n\right)\right)dt = \frac{K\nu!}{\left(1-A/n\right)^{\nu+1}}$$

and

$$\left|\left(W_n^{\beta,\gamma}f\right)(x)\right| \leq \frac{Kn}{n-A}\left(E_{1,\beta}^{\gamma}\left(nx\right)\right)^{-1}E_{1,\beta}^{\gamma}\left(\frac{n^2x}{n-A}\right).$$

At the point $x=0$, the operators (1.2) approximate the function f, provided that it is continuous from the right at $x=0$. In this case, we have $\left(W_n^{\beta,\gamma}f\right)(0) = \int_0^\infty e^{-t}f\left(\frac{t}{n}\right)dt \to f(0)$ as $n \to \infty$.

In the special case, $\gamma = 1$, the operator $W_n^{\beta,\gamma}$ was discussed by Patel [5], another variant was discussed in [7]. Further, the special case $\beta = \gamma \geq 1$, the operator $W_n^{\beta,\gamma}$ reduces to

$$\left(W_n^{\beta,\beta}f\right)(x) = e^{-nx}\int_0^\infty e^{-t}I_0\left(2\sqrt{nxt}\right)f\left(\frac{t}{n}\right)dt,$$

where I_0 denotes the modified Bessel function of the first kind given by

$$I_0\left(z\right) = \sum_{\nu=0}^\infty \frac{z^{2\nu}}{4^\nu\left(\nu!\right)^2}.$$

Our focus is to study the rate of convergence for the operators $W_n^{\beta,\gamma}$. Also, we derive a (pointwise) complete asymptotic expansion for the sequence $\left(\left(W_n^{\beta,\gamma}f\right)(x)\right)_{n=1}^\infty$ in the form

$$\left(W_n^{\beta,\gamma}f\right)(x) \sim f(x) + \sum_{k=1}^\infty a_k^{\beta,\gamma}\left(f,x\right)n^{-k} \qquad (n\to\infty), \tag{1.3}$$

provided that f admits derivatives of sufficiently high order at $x > 0$. Formula (1.3) means that, for all $q = 1, 2, \ldots$, it holds

$$\left(W_n^{\beta,\gamma}f\right)(x) = f(x) + \sum_{k=1}^q a_k^{\beta,\gamma}\left(f,x\right)\,n^{-k} + o(n^{-q}) \qquad (n\to\infty).$$

The coefficients $a_k^{\beta,\gamma}\left(f,x\right)$, which are independent of n, will be given in terms of coefficients of a certain power series. As a particular case $\gamma = 1$, we obtain the complete asymptotic expansion for the sequence of Patel's operators established in [5]. In the special case $q=1$, we complement his results on $W_n^{\beta,1}$ by a Voronovskaja-type theorem.

2. Main Results

The following theorem presents as our main result the (pointwise) complete asymptotic expansion for the integral generalization of the Mittag-Leffler operators $W_n^{\beta,\gamma}$.

Theorem 2.1. *Let $q \in \mathbb{N}$ and $x \in (0,\infty)$. For each function $f \in E\,[0,\infty)$ which is $2q$ times differentiable at x, the operators $W_n^{\beta,\gamma}$ possess the asymptotic expansion*

$$\left(W_n^{\beta,\gamma}f\right)(x) = f(x) + \sum_{k=1}^q a_k^{\beta,\gamma}\left(f,x\right)n^{-k} + o(n^{-q}) \qquad (n\to\infty) \tag{2.4}$$

with the coefficients

$$a_k^{\beta,\gamma}(f,x) = \sum_{s=1}^{2k} \frac{f^{(s)}(x)}{s!} x^{s-k} T_{k,s}(\gamma,\beta) \qquad (k=1,2,\ldots), \tag{2.5}$$

where the numbers $T_{k,s}(\alpha,\gamma,\beta)$ *and* $H_k(a,b,j)$ *are defined by*

$$T_{k,s}(\gamma,\beta) := \sum_{j=0}^{k} \sum_{r=j}^{s} (-1)^{s-r} \binom{s}{r} \binom{r}{j} r^{\underline{j}} H_{k-j}(\gamma,\beta,r-j) \tag{2.6}$$

and by equation (3.10)*, respectively.*

Before proceeding to the proof of the above theorems, we first establish several fundamental properties of the operators $W_n^{\beta,\gamma}$. In particular, we demonstrate their connections with other special functions, notably the confluent hypergeometric function. Throughout the paper, $z^{\underline{0}} = 1$, $z^{\underline{j}} = z(z-1)\cdots(z-j+1)$, $j \in \mathbb{N}$, denote the falling factorials.

Remark 2.1. *If* $f \in \bigcap_{q=1}^{\infty} K[q;x]$*, the operators* $W_n^{\beta,\gamma}$ *possess the complete asymptotic expansion*

$$\left(W_n^{\beta,\gamma} f\right)(x) = f(x) + \sum_{k=1}^{\infty} a_k^{\beta,\gamma}(f,x)\; n^{-k} \qquad (n \to \infty),$$

where the coefficients $a_k^{\beta,\gamma}(f,x)$ *are defined in* (2.5)*.*

In the case $q = 1$, an immediate consequence of Theorem 2.1 is the following Voronovskaja-type formula.

Corollary 2.1. *Let* $x \in (0,\infty)$*. For each function* $f \in K[2;x]$*, the operators* $W_n^{\beta,\gamma}$ *satisfy*

$$\lim_{n\to\infty} n\left(\left(W_n^{\beta,\gamma} f\right)(x) - f(x)\right) = (1+\gamma-\beta) f'(x) + x f''(x).$$

Remark 2.2. *For the convenience of the reader, we list the explicit expressions for the initial coefficients* $a_k^{\beta,\gamma}(f,x)$*:*

$$\begin{aligned}
a_1^{\beta,\gamma}(f,x) &= (1+\gamma-\beta) f'(x) + x f''(x) \\
a_2^{\beta,\gamma}(f,x) &= \frac{(\beta-\gamma)(\gamma-1)}{x} f'(x) \\
&+ \frac{(\gamma-\beta+1)(\gamma-\beta+2)}{2} f''(x) + \frac{\gamma-\beta+2}{6} x f^{(3)}(x) + \frac{x^2}{2} f^{(4)}(x) \\
a_3^{\beta,\gamma}(f,x) &= \frac{(\gamma-1)(\gamma-\beta)(2\gamma-\beta-2)}{x^2} f'(x) + \frac{(1-\gamma)(\gamma-\beta)(\gamma-\beta+1)}{x} f''(x) \\
&+ \frac{6(\gamma-\beta)^3 + (17-6\beta)(\gamma-\beta)^4}{6} f^{(3)}(x) + \frac{(\gamma-\beta+2)(\gamma-\beta+3)}{2} x f^{(4)}(x) \\
&+ \frac{\gamma-\beta+3}{2} x^2 f^{(5)}(x) + \frac{x^3}{6} f^{(6)}(x).
\end{aligned}$$

We emphasize the fact that $a_k^{\beta,\gamma}(f,x)$ contains only derivatives $f^{(s)}(x)$ of order $s \le 2k$. By equation (2.5), $a_k^{\beta,\gamma}(f,x)$, $k \in \mathbb{N}$, is a linear combination of the terms $x^{s-k} f^{(s)}(x)$, for $s = 1,\ldots,2k$. In the special case $\beta = \gamma$, we obtain the following corollary.

Corollary 2.2. *Let $q \in \mathbb{N}$ and $x \in (0,\infty)$. For each function $f \in E[0,\infty)$ which is $2q$ times differentiable at x, the operators $W_n^{\beta,\beta}$ possess the asymptotic expansion*

$$\left(W_n^{\beta,\beta} f\right)(x) = f(x) + \sum_{k=1}^{q} a_k^{\beta,\beta}(f,x)\, n^{-k} + o(n^{-q}) \qquad (n \to \infty)$$

with the coefficients

$$a_k^{\beta,\beta}(f,x) = \sum_{s=0}^{k} \binom{k}{s} \frac{x^s}{s!} f^{(k+s)}(x) \qquad (k = 1,2,\ldots).$$

3. Moments for the Operators $W_n^{\beta,\gamma}$

Throughout the paper, e_r denote the monomials $e_r(t) = t^r$, $(r = 0,1,2,\ldots)$ and for each real x, we put $\psi_x = e_1 - x e_0$.

To demonstrate our main findings on the operators $W_n^{\beta,\gamma}$, we will need the moments of the integral operators (1.2). The proof of Theorem 2.1 relies on many lemmas, which are compiled in this section.

Lemma 3.1. *For $r = 0,1,2,\ldots$, the moments of the operators $W_n^{\beta,\gamma}$ have the representation*

$$\left(W_n^{\beta,\gamma} e_r\right)(x) = n^{-r} \sum_{j=0}^{r} \binom{r}{j} r^{\underline{r-j}} (nx)^j \frac{\gamma^{\overline{j}}}{\beta^{\overline{j}}} \frac{{}_1F_1(\gamma+j;\beta+j;nx)}{{}_1F_1(\gamma;\beta;nx)}. \tag{3.7}$$

With the abbreviation

$$R_{n,j}^{\beta,\gamma}(x) = \frac{\gamma^{\overline{j}}}{\beta^{\overline{j}}} \frac{{}_1F_1(\gamma+j;\beta+j;nx)}{{}_1F_1(\gamma;\beta;nx)},$$

the first instances are given by

$$\begin{aligned}
W_n^{\beta,\gamma} e_0 &= e_0, \\
W_n^{\beta,\gamma} e_1 &= R_{n,1}^{\beta,\gamma} e_1, \\
W_n^{\beta,\gamma} e_2 &= R_{n,2}^{\beta,\gamma} e_2 + \frac{2}{n} R_{n,1}^{\beta,\gamma} e_1, \\
W_n^{\beta,\gamma} e_3 &= R_{n,3}^{\beta,\gamma} e_3 + \frac{9}{n} R_{n,2}^{\beta,\gamma} e_2 + \frac{27}{n^2} R_{n,1}^{\beta,\gamma} e_1, \\
W_n^{\beta,\gamma} e_4 &= R_{n,4}^{\beta,\gamma} e_4 + \frac{16}{n} R_{n,3}^{\beta,\gamma} e_3 + \frac{96}{n^2} R_{n,2}^{\beta,\gamma} e_2 + \frac{256}{n^3} R_{n,1}^{\beta,\gamma} e_1.
\end{aligned}$$

Proof. Since $\int_0^\infty e^{-t} t^\nu e_r\left(\frac{t}{n}\right) dt = n^{-r}(\nu+r)!$, we obtain

$$\left(W_n^{\beta,\gamma} e_r\right)(x) = n^{-r} \left(E_{1,\beta}^{\gamma}(nx)\right)^{-1} \sum_{\nu=0}^{\infty} \binom{\gamma+\nu-1}{\nu} \frac{(nx)^\nu (\nu+r)!}{\nu!\,\Gamma(\nu+\beta)}.$$

By Vandermonde convolution, we have

$$\frac{(\nu+r)!}{\nu!} = \sum_{j=0}^{r} \binom{r}{j} \nu^{\underline{j}} r^{\underline{r-j}}$$

which leads to

$$\left(W_n^{\beta,\gamma} e_r\right)(x) = n^{-r} \left(E_{1,\beta}^{\gamma}(nx)\right)^{-1} \sum_{j=0}^{r} \binom{r}{j} r^{\underline{r-j}} \sum_{\nu=j}^{\infty} \binom{\gamma+\nu-1}{\nu} \frac{(nx)^\nu}{\Gamma(\nu+\beta)} \nu^{\underline{j}}.$$

Since

$$\nu^{\underline{j}}\binom{\gamma+\nu-1}{\nu}=\gamma^{\overline{j}}\binom{\gamma+\nu-1}{\nu-j}\qquad(\nu\geq j),$$

we obtain

$$\left(W_n^{\beta,\gamma}e_r\right)(x)=n^{-r}\left(E_{1,\beta}^{\gamma}(nx)\right)^{-1}\sum_{j=0}^{r}\binom{r}{j}r^{\underline{r-j}}\gamma^{\overline{j}}(nx)^j E_{1,\beta+j}^{\gamma+j}(nx).$$

Taking advantage of (3.8) leads to

$$\left(W_n^{\beta,\gamma}e_r\right)(x)=n^{-r}\sum_{j=0}^{r}\binom{r}{j}r^{\underline{r-j}}\gamma^{\overline{j}}(nx)^j\frac{\Gamma(\beta)}{\Gamma(\beta+j)}\frac{{}_1F_1(\gamma+j;\beta+j;nx)}{{}_1F_1(\gamma;\beta;nx)},$$

which is the desired formula, since $\Gamma(\beta+j)/\Gamma(\beta)=\beta^{\overline{j}}$. □

For the subsequent discussion, we require the connection between the confluent hypergeometric function and the three-parameters Mittag-Leffler function (1.1), as stated in [2, Equation 5.1.39] as

$$E_{1,\beta}^{\gamma}(z)=\frac{1}{\Gamma(\beta)}\sum_{\nu=0}^{\infty}\frac{\gamma^{\overline{\nu}}}{\beta^{\overline{\nu}}}\frac{z^\nu}{\nu!}=\frac{1}{\Gamma(\beta)}\,{}_1F_1(\gamma;\beta;z),\tag{3.8}$$

where the confluent hypergeometric function is given by

$${}_1F_1(a;b;z)=\sum_{\nu=0}^{\infty}\frac{a^{\overline{\nu}}}{b^{\overline{\nu}}}\frac{z^\nu}{\nu!}.$$

It is well known that ${}_1F_1$ satisfies the asymptotic relation (see [3, Formula 13.7.1])

$${}_1F_1(a;b;z)\sim\frac{e^z z^{a-b}\Gamma(b)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{(b-a)^{\overline{k}}(1-a)^{\overline{k}}}{k!z^k}\qquad(z\to+\infty).\tag{3.9}$$

As in [1], we define the numbers $H_k(a,b,j)$ by the relation

$$\frac{1+\sum_{k=1}^{\infty}(b-a)^{\overline{k}}(1-a-j)^{\overline{k}}\frac{w^k}{k!}}{1+\sum_{k=1}^{\infty}(b-a)^{\overline{k}}(1-a)^{\overline{k}}\frac{w^k}{k!}}=\sum_{k=0}^{\infty}H_k(a,b,j)\,w^k,\tag{3.10}$$

where all series are interpreted as formal power series. The following consequence will become useful related to $H_k(a,b,j)$

$$\frac{{}_1F_1(a+j;b+j;z)}{{}_1F_1(a;b;z)}\sim\frac{b^{\overline{j}}}{a^{\overline{j}}}\sum_{k=0}^{\infty}\frac{H_k(a,b,j)}{z^k}\qquad(z\to+\infty).\tag{3.11}$$

For the convenience of the reader we list some explicit expressions, which were also mentioned in [1]:

$$\begin{aligned}H_0(a,b,j)&=1,\\H_1(a,b,j)&=(a-b)\,j,\\H_2(a,b,j)&=\frac{(a-b)\,j}{2}(3-3a+b-j+aj-bj),\end{aligned}$$

$$\begin{aligned}H_3(a,b,j) = &\frac{(a-b)j}{6}\left(22-42a+20a^2+18b-16ab+2b^2\right.\\&-3(a-b-1)(3a-b-4)j\\&\left.+(a-b-2)(a-b-1)j^2\right).\end{aligned}$$

Lemma 3.2. *For $r=0,1,2,\ldots$, the moments of the operators $W_n^{\beta,\gamma}$ satisfy the asymptotic relation*

$$\left(W_n^{\beta,\gamma}e_r\right)(x)\sim\sum_{k=0}^{\infty}\frac{1}{n^k}\sum_{j=0}^{\min\{k,r\}}\binom{r}{j}r^{\underline{j}}H_{k-j}(\gamma,\beta,r-j)\,x^{r-k}\qquad(n\to\infty),$$

where $H_k(a,b,j)$ is defined in (3.10).

Proof. The starting point is the Equation (3.7) in Lemma 3.1, i.e.,

$$\left(W_n^{\beta,\gamma}e_r\right)(x)=n^{-r}\sum_{j=0}^{r}\binom{r}{j}r^{\underline{r-j}}(nx)^j\,\frac{\gamma^{\overline{j}}}{\beta^{\overline{j}}}\frac{{}_1F_1(\gamma+j;\beta+j;nx)}{{}_1F_1(\gamma;\beta;nx)}.$$

Hence, by (3.11),

$$\left(W_n^{\beta,\gamma}e_r\right)(x)\sim n^{-r}\sum_{j=0}^{r}\binom{r}{j}r^{\underline{r-j}}(nx)^j\sum_{\ell=0}^{\infty}\frac{H_\ell(\gamma,\beta,j)}{(nx)^\ell}\qquad(n\to\infty).$$

Finally,

$$\left(W_n^{\beta,\gamma}e_r\right)(x)\sim\sum_{j=0}^{r}\binom{r}{j}r^{\underline{j}}n^{-j}x^{r-j}\sum_{\ell=0}^{\infty}\frac{H_\ell(\gamma,\beta,r-j)}{(nx)^\ell}\qquad(n\to\infty)$$

and collecting all terms with $j+\ell=k$, we obtain the desired formula. $\square$

Lemma 3.3. *For $s=0,1,2,\ldots$, the central moments of the operators $W_n^{\beta,\gamma}$ have the representation*

$$\left(W_n^{\beta,\gamma}\psi_x^s\right)(x)\sim\sum_{k=\lfloor(s+1)/2\rfloor}^{\infty}\frac{x^{s-k}}{n^k}T_{k,s}(\gamma,\beta)\qquad(n\to\infty),\tag{3.12}$$

where $T_{k,s}(\gamma,\beta)$ is defined by the Equation (2.6).

A direct consequence of Lemma 3.3 is the asymptotic relation

$$\left(W_n^{\beta,\gamma}\psi_x^s\right)(x)=O\left(n^{-\lfloor(s+1)/2\rfloor}\right)\qquad(n\to\infty).\tag{3.13}$$

Proof. By the binomial formula, we have

$$\left(W_n^{\beta,\gamma}\psi_x^s\right)(x)=\sum_{r=0}^{s}(-x)^{s-r}\binom{s}{r}\left(W_n^{\beta,\gamma}e_r\right)(x).$$

Application of Lemma 3.2 yields

$$\begin{aligned}\left(W_n^{\beta,\gamma}\psi_x^s\right)(x)&\sim\sum_{k=0}^{\infty}\frac{x^{s-k}}{n^k}\sum_{r=0}^{s}(-1)^{s-r}\binom{s}{r}\sum_{j=0}^{\min\{k,r\}}\binom{r}{j}r^{\underline{j}}H_{k-j}(\gamma,\beta,r-j)\\&=\sum_{k=0}^{\infty}\frac{x^{s-k}}{n^k}T_{k,s}(\gamma,\beta)\qquad(n\to\infty),\end{aligned}$$

where $T_{k,s}(\gamma,\beta)$ is defined in equation (2.6). It remains to prove that $T_{k,s}(\gamma,\beta)=0$ if $2k<s$. Using the binomial identity

$$\binom{s}{r}\binom{r}{j}=\binom{s}{j}\binom{s-j}{r-j},$$

we obtain

$$\begin{aligned} T_{k,s}(\gamma,\beta) &= \sum_{j=0}^{k}\binom{s}{j}\sum_{r=j}^{s}(-1)^{s-r}\binom{s-j}{r-j}r^{\underline{j}}H_{k-j}(\gamma,\beta,r-j) \\ &= \sum_{j=0}^{k}\binom{s}{j}\sum_{r=0}^{s-j}(-1)^{s-j-r}\binom{s-j}{r}(r+j)^{\underline{j}}H_{k-j}(\gamma,\beta,r). \end{aligned}$$

Equation (3.10) reveals that $H_k(a,b,z)$ is a polynomial in the variable z of degree at most k. This implies that

$$\sum_{r=0}^{s-j}(-1)^{s-j-r}\binom{s-j}{r}(r+j)^{\underline{j}}H_{k-j}(\gamma,\beta,r)=0$$

if $k<s-j$, i.e., if $k+j<s$. Noting that $0\le j\le k$, we conclude that $T_{k,s}(\gamma,\beta)=0$ if $2k<s$. This completes the proof. □

4. Proof of Theorem 2.1

In order to derive Theorem 2.1, a general approximation theorem due to Sikkema [9, Theorem 3] will be applied. To this end, we need some notation. Let I be a real interval and $x\in I$. For $s\in\mathbb{N}$, let $H^{(s)}(x)$ denote the class of all locally bounded real functions $f:I\to\mathbb{R}$, which are s times differentiable at x. In the case that I is an infinite interval, f has to satisfy the additional condition $f(t)=O(|t|^s)$ as $t\to+\infty$ or $t\to-\infty$, respectively, or both if $I=\mathbb{R}$. An inspection of the proof of Sikkema's result reveals that it can be stated in the following form which is more appropriate for our purposes.

Lemma 4.4. *Let $q\in\mathbb{N}$ and let $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators, $L_n: H^{(2q)}(x)\to C[c,d]$, $x\in[c,d]$. Suppose that the operators L_n apply to ψ_x^{2q+1} and to ψ_x^{2q+2}. Then the condition*

$$(L_n\psi_x^s)(x)=O\left(n^{-\lfloor (s+1)/2\rfloor}\right)\qquad (n\to\infty),\qquad \text{for } s=0,1,\ldots,2q+2,$$

implies, for each function $f\in H^{(2q)}(x)$, the asymptotic relation

$$(L_nf)(x)=\sum_{s=0}^{2q}\frac{f^{(s)}(x)}{s!}(L_n\psi_x^s)(x)+o\left(n^{-q}\right)\qquad (n\to\infty).$$

Since Lemma 4.4 deals with functions of polynomial growth, we cannot apply it directly to functions $f\in E[0,\infty)$. First, we note that the definition of $E_A[0,\infty)$ implies that the functions of this class are bounded on each finite interval $[0,r]$, $r>0$. In order to prove Theorem 2.1 for functions of (at most) exponential growth, we need a localization result (Lemma 4.6) for the operators $W_n^{\beta,\gamma}$. The next result will be applied in its proof.

For real numbers A, define $\exp_A(t)=e^{At}$.

Lemma 4.5. *Let A be a real number and $x>0$. Then, $\left(W_n^{\beta,\gamma}\exp_A\right)(x)$ is well defined, for all positive integers $n>A$, with*

$$\left(W_n^{\beta,\gamma}\exp_A\right)(x)=\frac{n}{n-A}\,\frac{{}_1F_1\left(\gamma;\beta;\dfrac{n}{n-A}nx\right)}{{}_1F_1(\gamma;\beta;nx)} \tag{4.14}$$

and has $\lim_{n\to\infty}\left(W_n^{\beta,\gamma}\exp_A\right)(x)=\exp_A(x)$.

Proof. Let $x>0$. An elementary calculation and application of (3.8) shows that, for $n>A$,

$$\begin{aligned}\left(W_n^{\beta,\gamma}\exp_A\right)(x)&=\left(E_{1,\beta}^{\gamma}(nx)\right)^{-1}\sum_{\nu=0}^{\infty}\binom{\gamma+\nu-1}{\nu}\frac{(nx)^{\nu}}{\nu!\Gamma(\nu+\beta)}\int_0^{\infty}e^{-t}t^{\nu}e^{\frac{At}{n}}dt\\&=\frac{n}{n-A}\frac{E_{1,\beta}^{\gamma}\left(\frac{n^2x}{n-A}\right)}{E_{1,\beta}^{\gamma}(nx)}\\&=\frac{n}{n-A}\frac{{}_1F_1\left(\gamma;\beta;\frac{n^2x}{n-A}\right)}{{}_1F_1(\gamma;\beta;nx)}.\end{aligned}$$

By (3.9), we conclude that

$$\begin{aligned}\left(W_n^{\beta,\gamma}\exp_A\right)(x)\sim&\frac{n}{n-A}\frac{\frac{e^{z_1}z_1^{\gamma-\beta}\Gamma(\beta)}{\Gamma(\gamma)}\left(1+O\left(z_1^{-1}\right)\right)}{\frac{e^{z_2}z_2^{\gamma-\beta}\Gamma(\beta)}{\Gamma(\gamma)}\left(1+O\left(z_2^{-1}\right)\right)}\\=&\frac{n}{n-A}e^{z_1-z_2}\left(\frac{z_1}{z_2}\right)^{\gamma-\beta}\left(1+O\left(z_1^{-1}\right)-O\left(z_2^{-1}\right)\right),\end{aligned}$$

where

$$z_1=\frac{n^2x}{n-A}=nx\frac{n}{n-A},\qquad z_2=nx.$$

As n tends to infinity, we have

$$\frac{n}{n-A}=1+\frac{A}{n}+O\left(n^{-2}\right),\qquad z_1=nx+Ax+O\left(n^{-1}\right).$$

Hence

$$z_1-z_2=Ax+O\left(n^{-1}\right),\qquad e^{z_1-z_2}=e^{Ax}\left(1+O\left(n^{-1}\right)\right).$$

Similarly,

$$\left(\frac{z_1}{z_2}\right)^{\gamma-\beta}=\left(\frac{n}{n-A}\right)^{\gamma-\beta}=1+(\gamma-\beta)\frac{A}{n}+O\left(n^{-2}\right).$$

Therefore

$$\left(W_n^{\beta,\gamma}\exp_A\right)(x)\sim e^{Ax}\left(1+\frac{A}{n}\right)\left(1+(\gamma-\beta)\frac{A}{n}\right)\left(1+O\left(n^{-1}\right)\right).$$

Hence, we have $\lim_{n\to\infty}\left(W_n^{\beta,\gamma}\exp_A\right)(x)=\exp_A(x)$. □

Lemma 4.6 (Localization Theorem). *Let* $\delta>0$ *and fix* $x>0$. *If a function* $f\in E_A[0,\infty)$ *vanishes on the interval* $(x-\delta,x+\delta)\cap[0,\infty)$, *then* $\left(W_n^{\beta,\gamma}f\right)(x)=O\left(n^{-m}\right)$ *as* $n\to\infty$, *for arbitrarily large* $m>0$.

Proof. Let $m \in \mathbb{N}$. For a certain positive constant K, we have

$$\left(E_{1,\beta}^{\gamma}\left(nx\right)\right)\left|\left(W_n^{\beta,\gamma}f\right)\left(x\right)\right| \leq K \sum_{\substack{\nu\geq 0,\\ \left|\nu/n-x\right|\geq\delta}} \binom{\gamma+\nu-1}{\nu}\frac{1}{\Gamma\left(\nu+\beta\right)}\left(\frac{n^2x}{n-A}\right)^{\nu}$$
$$\leq K\delta^{-2m}\sum_{\nu=0}^{\infty}\binom{\gamma+\nu-1}{\nu}\frac{1}{\Gamma\left(\nu+\beta\right)}\left(\frac{n^2x}{n-A}\right)^{\nu}\left(\frac{\nu}{n}-x\right)^{2m}.$$

Application of the Schwarz inequality yields

$$\left|\left(W_n^{\beta,\gamma}f\right)\left(x\right)\right| \leq K\delta^{-2m}\sqrt{\left(W_n^{\beta,\gamma}\exp_{2A}\right)\left(x\right)}\sqrt{\left(W_n^{\beta,\gamma}\psi_x^{4m}\right)\left(x\right)}.$$

By Lemma 4.5, the first square root has the finite limit $\exp_A\left(x\right)$ as $n\to\infty$. By relation (3.13), the second root satisfies $\sqrt{\left(W_n^{\beta,\gamma}\psi_x^{4m}\right)\left(x\right)} = O\left(n^{-m}\right)$ as $n\to\infty$. This completes the proof. □

Now we are in position to prove the asymptotic expansion.

Proof of Theorem 2.1. Let $x>0$ and suppose that $f^{(2q)}\left(x\right)$ exists. Firstly, we assume that $f \in H^{(2q)}\left(x\right)$. By the Equation (3.13), we have $\left(W_n^{\beta,\gamma}\psi_x^{2s}\right)\left(x\right) = O\left(n^{-s}\right)$ as $n\to\infty$. Lemma 4.4 implies that

$$\left(W_n^{\beta,\gamma}f\right)\left(x\right) = \sum_{s=0}^{2q}\frac{f^{(s)}\left(x\right)}{s!}\left(W_n^{\beta,\gamma}\psi_x^{s}\right)\left(x\right) + o\left(n^{-q}\right) \qquad \left(n\to\infty\right).$$

Now, we suppose that $f\in E\left[0,\infty\right)$ and $f^{(2q)}\left(x\right)$ exists. Note that the above asymptotic expansion depends only on the behaviour of f in an arbitrary small neighborhood of x. Therefore, by the localization theorem, we can assume, without loss of generality, that $f\in H^{(2q)}\left(x\right)$. By Lemma 3.3 and noting that $W_n^{\beta,\gamma}e_0 = e_0$, we infer that

$$\sum_{s=0}^{2q}\frac{f^{(s)}\left(x\right)}{s!}\left(W_n^{\beta,\gamma}\psi_x^{s}\right)\left(x\right) \sim f\left(x\right) + \sum_{k=1}^{\infty}\frac{1}{n^k}\sum_{s=1}^{2k}\frac{f^{(s)}\left(x\right)}{s!}x^{s-k}T_{k,s}\left(\gamma,\beta\right)$$

as $n\to\infty$. Finally, we obtain

$$\left(W_n^{\beta,\gamma}f\right)\left(x\right) = f\left(x\right) + \sum_{k=1}^{q}\frac{1}{n^k}\sum_{s=1}^{2k}\frac{f^{(s)}\left(x\right)}{s!}x^{s-k}T_{k,s}\left(\gamma,\beta\right) + o\left(n^{-q}\right)$$

as $n\to\infty$. This is the desired expansion. □

Proof of Corollary 2.2. Let $x>0$ and suppose that $f^{(2q)}\left(x\right)$ exists. Since ${}_1F_1\left(a;a;z\right) = e^z$, Equation (3.10) reveals that $H_0\left(a,a,j\right)=1$ and $H_k\left(a,a,j\right)=0$ $\left(k\in\mathbb{N}\right)$. By (2.6), we obtain

$$T_{k,s}\left(\beta,\beta\right) = \sum_{r=k}^{s}\left(-1\right)^{s-r}\binom{s}{r}\binom{r}{k}r^{\underline{k}}$$

and Equation (2.5) yields

$$a_k^{\beta,\beta}(f,x) = \sum_{s=1}^{2k} \frac{f^{(s)}(x)}{s!} x^{s-k} \sum_{r=k}^{s} (-1)^{s-r} \binom{s}{r}\binom{r}{k} r^{\underline{k}}$$
$$= \sum_{s=1}^{2k} \binom{s}{k} \frac{f^{(s)}(x)}{s!} x^{s-k} \sum_{r=k}^{s} (-1)^{s-r} \binom{s-k}{r-k} r^{\underline{k}} \qquad (k = 1, 2, \ldots).$$

Observing that

$$\sum_{r=k}^{s} (-1)^{s-r} \binom{s-k}{r-k} r^{\underline{k}} = \sum_{r=0}^{s-k} (-1)^{s-k-r} \binom{s-k}{r} (r+k)^{\underline{k}}$$
$$= \sum_{r=0}^{s-k} (-1)^{s-k-r} \binom{s-k}{r} \left(\left(\frac{d}{dy}\right)^k y^{r+k} \right)\Bigg|_{y=1}$$
$$= \left(\left(\frac{d}{dy}\right)^k y^k (y-1)^{s-k} \right)\Bigg|_{y=1}$$
$$= \binom{k}{s-k} \frac{k!}{(s-k)!} (s-k)! = k! \binom{k}{s-k}$$

(note that $\binom{k}{s-k} = 0$ if $2 \leq 2k < s$), we conclude that

$$a_k^{\beta,\beta}(f,x) = \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{(s-k)!} \binom{k}{s-k} x^{s-k} = \sum_{s=0}^{k} \binom{k}{s} \frac{x^s}{s!} f^{(k+s)}(x).$$

This completes the proof of the corollary. □

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Research Article

Approximation with rational and rescaled kernels: New steps forward

Dedicated to Professor Ioan Raşa, on the occasion of his 75th birthday

STEFANO DE MARCHI* AND ELISABETH LARSSON

ABSTRACT. The aim of this paper is to present new insights into Rational Radial Basis Function (RRBF) approximations and their localized rescaled counterparts. After some necessary notation, we briefly recall the essence of stable computational techniques for overcoming the ill-conditioning of the kernel matrices, as well as the ones that explore stable bases. RRBFs are then introduced, and their benefits are described. We discuss Variably Scaled Kernels, which provide a more flexible tool for approximating with RBFs, substituting the shape parameter with a scaling function. This scaling function is then applied to the RRBFs to obtain a new family of RBFs that are more appropriate for discontinuous functions. Finally, the localized version of the RRBFs is recalled, for which we provide new insights into an open conjecture regarding the sum of the cardinal function associated with the approximant in rescaled form.

Keywords: Radial basis function (RBF), variably scaled kernel, rational approximation, rescaled localized RBF.

2020 Mathematics Subject Classification: 41A20, 65D12, 65D15.

1. INTRODUCTION

Radial Basis Functions or RBF are functions that depend only on the Euclidean distance from a center point, i.e., $\phi(\|x-c\|)$, where x is a point in a given multidimensional domain and c is the center. Common RBFs include Gaussian, multiquadric, inverse multiquadric, and polyharmonic splines. They can be globally supported, locally supported, with different smoothness properties. Among the fundamental references, we recall these books [22, 24, 64].

Suppose we wish to conduct a literature search on platforms such as Google Scholar or specialized academic databases (e.g., IEEE Xplore, SpringerLink, or Elsevier's ScienceDirect). In that case, we can identify several thousand papers on RBFs across these fields. For example, a Google Scholar search for Radial Basis Functions yields more than 10^6 papers, including applications in machine learning, numerical analysis, and applied mathematics.

We can approximate functions or data at scattered points in $\mathbb{R}^M, M \geq 1$ by the following setting: Let $\Omega \subseteq \mathbb{R}^M$ be a bounded set, let $\mathcal{X}_N = \{\boldsymbol{x}_i, i = 1, \ldots, N\} \subseteq \Omega$ be a set of distinct data points (also called data sites or nodes) and let $\mathcal{F}_N = \{f_i = f(\boldsymbol{x}_i), i = 1, \ldots, N\}$ be a set of data values (or measurements or function values). The approximation problem consists in finding a function $P_f : \Omega \longrightarrow \mathbb{R}$ such that $P_f(\boldsymbol{x}_i) \approx f_i$, $i = 1, \ldots, N$. When equality holds we talk about interpolation. To this end, we consider $P_f \in \text{span}\{\Phi(\cdot, \boldsymbol{x}_i), \boldsymbol{x}_i \in \mathcal{X}_N\}$, where $\Phi : \Omega \times \Omega \longrightarrow \mathbb{R}$ is a strictly positive definite and symmetric kernel, for which the associated interpolation matrix A with entries $A_{i,j} = \Phi(x_i, x_j)$ is strictly positive definite. The interpolant then assumes the

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form

$$P_f(\boldsymbol{x}) = \sum_{k=1}^{N} \alpha_k \Phi(\boldsymbol{x}, \boldsymbol{x}_k), \quad \boldsymbol{x} \in \Omega. \tag{1.1}$$

The coefficients $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)^T$ in (1.1) are found by solving the linear system $A\boldsymbol{\alpha} = \boldsymbol{f}$, with $\boldsymbol{f} = (f_1, \ldots, f_N)^T$, where A being strictly positive definite and symmetric ensures the uniqueness of the solution.

Definition 1.1. *A function $\sigma : \mathbb{R}^M \to \mathbb{R}$ is called radial, if there exists a continuous function $\phi : [0, \infty) \to \mathbb{R}$ such that $\sigma(\boldsymbol{x}) = \phi(r)$ with $r = \|\boldsymbol{x}\|_2$.*

Using this definition, given a basic function ϕ and the (symmetric positive definite) kernel Φ, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^M$ we have a radial kernel just by considering $\Phi(\boldsymbol{x}, \boldsymbol{y}) = \phi(\|\boldsymbol{x} - \boldsymbol{y}\|_2)$. We can then indifferently use Φ or ϕ by referring to Definition 1.1. An example of a RBF is the well-known Gaussian function $\phi(r) = \mathrm{e}^{-\varepsilon^2 r^2}$, where ε is the shape parameter, which is indeed a scale parameter.

Many papers focus on using RBFs for the numerical solution of PDEs within a meshless framework. These methods include RBF collocation (cf. e.g. [9, 34, 41, 42]), RBF-generated finite differences (RBF-FD) (cf. e.g. [2, 25, 58, 59, 60]), and RBF partition of unity methods (RBF-PUM) (cf. e.g. [17, 38, 46]) with applications to elliptic, parabolic, and hyperbolic PDEs (especially in fluid dynamics, electromagnetics, and material science) (cf. e.g. [3, 5, 61, 63, 69]), option problems in the financial market [46, 57]. A significant portion of the research focuses on the use of RBFs for function interpolation and data fitting, which is fundamental to scattered data approximation. In machine learning, RBFs are used as kernels in algorithms like Support Vector Machines (SVMs) and in neural networks (Radial Basis Function Networks - RBFNs) (cf. e.g. [31, 43, 56]). These applications contribute to the large number of papers on RBFs in pattern recognition, classification, and regression. A large body of theoretical work deals with the analysis of convergence rates, error estimates, and the properties of different types of RBFs (e.g., Gaussian, multiquadric, inverse multiquadric) (see e.g. [10, 11, 12, 29, 39, 45, 49, 50, 51, 52, 54, 55, 65, 66, 67, 68]).

It is well-known that rational approximation (by using the ratio of polynomials) is more robust than the standard polynomial interpolation, in particular for functions characterized by steep gradients. However, this is a mesh-dependent approach and, as a consequence, extending polynomial approximation to higher dimensions is always quite hard (see, e.g., [32]). Moreover, there are many challenging issues associated with rational approximation, such as the computational cost (the number of coefficients to be determined is large) and the need to avoid singularities in the denominator (in higher dimensions, these are curves or manifolds).

In this paper, we would like to deepen the rescaled localized RBF approach, firstly used in [19] for the solution of PDEs with a meshless approach with compactly supported radial functions (see also [15, 16]), which represents a particular instance of rational RBFs (cf. [18, 33]).

The paper is organized as follows. In Section 2, we present preliminaries on stability and error analysis for RBFs. In Section 3, we recall some stable computational techniques, introduce Rational RBF, Variably Scaled Kernels and show that mixing them we can get a more stable approximation. In Section 4, we discussed the so-called Rescaled Localized RBF and presented new results on an open problem concerning the sum of the cardinals in this framework. We make some conclusions in Section 5.

2. Preliminaries on Stability and Error Analysis

It is well known that the ℓ_2 conditioning of a positive definite matrix K can be expressed

$$\text{cond}(\mathsf{K}) = \frac{\lambda_{\max}}{\lambda_{\min}}, \tag{2.2}$$

where $\lambda_{\max}, \lambda_{\min}$ are the largest and smallest eigenvalues of K respectively.

Remark 2.1. *The definition of ℓ_2 conditioning in (2.2) is valid as long as we deal with a square normal matrix K, i.e., it commutes with its conjugate transpose K^{H}. Such a definition is a particular case of the following formulation*

$$cond(\mathsf{A}) = \frac{\sigma_{\max}}{\sigma_{\min}},$$

which is true for any square matrix A. Here, $\sigma_{\max},\ \sigma_{\min}$ are the largest and smallest singular values of A respectively.

Concerning the largest eigenvalue, if K is the $N \times N$ positive definite kernel matrix related to a translational invariant kernel $\Phi(\boldsymbol{x}, \boldsymbol{y}) = \phi(\boldsymbol{x} - \boldsymbol{y})$, an immediate consequence of Gershgorin's theorem is that

$$\lambda_{\max} \le N\phi(\mathbf{0}).$$

For error analysis, we sometimes need the approximant to be expressed in terms of cardinal functions (cf. [22, Section 14.2]).

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^d$ and let Φ be a strictly positive definite kernel. Then, for every set $X = \{\boldsymbol{x}_k,\ k = 1, \ldots, N\} \subset \Omega$ of distinct node points there exist functions $u_k^* \in span\{\Phi(\cdot, \boldsymbol{x}_k),\ k = 1, \ldots, N\}$ such that $u_i^*(\boldsymbol{x}_j) = \delta_{ij}$, where δ_{ij} is the well-known Kronecker delta.*

By means of the so-defined functions u_k^*, $k = 1, \ldots, N$, also called cardinal functions, we can write the interpolant in (1.1) of a given function $f : \Omega \longrightarrow \mathbb{R}$ in its Lagrangian form [71], that is,

$$P_f(\boldsymbol{x}) = \sum_{k=1}^{N} f(\boldsymbol{x}_k)\, u_k^*(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{2.3}$$

Let $\Phi \in \mathcal{C}(\Omega \times \Omega)$. For any set of distinct nodes X_N and vector $\boldsymbol{u} \in \mathbb{R}^N$ we can consider the quadratic form

$$Q(\boldsymbol{u}) := \Phi(\boldsymbol{x}, \boldsymbol{x}) - 2\sum_{k=1}^{N} u_k \Phi(\boldsymbol{x}, \boldsymbol{x}_k) + \sum_{i=1}^{N}\sum_{j=1}^{N} u_i u_j \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j).$$

Then, we give the following definition [22, Section 14.3].

Definition 2.2. *Let $\Omega \subseteq \mathbb{R}^d$ and let $\Phi \in C(\Omega \times \Omega)$ be a strictly positive definite kernel. Then, for any set of distinct points $X = \{\boldsymbol{x}_k,\ k = 1, \ldots, N\} \subseteq \Omega$ we define the power function*

$$[P_{X,\Phi}(\boldsymbol{x})]^2 := Q(\boldsymbol{u}^*(\boldsymbol{x})),$$

where $\boldsymbol{u}^*$ is the vector of cardinal functions (see Theorem 2.1).

The power function can also be computed as

$$P_{X,\Phi}(\boldsymbol{x}) = \sqrt{\Phi(\boldsymbol{x}, \boldsymbol{x}) - (\boldsymbol{b}(\boldsymbol{x}))^{\mathsf{T}} \mathsf{K}^{-1} \boldsymbol{b}(\boldsymbol{x})},$$

where $\boldsymbol{b} = (\Phi(\cdot, \boldsymbol{x}_1), \ldots, \Phi(\cdot, \boldsymbol{x}_N))^\intercal$ and K is the kernel matrix defined previously. Moreover, letting $\mathsf{K}^{\boldsymbol{y}}$ be the kernel matrix related to the augmented dataset $X \cup \{\boldsymbol{y}\}$, we can also express the power function as (see [13])

$$P_{X,\Phi}(\boldsymbol{y}) = \sqrt{\frac{\det \mathsf{K}^{\boldsymbol{y}}}{\det \mathsf{K}}}. \tag{2.4}$$

Remark 2.2. *The power function vanishes if it is evaluated at the interpolation nodes, as emphasized by the formulation in* (2.4).

Thanks to the power function definition, we can now provide the following error bound for the kernel-based interpolant in (2.3) [22, Section 14.4].

Theorem 2.2. *Let $\Omega \subseteq \mathbb{R}^d$ and let $\Phi \in \mathcal{C}(\Omega \times \Omega)$ be a strictly positive definite kernel. Moreover, let $X = \{\boldsymbol{x}_k, k = 1, \ldots, N\} \subset \Omega$ be a set of distinct nodes. Then*

$$|f(\boldsymbol{x}) - P_f(\boldsymbol{x})| \leq P_{X,\Phi}(\boldsymbol{x}) \, \|f\|_{\mathcal{N}_\Phi(\Omega)}, \quad \boldsymbol{x} \in \Omega,$$

where $f \in \mathcal{N}_\Phi(\Omega)$.

Despite the dependency on the power function, and thus on the chosen set of nodes, we observe that Theorem 2.2 provides a pointwise error bound that does not give any indication concerning the mesh size, a term used in abuse of notation in the scattered data interpolation context. To obtain further details in this direction, we introduce the so-called separation distance and the fill distance [22, Section 14.1].

Definition 2.3. *The separation distance is defined as*

$$q_X := \frac{1}{2} \min_{i \neq j} \|\boldsymbol{x}_i - \boldsymbol{x}_j\|.$$

Definition 2.4. *The fill distance is given by*

$$h_{X,\Omega} := \sup_{\boldsymbol{x} \in \Omega} \min_{\boldsymbol{x}_k \in X} \|\boldsymbol{x} - \boldsymbol{x}_k\|.$$

Remark 2.3. *Geometrically, the quantity q_X represents the radius of the largest ball that can be centered at every point in X so that there is no overlap. At the same time, $h_{X,\Omega}$ is the radius of the largest empty ball that can be placed among the data sites in Ω.*

3. Stable Computation Techniques

One of the most investigated topics is the behaviour of the kernel interpolant in the *flat* limit, i.e., when $\varepsilon \longrightarrow 0$ (cf. e.g. [20, 36, 39, 53]) and associated methods for stable evaluation of the kernel interpolant in the limit (cf. e.g. [26, 70]). In agreement with the trade-off principles [50], we remark the following:

- In some cases, the approximants may suffer from instability because of the ill-conditioning of the interpolation matrices as the shape parameter becomes small, corresponding to the so-called flat RBFs [26]. This problem depends on both the smoothness order of the basis function and the node distribution. More specifically, if one keeps the number of nodes fixed and considers smooth basis functions, then the problem of instability becomes evident for small values of the shape parameter.
- A radial basis function with a finite order of smoothness can be used to improve the conditioning, but the accuracy of the fit gets worse. For this reason, recent research has moved towards the study of stable bases (see e.g. [14, 23, 27, 28]).

Among the most popular methods for stabilizing the interpolant, many belong to the following categories.

(1) **RBF-QR Method**. It is rooted in a particular decomposition of the kernel, and it has been developed so far to treat the Gaussian kernel [23, 27, 28, 35].
(2) **Hilbert-Schmidt Singular Value Decomposition (HS-SVD)**. It has been developed to stably compute the RBF interpolants [8, 24]. In principle, this technique can be applied to any kernel, provided that the HS eigenvalues and eigenvectors are known. However, these quantities are far from being easy to compute, and in practice, they only work for the Gaussian function.
(3) **WSVD Bases**. It is a more general approach that applies to any RBF, consisting of computing a weighted SVD decomposition which produces stable bases [14].

3.1. **Rational Radial Basis Functions (RRBFs).** Aiming to modify the kernel basis to improve computational stability, we here present a recent research topic which develop rational RBF approximation. We start by recalling the main theoretical and computational aspects studied in [6]. The method reported here is the so-called eigen-rational kernel-based scheme. It consists of a fractional RBF expansion, with the denominator depending on the eigenvector associated to the largest eigenvalue of the kernel matrix.

We have already seen that the spaces spanned by kernels provide beneficial properties for multivariate approximation. Nevertheless, it might be advantageous to study approximants in non-linear spaces generated by RBFs. A few examples of these approaches already exist in the literature and are known as rational RBFs, introduced in [33] and further developed in [16, 47]. The main disadvantage of the scheme proposed in these papers is that the rational basis is constructed from function values, and consequently, it depends not only on the data sites. A solution comes from the eigen-rational basis.

We recall that rational real functions are expressed as ratios of polynomials. That is, at each point $x \in \mathbb{R}$

$$R(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots b_1 x + b_0}\,.$$

The collection $\{a_n, a_{n-1}, \ldots, a_0; b_m, b_{m-1}, \ldots, b_0\}$ is the set of the coefficients of R, that can be computed imposing some interpolation conditions. It is also usually assumed that the polynomials P_n and Q_m are relatively prime, that is, common factors are canceled.

Rational RBFs (RRBFs) are an extension of classical radial basis functions (RBFs) that, starting from two RBF approximations, say R_1, R_2 expressed as in (1.1), can be similarly expressed as the ratio of them. It has proven to be an effective tool for solving both interpolation problems and partial differential equations (PDEs) (cf. e.g. [6, 16]).

Fix the integers m, n, k and j so that $m, n \le N$, $1 \le k \le N + m - 1$ and $1 \le j \le N + n - 1$. Letting $\mathcal{X}_m = \{\boldsymbol{x}_i, i = k, \ldots, k+m-1\}$ and $\mathcal{X}_n = \{\boldsymbol{x}_i, i = j, \ldots, j+n-1\}$ two non-empty subsets of $\mathcal{X}_N \subseteq \Omega(\subseteq \mathbb{R}^M)$, a natural extension of the classical RBF approximation to the rational case, is then

$$\mathscr{R}(\boldsymbol{x}) = \frac{R^{(1)}(\boldsymbol{x})}{R^{(2)}(\boldsymbol{x})} = \frac{\sum_{i_1=k}^{k+m-1} \alpha_{i_1}\Phi(\boldsymbol{x}, \boldsymbol{x}_{i_1})}{\sum_{i_2=j}^{j+n-1} \beta_{i_2}\Phi(\boldsymbol{x}, \boldsymbol{x}_{i_2})}, \tag{3.5}$$

provided $R^{(2)}(\boldsymbol{x}) \neq 0$, $\boldsymbol{x} \in \Omega$. Details concerning the well-posedness and the solution of the interpolation problem, as well as the error analysis, can be found in [16, 33]. In particular, the well-posedness as discussed in [16, §3.2] for $n = m = N$, is equivalent to finding the eigenvector, say $\mathbf{q}$, associated with the smallest eigenvalue in modulus of a generalized eigenvalue problem:

$$\Lambda \boldsymbol{q} = \lambda \Theta \boldsymbol{q},$$

with

$$\Lambda = \frac{1}{||\boldsymbol{f}||_2^2} D^T A^{-1} D + A^{-1}, \quad \text{and} \quad \Theta = \frac{1}{||\boldsymbol{f}||_2^2} D^T D + I_N,$$

where I_N is the $N \times N$ identity matrix, A is the kernel matrix defined above and $D = \texttt{diag}(f_1, \ldots, f_n)$. Hence, we have to simply define $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ as standard RBF interpolants of $\boldsymbol{p}$ and $\boldsymbol{q}$, respectively. Some known benefits of using Rational RBFs:

- **Improved Stability**. Rational RBFs are significantly more accurate than standard RBFs for functions with steep gradients or discontinuous behavior, both in the neighborhood of the discontinuity and away from it and for solving PDEs with such solutions (cf. [47]). Moreover, when using flat, infinitely smooth kernels (like Gaussians), which are desirable for high accuracy but cause severe ill-conditioning, rational RBF methods can be implemented in a stable manner (e.g., using Variably Scaled Kernels or RBF-QR), preventing the system matrix from becoming overly unstable (cf.[16, 37]).
- **Multiscale Behavior**. Rational RBFs can handle multiscale phenomena, which are common in many PDE problems (e.g., fluid dynamics, wave propagation) (cf. e.g. [62]).
- **Boundary Conditions**. Rational RBFs can better handle boundary conditions in PDEs by providing more accurate approximations near the boundaries of the domain (cf. [21]).

3.2. **Variably Scaled Kernels.** This subsection briefly describes an interesting approach to the shape parameter issues, called Variably Scaled Kernels, shortly VSK. We recall the definition as in [4, Definition 2.1].

Definition 3.5. *Let $\Phi : \mathbb{R}^{(d+1)\times(d+1)} \longrightarrow \mathbb{R}$ be a continuous strictly positive definite radial kernel and let $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a scaling function. A variably scaled kernel Φ^Ψ on $\mathbb{R}^{d\times d}$ is defined as*

$$\Phi^\Psi(\boldsymbol{x}, \boldsymbol{y}) = \Phi((\boldsymbol{x}, \psi(\boldsymbol{x})), (\boldsymbol{y}, \psi(\boldsymbol{y}))), \tag{3.6}$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$.

We note that the shape parameter of the underlying kernel Φ in Definition 3.5 is chosen equal to 1. We do not include this information in the notation for simplicity and to avoid overly complicated notation.

Remark 3.4. *Often we consider radial kernels: the VSK for radial kernels becomes*

$$\Phi^\Psi(\boldsymbol{x}, \boldsymbol{y}) = \phi\left(\sqrt{\|\boldsymbol{x} - \boldsymbol{y}\|^2 + |\psi(\boldsymbol{x}) - \psi(\boldsymbol{y})|^2}\right),$$

where ϕ is the usual univariate function related to Φ. In other words, if Φ is radial on $\mathbb{R}^{(d+1)\times(d+1)}$ so is Φ^Ψ on $\mathbb{R}^{d\times d}$. Moreover, if Φ is (strictly) positive definite, so is Φ^Ψ.

By considering the map $\Psi(\boldsymbol{x}) = (\boldsymbol{x}, \psi(\boldsymbol{x}))$ on Ω and in analogy with (1.1), we can express the VSK interpolant at the nodes $\tilde{X} = \{(\boldsymbol{x}_k, \psi(\boldsymbol{x}_k)),\ \boldsymbol{x}_k \in X\}$ as

$$R(\Psi(\boldsymbol{x})) = \sum_{k=1}^{N} c_k \Phi(\Psi(\boldsymbol{x}), \Psi(\boldsymbol{x}_k)), \tag{3.7}$$

with $\boldsymbol{x} \in \Omega,\ \boldsymbol{x}_k \in X$. Therefore, in order to obtain a VSK interpolant R^{Ψ} at the set of nodes X on Ω, it is sufficient to project back the interpolant in (3.7), that is,

$$R^{\Psi}(\boldsymbol{x}) = \sum_{k=1}^{N} c_k \Phi^{\Psi}(\boldsymbol{x}, \boldsymbol{x}_k) = \sum_{k=1}^{N} c_k \Phi(\Psi(\boldsymbol{x}), \Psi(\boldsymbol{x}_k)) = R(\Psi(\boldsymbol{x})).$$

An important consequence of this construction is that the analysis of the variably scaled setting is fully understood in terms of the analysis of the underlying standard kernel (which is composed with Ψ). The following proposition in [4] states some additional properties of VSKs, which are fundamental to understanding why VSKs improve the stability of the approximation process.

Proposition 3.1. *Let Φ, Φ^{Ψ}, and ψ be as in Definition 3.5. We have*

(i) If Φ and ψ are continuous, so is Φ^{Ψ}. Moreover, if $\psi : \Omega \to \psi(\Omega)$ is a bijection, then Φ^{ψ} inherits the positiveness properties of Φ.

(ii) Let q_X be the separation distance between the centers (see Definition 2.3 above), we have

$$q_X \le q_{\Psi(X)}$$

for any choice of the scaling map ψ. Indeed, by the definition of the Euclidean norm, we have

$$\|\Psi(\boldsymbol{x}_i) - \Psi(\boldsymbol{x}_j)\|^2 = \|\boldsymbol{x}_i^2 - \boldsymbol{x}_j^2\|^2 + (\psi(\boldsymbol{x}_i) - \psi(\boldsymbol{x}_j))^2 \le \|\boldsymbol{x}_i^2 - \boldsymbol{x}_j^2\|^2(1+L)^2,$$

where L is the Lipschitz constant of ψ.

(iii) Let $G_{\Psi}(\Omega) = \{(\boldsymbol{x}, \psi(\boldsymbol{x})) \mid \boldsymbol{x} \in \Omega\} \subset \Omega \times \mathbb{R}$ be the graph of the scaling function ψ. Then, the native spaces $\mathcal{N}_{\Phi}(G_{\Psi}(\Omega))$ and $\mathcal{N}_{\Phi^{\Psi}}(\Omega)$ are isometrically isomorphic (cf. [4, Theorem 2]*).*

Remark 3.5. *As a consequence of Proposition 3.1 (ii), the variably scaled setting might improve the stability of the interpolation process by increasing the separation distance.*

Remark 3.6. *The interpolation via Variably Scaled Kernels (VSKs) depends on the definition of an appropriate scaling function, but no fixed theoretical or numerical recipes for its construction have been provided. Recently, in* [1]*, a user-independent tool for learning the scaling function using Discontinuous Neural Networks (called δ-NN) was presented, which partially addresses this gap.*

3.3. **Rational Kernel-Based Approximation.** In [6], an eigen-rational kernel-based scheme was proposed for multivariate interpolation using meshfree methods. Let us suppose that $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a conditionally positive definite kernel and $\bar{\Phi}$ its associate positive definite one (for example, Φ can be a generalized multiquadrics of order 2, $\bar{\Phi}$ is the inverse multiquadrics). We note that if Φ is strictly positive definite, we set $\hat{\Phi} = \Phi$ so that we use the same kernel matrix for both the numerator and the denominator (for example, consider Φ as the Matérn kernel). We can define the interpolant of a function f

$$\hat{P}_f(\boldsymbol{x}) = \frac{\sum_{i=1}^{N} \alpha_i \Phi(\boldsymbol{x}, \boldsymbol{x_i}) + \sum_{m=1}^{L} \gamma_m p_m(\boldsymbol{x})}{\sum_{k=1}^{N} \beta_k \bar{\Phi}(\boldsymbol{x}, \boldsymbol{x_k})} = \frac{P_g(\boldsymbol{x})}{P_h(\boldsymbol{x})}, \tag{3.8}$$

defined for some function values $g_i, h_i,\ i = 1, \dots, N$. Roughly speaking, once we provide the function values h_i, we can construct P_g in the standard way, i.e., such that it interpolates $\boldsymbol{g} = (f_1 h_1, \dots, f_N h_N)^{\intercal}$. Then, obviously, $\hat{P}_f$ interpolates the given function values $\mathcal{F}_N$ at the nodes $\mathcal{X}_N$. This definition, for the one given above in (3.5), for constructing P_g, adapts the more general case of conditionally positive definite kernels, still restricting to the case of strictly positive kernels for P_h. This enables us to make the eigen-rational interpolant well-defined, i.e. such that $P_h(\boldsymbol{x}) \neq 0$, for all $\boldsymbol{x} \in \Omega$. We recall that a similar approach for the class of polyharmonic kernels considered both in the numerator and denominator of (3.8), has been investigated in [21].

Proposition 3.2. *The method consists of a fractional RBF expansion, with the denominator depending on the largest eigenvalue of the kernel matrix.*

Proof. We need to define $\hat{P}_f$ by using the cardinal functions, $\hat{u}_j$ (which form a partition of unity) (cf. [6, Theorem 2.3]). If the kernel $\bar{K}$ is strictly positive definite, the same argument holds for functions $\bar{u}_k \in \text{span}\{\bar{K}(\cdot, \boldsymbol{x}_j), \;\; j = 1, \ldots, N\}$. Therefore,

$$P_g\left(\boldsymbol{x}\right) = \sum_{k=1}^{N} g\left(\boldsymbol{x}_k\right) u_k\left(\boldsymbol{x}\right), \quad P_h\left(\boldsymbol{x}\right) = \sum_{k=1}^{N} h\left(\boldsymbol{x}_k\right) \bar{u}_k\left(\boldsymbol{x}\right), \quad \boldsymbol{x} \in \Omega.$$

Thus, the resulting eigen-rational interpolant in cardinal form is given by

$$\begin{aligned} \hat{P}_f(\boldsymbol{x}) = & \frac{\sum_{j=1}^{N} g_j u_j(\boldsymbol{x})}{\sum_{k=1}^{N} h_k \bar{u}_k(\boldsymbol{x})} = \frac{\sum_{j=1}^{N} h_j f_j u_j(\boldsymbol{x})}{\sum_{k=1}^{N} h_k \bar{u}_k(\boldsymbol{x})} \\ = & \sum_{j=1}^{N} f_j \frac{h_j u_j(\boldsymbol{x})}{\sum_{k=1}^{N} h_k \bar{u}_k(\boldsymbol{x})} =: \sum_{j=1}^{N} f_j \hat{u}_j(\boldsymbol{x}), \end{aligned}$$

and, furthermore, $\hat{u}_i\left(\boldsymbol{x}_i\right) = \delta_{ik}$, $\boldsymbol{x}_i \in \mathcal{X}_N$. Moreover, if $K = \bar{K}$ is strictly positive definite, $\{\hat{u}_j\}_{j=1}^{N}$ form a partition of unity. Indeed for $\boldsymbol{x} \in \Omega$

$$\sum_{j=1}^{N} \hat{u}_j(\boldsymbol{x}) = \sum_{j=1}^{N} h_j \frac{\bar{u}_j(\boldsymbol{x})}{\sum_{k=1}^{N} h_k \bar{u}_k(\boldsymbol{x})} = \frac{\sum_{j=1}^{N} h_j \bar{u}_j(\boldsymbol{x})}{\sum_{k=1}^{N} h_k \bar{u}_k(\boldsymbol{x})} = 1.$$

Considering

$$\sum_{j=1}^{N} (\hat{u}_j(\boldsymbol{x}))^2 = \sum_{j=1}^{N} \left(\frac{h_j u_j(\boldsymbol{x})}{\sum_{k=1}^{N} h_k \bar{u}_k(\boldsymbol{x})} \right)^2 \leq \frac{||\boldsymbol{h}^2||_\infty}{P_h^2(\boldsymbol{x})} \sum_{j=1}^{N} (u_j(\boldsymbol{x}))^2,$$

and from [64, Theorem 12.1, p. 208], we know that

$$1 + \sum_{j=1}^{N} (u_j(\boldsymbol{x}))^2 \leq \frac{\mathcal{P}^2_{K, \mathcal{X}_N}(\boldsymbol{x})}{\omega}, \tag{3.9}$$

where ω is the smallest eigenvalue of the kernel matrix constructed on the node set $\mathcal{X}_N \cup \{\boldsymbol{x}\}$ with $\omega > 0$. This concludes the proof. □

This also provides an upper bound for the Lebesgue function $\sum_{j=1}^{N} |\hat{u}_j|$. Classical bounds in terms of Lebesgue constants and convergence rates with respect to the mesh size of the eigen-rational interpolant showed to be comparable with the ones of the classical kernel-based methods, as discussed in [6].

3.4. **Variably Scaled Rational Kernels (VSRK).** Both strategies, Rational RBF plus VSK, can be combined to form a new family of kernels, which we denote as Variably Scaled Rational Kernels (VSRK). The idea is to take advantage of the flexibility of the VSK and their stability properties, which, combined with those of the rational kernels, can yield even better approximation results. At the moment, we have not developed a complete analysis of these claims (it will appear in a forthcoming paper). In fact, the variably scaled kernel injects strictly positive spectral mass, preventing eigenvalues from collapsing to zero, which directly improves conditioning and therefore stability. Let $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$, $K_r : X \times X \to \mathbb{R}$ be a rational positive definite kernel, and $K_s : X \times X \to \mathbb{R}$ be a variably scaled positive definite kernel. Define the combined kernel:

$$K_\gamma(x,y) := K_r(x,y) + \gamma K_s(x,y), \quad \gamma > 0.$$

Theorem 3.3. *Let $\mathbf{K}_r, \mathbf{K}_s, \mathbf{K}_\gamma \in \mathbb{R}^{N\times N}$ be the corresponding kernel matrices. If $\mathbf{K}_s$ is strictly positive definite, then $\mathbf{K}_\gamma$ is strictly positive definite, and we have*

$$\lambda_{\min}(\mathbf{K}_\lambda) \geq \lambda_{\min}(\mathbf{K}_r) + \gamma\lambda_{\min}(\mathbf{K}_s), \tag{3.10}$$

$$\kappa(\mathbf{K}_\gamma) \leq \frac{\lambda_{\max}(\mathbf{K}_r) + \gamma\lambda_{\max}(\mathbf{K}_s)}{\lambda_{\min}(\mathbf{K}_r) + \gamma\lambda_{\min}(\mathbf{K}_s)}. \tag{3.11}$$

Proof. The proof goes as follows:

(1) **Positive Definiteness.** Since K_r and K_s are positive definite kernels, their matrices satisfy: $\mathbf{K}_r \succeq 0, \quad \mathbf{K}_s \succ 0$. For any nonzero vector $c \in \mathbb{R}^N$, $c^T\mathbf{K}_\gamma c = c^T\mathbf{K}_r c + \gamma c^T\mathbf{K}_s c$. Because $c^T\mathbf{K}_r c \geq 0$ and $c^T\mathbf{K}_s c > 0$, it follows that $c^T\mathbf{K}_\gamma c > 0$, hence $\mathbf{K}_\gamma \succ 0$.

(2) **Lower Bound on Eigenvalues.** By Weyl's inequality for symmetric matrices, that is $\lambda_{\min}(\mathbf{A}+\mathbf{B}) \geq \lambda_{\min}(\mathbf{A}) + \lambda_{\min}(\mathbf{B})$, and applying this to $\mathbf{K}_r$ and $\gamma\mathbf{K}_s$, we get formula (3.10)

$$\lambda_{\min}(\mathbf{K}_\gamma) \geq \lambda_{\min}(\mathbf{K}_r) + \gamma\lambda_{\min}(\mathbf{K}_s).$$

(3) **Upper Bound on Eigenvalues.** Similarly,

$$\lambda_{\max}(\mathbf{K}_\gamma) \leq \lambda_{\max}(\mathbf{K}_r) + \gamma\lambda_{\max}(\mathbf{K}_s).$$

(4) **Condition Number Bound.** By definition, $\kappa(\mathbf{K}_\gamma) = \frac{\lambda_{\max}(\mathbf{K}_\lambda)}{\lambda_{\min}(\mathbf{K}_\lambda)}$. Hence, by using the bounds above,

$$\kappa(\mathbf{K}_\gamma) \leq \frac{\lambda_{\max}(\mathbf{K}_r) + \gamma\lambda_{\max}(\mathbf{K}_s)}{\lambda_{\min}(\mathbf{K}_r) + \gamma\lambda_{\min}(\mathbf{K}_s)}.$$

(5) **Stability Improvement.** If $\lambda_{\min}(\mathbf{K}_r)$ is small (ill-conditioning), then the denominator is increased by $\gamma\lambda_{\min}(\mathbf{K}_s) > 0$, leading to a reduced condition number and improved numerical stability.

This concludes the proof. □

Remark 3.7. *If, instead, one defines*

$$K(x,y) = K_r(x,y)K_s(x,y), \tag{3.12}$$

then the corresponding kernel matrix is the Hadamard product $\mathbf{K} = \mathbf{K}_r \circ \mathbf{K}_s$. By the Schur product theorem, $\mathbf{K}$ is positive definite. Moreover, under mild assumptions,

$$\lambda_{\min}(\mathbf{K}) \geq \min_i(\mathbf{K}_s)_{ii}\,\lambda_{\min}(\mathbf{K}_r)$$

showing that K_s acts as a local preconditioner improving stability.

Here, we provide a couple of examples that demonstrate the effectiveness of the new approach. In both examples, we consider the centers to be a set X_c of $N = 81$ Halton points, and the evaluation points to be the set X_e of $M = 100$ equally spaced points. Moreover, we consider $\hat{\Phi} = \Phi$, in particular we take the C^6 Matérn kernel. The Rational+VSK interpolant is obtained by evaluating the rational interpolant $\hat{P}_f$ whose coefficients are constructed by solving linear systems with matrices constructed at the sets $\psi(X_c)$ and $\psi(X_e)$.

Here, we outline the algorithm for constructing the VSRK interpolant.

(1) Given the function Φ, ψ and f the sets of the centers, C and the evaluation points, E. Consider $C_\psi = (C, \psi(C)), E_\psi = (E, \psi(E))$. Let $B1$ and $B2$ be the corresponding kernel matrices
(2) Find, for instance with the power method, the eigenvector $\tilde{x}$ corresponding to maximum eigenvalue of the matrix $B2$, and set $a = B\tilde{x}$
(3) Let $p = f\ \tilde{x}$, where $f_i = f(\boldsymbol{c}_i)$ with $\boldsymbol{c}_i$ the $i-th$ center
(4) Find the coefficients $\boldsymbol{\alpha}$ of the interpolant by solving the linear system $B1\,\boldsymbol{\alpha} = p$
(5) Hence, the VSRK interpolant is

$$\hat{P}_f^\psi = \frac{E\boldsymbol{\alpha}}{E\hat{x}}$$

with E the kernel matrix at the evaluation points mapped by ψ.

Example 3.1. *The first numerical example consists of the function*

$$f(x) = \begin{cases} \sin(x) + \cos(x), & x \geq \pi/4 \\ \sqrt{2}, & 0 \leq x < \pi/4 \end{cases} . \tag{3.13}$$

This is a continuous function that we approximate using a C^6 Matérn rational VSK kernel with $\psi(x) = 2\log(x)/\pi$. The function ψ and the resulting plot of f and the comparison with the rational approximant $\hat{P}_f$ and $\hat{P}_f$+VSK are displayed in Fig. 1.

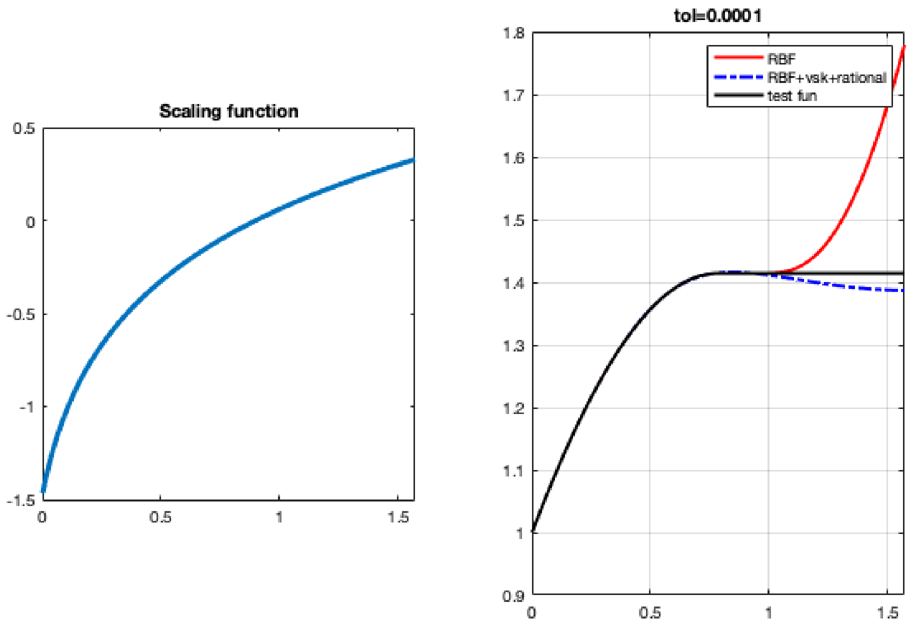

FIGURE 1. Left: The scaling function. Right: The function (3.13), the approximation with classical rational and the rational+VSK

Example 3.2. *The second numerical example consists of a function*

$$f(x) = \begin{cases} 0.4 + x^8/\tan(1+x^2) + 0.5, & x > 1/2 \\ \sin(x), & 0 \leq x \leq 1/2 \end{cases} . \tag{3.14}$$

This is a discontinuous function that we approximate using a Matérn rational discontinuous VSK kernel with

$$\psi(x) = \begin{cases} 1, & x > 1/2 \\ -1, & 0 \leq x \leq 1/2 \end{cases} .$$

The results are displayed in Fig. 2.

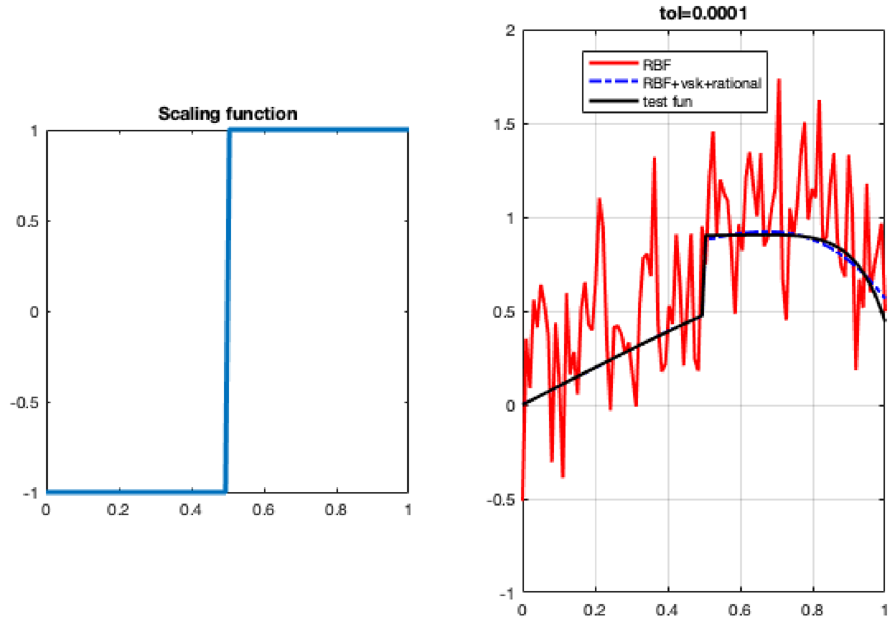

FIGURE 2. Left: The scaling function. Right: The function (3.14), the approximation with classical rational and the rational+VSK

4. RESCALED LOCALIZED RBFS

The idea of the RL-RBF, introduced in [19], is that of computing the RBF interpolant of a function f, based on a compactly supported ϕ, that we call P_f, and the interpolant of the constant function $g \equiv 1$, say P_1, and form their quotient. The new approximation $\hat{P}_f$ is then

$$\hat{P}_f := \frac{P_f}{P_1}. \tag{4.15}$$

The Rescaled Localized Radial Basis Function (RL-RBF) interpolant is then a simple rational approximation method (see e.g. [15, 18]). The construction simply goes as follows.

Let $\mathcal{X}_N = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset \Omega \subset \mathbb{R}^d$ and consider two functions $f, g : \mathbb{R}^d \to \mathbb{R}$, s.t. $g(\mathbb{R}^d) = 1$ and consider the function

$$\hat{\mathcal{P}}_f(\mathcal{X}_N; \boldsymbol{x}) = \frac{\mathcal{P}_f(\mathcal{X}_N; \boldsymbol{x})}{\mathcal{P}_g(\mathcal{X}_N; \boldsymbol{x})}$$

where $\mathcal{P}_{\cdot}(\mathcal{X}_N; \boldsymbol{x})$ is a standard, generic RBF interpolant. We notice that $\hat{\mathcal{P}}_f(\mathcal{X}_N; \boldsymbol{x})$ is still an interpolant to f, since

$$\hat{\mathcal{P}}_f(\mathcal{X}_N; \mathcal{X}_N) = \frac{\mathcal{P}_f(\mathcal{X}_N; \mathcal{X}_N)}{\mathcal{P}_g(\mathcal{X}_N; \mathcal{X}_N)} = \frac{f(\mathcal{X}_N)}{g(\mathcal{X}_N)} = f(\mathcal{X}_N).$$

Heuristically, it has been showed in [15, 19] that

- $\hat{\mathcal{P}}_f$ is powerful when compactly supported RBF are chosen.
- If ϵ is the shape parameter, there exists $\hat{\epsilon}$ and $(\hat{\epsilon} - \delta_1, \hat{\epsilon} + \delta_2)$ s.t. if ϵ stands in the left side, $\hat{\mathcal{P}}_f$ behaves better than $\mathcal{P}_f$, while on the right side it shows the opposite.
- $\hat{\mathcal{P}}_f$ is, in general, much less sensitive to the setting of ϵ than $\mathcal{P}_f$.
- $\hat{\mathcal{P}}_f$ can be not defined if $\mathcal{X}_N$ doesn't have a sufficiently homogeneous density in Ω.

About the associated native space, we recall the classical result from functional analysis.

Theorem 4.4 (Aronszajn). *Let $K : \Omega \times \Omega \to \mathbb{R}$ be a (strictly) positive definite kernel. Let $s : \Omega \to \mathbb{R}$ be a continuous and nonvanishing function on Ω. Then*

$$K_s(\boldsymbol{x}, \boldsymbol{y}) = s(\boldsymbol{x})s(\boldsymbol{y})K(\boldsymbol{x}, \boldsymbol{y})$$

is (strictly) positive definite.

Letting $s(\cdot) = \frac{1}{\mathcal{P}_g(\mathcal{X}_N, \cdot)}$, if $\mathcal{X}_N$ is s.t. $\mathcal{P}_g(\Omega) \neq 0$, then

$$K_r(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\mathcal{P}_g(\mathcal{X}_N, \boldsymbol{x})} \frac{1}{\mathcal{P}_g(\mathcal{X}_N, \boldsymbol{y})} K(\boldsymbol{x}, \boldsymbol{y})$$

is a kernel, and by considering the usual inner product, we can build the associated Native Space $\mathcal{N}_{K_r}$. Furthermore, let $\{u_j(\boldsymbol{x}_i) = \delta_{i,j}\}_j$ be the cardinals for $\mathcal{P}_f$,

$$\hat{\mathcal{P}}_f(\mathcal{X}_N, \boldsymbol{x}) = \frac{\sum_{j=1}^N f(\boldsymbol{x}_j) u_j}{\sum_{k=1}^N u_k} = \sum_{j=1}^N f(\boldsymbol{x}_j) \frac{u_j}{\sum_{k=1}^N u_k}$$

leads to a natural definition of $\frac{u_j}{\sum_{k=1}^N u_k} := \hat{u}_j$.

Theorem 4.5. *The rescaled interpolation method is a Shepard's method, where the weight functions are defined as $\hat{u}_j = u_j / \left(\sum_{k=1}^N u_k\right)$, $\{u_j\}_j$ being the cardinal basis of span$\{K(\cdot, x), x \in X\}$.*

The Lebesgue function and constant are, then,

$$\hat{\Lambda}_N(\boldsymbol{x}) := \sum_{j=1}^N |\hat{u}_j(\boldsymbol{x})|, \quad \hat{\lambda}_N := \|\hat{\Lambda}_N\|_{\infty,\Omega}$$

that gives an estimate for the stability,

$$\|\hat{P}_f\|_{\infty,\Omega} \leq \hat{\lambda}_N \|f\|_{\infty,X}.$$

A comparison of the Lebesgue constants is given in Figure 3.

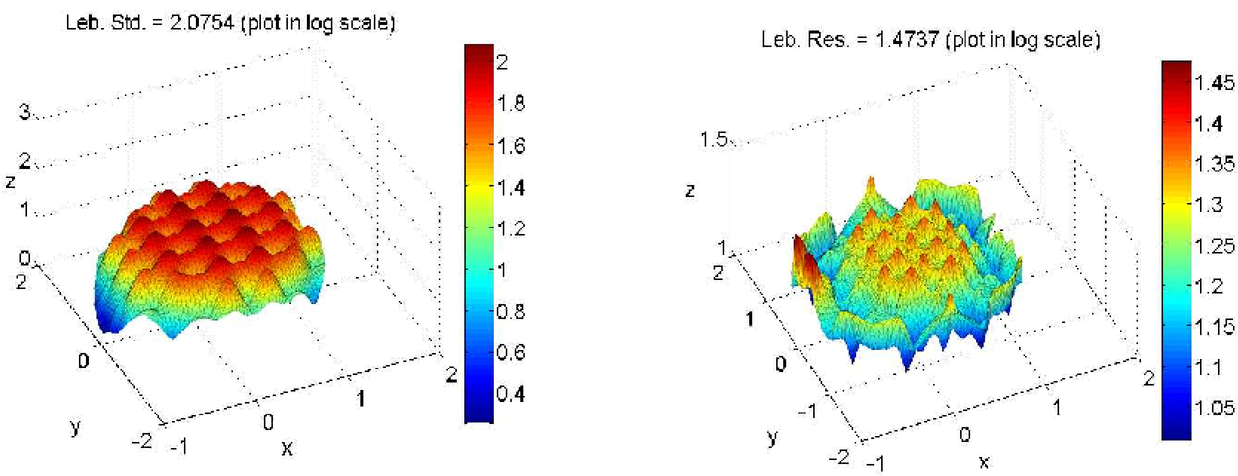

FIGURE 3. Standard basis (Left) and rescaled one (Right), $\mathcal{C}^2$ Wendland kernel on the square, $\varepsilon = 3.85$

In [18], the convergence of the RL-RBF method in the case of quasi-uniform data and stationary scaling has been discussed. As the method is not only interpolatory but also reproduces constants exactly (in fact, it is equivalent to the Shepard's method), linear convergence is expected. We showed that this linear convergence holds up to a certain conjecture about the sum of the rescaled cardinal functions. Concerning the convergence, the following Lemma posed an *open problem* on the sum of the cardinals.

Lemma 4.1. *Under the assumptions of* [18, Theorem 2.3], *there is a constant* $c > 0$ *such that*

$$\sum_{j=1}^{N} u_j(\boldsymbol{x}) \geq c, \qquad \boldsymbol{x} \in \Omega. \tag{4.16}$$

Remark 4.8. *Unfortunately, this Lemma has not been proved. One can easily check, by resorting to the fact that each cardinal is a ratio of two strictly positive determinants (similarly to the Lagrange elementary functions in the polynomial case), that it holds for* $n = 2$*. For* $n > 2$*, in* [18] *it was extensively checked by many numerical examples, even in the case of standard approximation, and in all cases it was confirmed. In Figure 4, we report the numerical results obtained from stable computations using the RBF-QR method for the Gaussian RBF.*

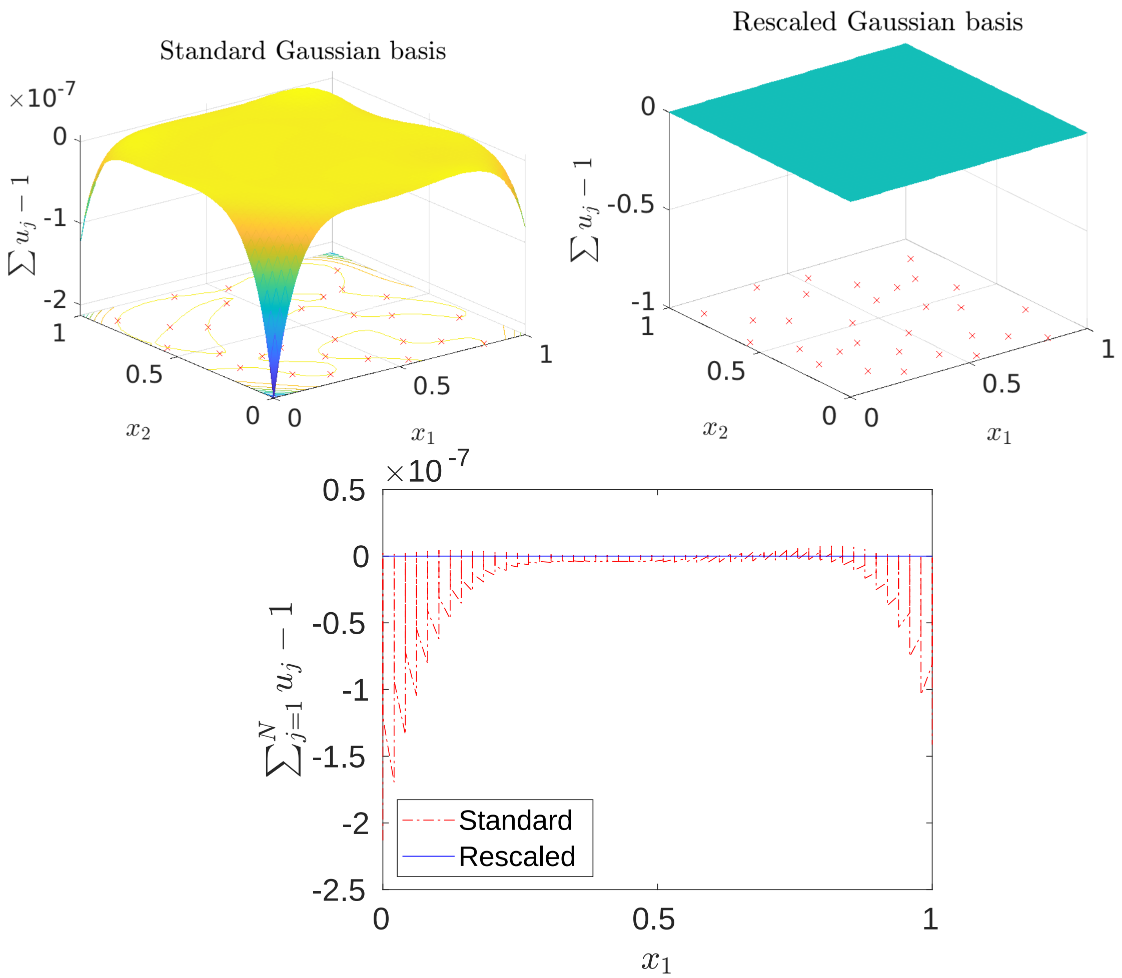

FIGURE 4. RBF-QR. Sum of the cardinals on a grid 50×50 centered at 30 Halton points: Standard, rescaled, and the comparison of their 1-d profiles, for the Gaussian function with $\varepsilon = \sqrt{2}/4$

4.1. **The Case of the Gaussian.** In the forthcoming publication [40], it is shown that for small enough values of the shape parameter ε, and for a unisolvent node set $X = \{\boldsymbol{x}_i\}_{i=1}^N$ in d dimensions, the Gaussian interpolant $s(x, \varepsilon)$ to the data vector $f(X)$ can be expressed on the following closed form that is not ill-conditioned as a function of the shape parameter

$$s(\boldsymbol{x}, \varepsilon) = P_K(\boldsymbol{x}) + (Q(\boldsymbol{x}, X) - vV^{-1}Q(X, X))(I + Q(X, X))^{-1} f(X), \tag{4.17}$$

where P_K is the unique interpolation polynomial of degree K, which means that $N = \binom{K+d}{d}$, V is the (multivariate) Vandermonde matrix for the given nodes, and v is a row vector evaluating the monomial basis at an arbitrary point $\boldsymbol{x}$. For the details of the function Q, we refer to [40]. When the data $f(X)$ is sampled from a polynomial of degree $\le K$, the interpolation polynomial $P_K(\boldsymbol{x})$ is the exact solution and the second term in (4.17) is the interpolation error. In particular, if the data is constant, such that $f_0(X) \equiv 1$, then the error is given by

$$E_0(\boldsymbol{x}, \varepsilon) = (Q(\boldsymbol{x}, X) - vV^{-1}Q(X, X))(I + Q(X, X))^{-1} f_0(X). \tag{4.18}$$

We note that the first factor is the polynomial interpolation error of Q, and we also note that the second factor alternatively can be expressed as the power series $I - Q + Q^2 - \cdots$.

4.1.1. *Error in the 1-d Case.* A detailed analysis of the full error expression has been performed for the 1-d case. When the shape parameter $\varepsilon \lesssim 1$, $\|Q\| < 1$ and the second factor can be approximated by I. Hence, we focus on the interpolation error of $Q(x, X)f_0(X)$. By analyzing the leading order terms of Qf_0 and using the standard error estimate for 1-d polynomial interpolation, we find the approximate error to be

$$E_0(x, \varepsilon) \approx \frac{\prod_{i=0}^{K}(x - x_i)}{(K+1)!} \frac{\varepsilon^{2\lfloor \frac{K+2}{2} \rfloor}}{(\lfloor \frac{K+2}{2} \rfloor)!}. \tag{4.19}$$

The analytical error estimate is compared with the actual errors in Figure 5, showing excellent agreement across the different types of node sets and numbers of points tested.

The estimate (4.19) illustrates the separation of the dependence of the error on the nodes and on the shape parameter. A large fill distance h due to an uneven node distribution increases the error, and uniform nodes lead to a larger error at the boundary, especially for large K, just as for polynomial interpolation in general. For interpolation of a constant (or other monomials of degree $\le K$), the error decreases towards zero as an even power of the shape parameter.

In the detailed analysis of Q, we find that it is beneficial for the approximation to have nodes symmetric about the origin. In the experiments that are shown in Figure 5, all node sets are scaled to cover the full interval [-1,1]. The uniform and Chebyshev nodes are symmetric, while the Halton nodes are not.

The theoretical estimate as well as the numerical results show that for Gaussian RBFs in 1-d, we can provide sufficient conditions on the number of nodes, the distribution of nodes, and the shape parameter ε, such that Lemma 4.1 is satisfied for a given c, showing that the RL-RBF method is well defined and has at least linear convergence in this case.

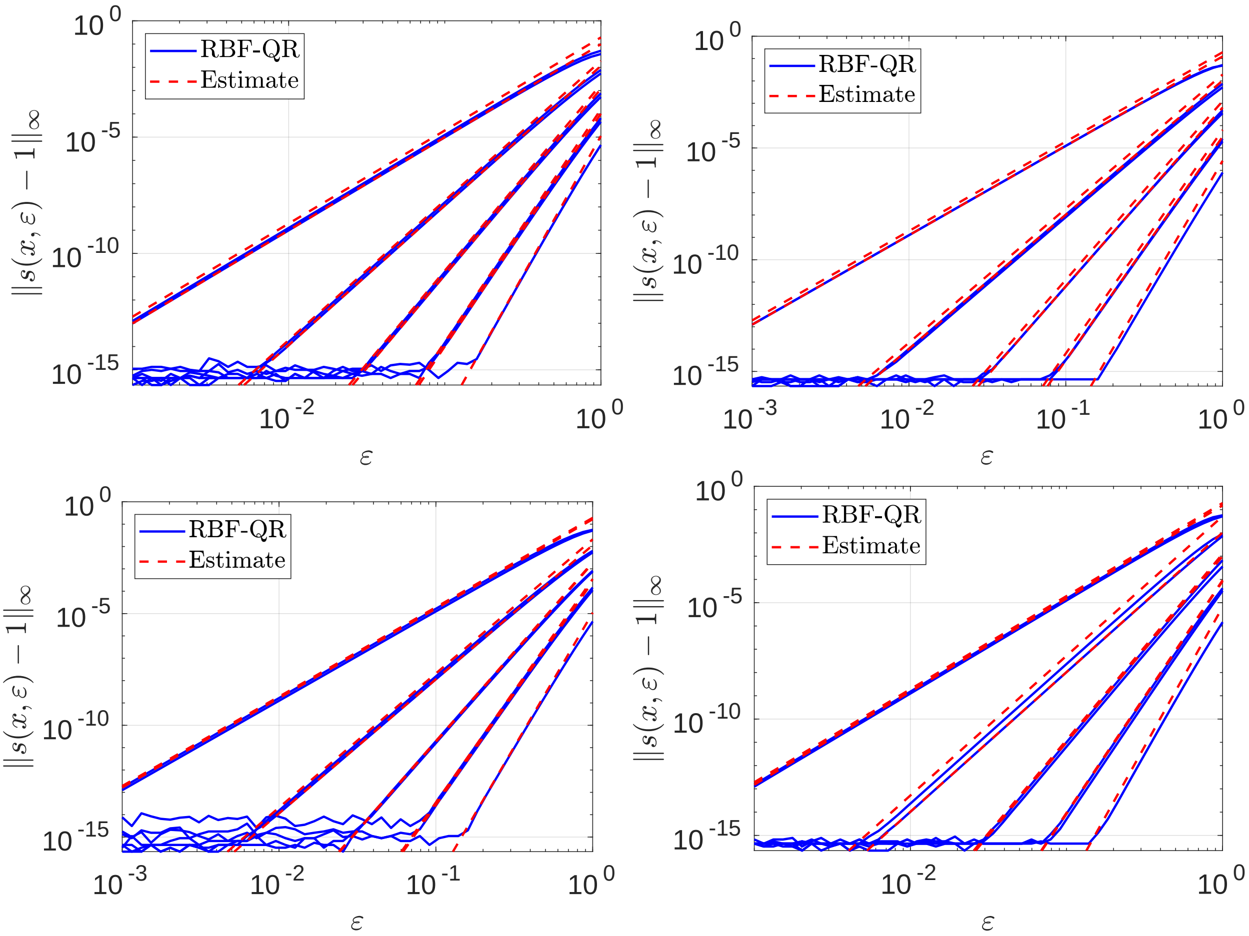

FIGURE 5. The actual Gaussian interpolation errors computed using the RBF-QR method and the error approximation (4.19) for constant data. The node points used were uniform (top left), Chebyshev (top right), Halton (bottom left), and Halton nodes clustered toward the boundary (bottom right) for polynomial degrees $K = 2, \dots, 10$. Note that curves for $K = 2s$ and $K = 2s + 1$ are similar in all plots and can be hard to distinguish

5. CONCLUSION AND FUTURE WORK

In this paper, we present new insights into rational RBFs and their simple counterparts for compactly supported RBFs that reproduce constant functions, denoted Rescaled Localized RBFs. We have also presented a new error estimate for the Gaussians that splits into two parts: one depending on the choice of nodes and the other on the shape parameter. This allows, at least in 1-d, to provide sufficient conditions for the well-definedness of the RL-RBF and its linear convergence.

As future work, we observe that polynomial reproduction plays a relevant role in deriving error estimates for various approximation schemes. Local reproduction in a quasi-uniform setting is a significant factor in error estimation and stability assessment, but for some computationally relevant schemes, such as the RL-RBF, it becomes a limitation. To facilitate the study of a wider range of approximation methods in a unified and efficient manner, a framework based on fast-decaying polynomial reproduction has been proposed in the master's thesis [7]. That is, we do not restrict the method to using compactly supported basis functions, but we instead allow the basis functions to decay to infinity as a function of the separation distance.

The implementation of fast-decaying polynomial reproduction yields stable, convergent methods that are smooth when approximated by moving least squares or very efficient for linear programming problems.

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APPENDIX A. MATLAB CODE

This is the core of the Matlab code for the Variably Scaled Rational Kernel.

```
dsites1 = [dsites psi(dsites,b)];
epoints1 = [epoints psi(epoints,b)];
ctrs1 = dsites1;
DM_data = DistanceMatrix(dsites1,ctrs1);
DM_eval = DistanceMatrix(epoints1,ctrs1);

IM1 = rbf1(ep,DM_data);
B1=IM1+mu.*eye(N);
IM2 = rbf2(ep,DM_data);
B2=IM1+mu.*eye(N);
x0=ones(N,1);
[x1]= powermethod(x0,IM2,1e-6);
int=(B2*x1);
p = rhs.*x1; % Define the vector p
coef = [B1\(rhs.*int)  x1]; % solve linear system B*pAlpha = p
EM1 = rbf1(ep,DM_eval);
ap_rational_vsk=(EM1*coef(:,1))./(EM1*x1);
```

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