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*Research Article*

# An asymptotic expansion for an integral variant of the Wright operators

*Dedicated to Professor Ioan Raşa, on the occasion of his 75th birthday*

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ABSTRACT. In this work, we investigate an integral variant of the Wright operators arising in approximation theory. Our main contribution is the derivation of a complete pointwise asymptotic expansion for these operators $W_n^{\beta,\gamma}$. To establish this expansion, we first develop several structural properties of the operators, emphasizing their intrinsic connections with special functions, including the Mittag-Leffler and confluent hypergeometric functions. The analysis requires a careful study of the moments of the associated integral operators and relies on a collection of auxiliary results, such as a localization theorem. The obtained asymptotic formula provides refined convergence estimates and yields deeper insight into the limiting behavior of the operators as $n \to \infty$.

**Keywords:** Mittag-Leffler functions, asymptotic expansion, hypergeometric series, pointwise estimations.

**2020 Mathematics Subject Classification:** 41A60, 41A30, 41A36.

## 1. INTRODUCTION

Positive linear operators constitute a fundamental tool in approximation theory, providing effective schemes for approximating continuous functions and for studying convergence and asymptotic behavior. In recent years, there has been growing interest in operator sequences constructed via special functions arising in fractional calculus and probability theory, owing to their rich analytical structure and enhanced flexibility. Among these, operators associated with the Wright function have attracted considerable attention due to their close connection with fractional diffusion processes and their ability to model non-local effects. The present paper is devoted to the study of an integral variant of the Wright operators, with particular emphasis on deriving a precise asymptotic expansion that yields refined convergence estimates and deeper insight into the limiting behavior of the operators.

The three-parameter Mittag-Leffler function $E_{\alpha,\beta}^{\gamma}$ (also known as the Prabhakar function), introduced by T. R. Prabhakar in 1971 [8], is defined as

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{\nu=0}^{\infty} \binom{\gamma+\nu-1}{\nu} \frac{z^\nu}{\Gamma(\alpha\nu+\beta)}, \qquad (z \in \mathbb{C},\ \Re(\alpha) > 0), \tag{1.1}$$

where $\binom{\gamma+\nu-1}{\nu} = \frac{\Gamma(\gamma+\nu)}{\nu!\Gamma(\gamma)} = \frac{\gamma^{\overline{\nu}}}{\nu!}$ and $\Gamma(\cdot)$ denotes the Euler gamma function. Here, $\gamma^{\overline{0}} = 1$ and $\gamma^{\overline{\nu}} = \gamma(\gamma+1)\cdots(\gamma+\nu-1)$, for $\nu \in \mathbb{N}$, denotes the rising factorial. Although the three parameters $\alpha$, $\beta$ and $\gamma$ are allowed to assume values in $\mathbb{C}$, we will use the function $E_{\alpha,\beta}^{\gamma}$ only for

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real values of $\alpha$, $\beta$ and $\gamma$. We mention the special instances $E_{1,\beta}^{\beta}(z) = \dfrac{e^z}{\Gamma(\beta)}$ and $E_{1,1}^{1}(z) = e^z$. In this paper, we focus on the special case $\alpha = 1$. The asymptotic formulae for the three parametric Mittag-Leffler functions for large values of indices were established in [4]. The positive linear operator using the terms of the Prabhakar function $E_{1,\beta}^{\gamma}(nx)$ as basis functions,

$$\left(L_n^{1,\beta,\gamma} f\right)(x) = \left(E_{1,\beta}^{\gamma}(nx)\right)^{-1} \sum_{\nu=0}^{\infty} \binom{\gamma+\nu-1}{\nu} \frac{(nx)^{\nu}}{\Gamma(\nu+\beta)} f\left(\frac{\nu}{n}\right)$$

was introduced and studied in [1].

Closely related to the Mittag-Leffler function is the Wright function [10], introduced by E. M. Wright as a natural generalization of exponential-type functions in the theory of entire functions. It is defined by the series

$$\phi_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(\alpha k+\beta)}, \qquad \alpha > -1,\ \beta \in \mathbb{C}.$$

The Wright function plays a significant role in fractional calculus, as it appears in the fundamental solutions of time-fractional and space-fractional diffusion equations. It is also closely connected with the Mittag-Leffler function, Lévy stable distributions, and probability density functions describing anomalous diffusion processes. Furthermore, in approximation theory and applied analysis, Wright-type functions are employed in the construction of positive linear operators [6] and in the study of their asymptotic behavior and convergence properties, highlighting their importance in both theoretical and applied contexts.

The Prabhakar function $E_{\alpha,\beta}^{\gamma}$ and the Wright function $\phi_{\alpha,\beta}$ play a central role in several areas of fractional calculus, probability theory, and anomalous relaxation phenomena. The Prabhakar function, as a three-parameter generalization of the classical Mittag-Leffler function, gives rise to the Prabhakar fractional integral operator, which extends the classical Riemann–Liouville and Caputo fractional integrals. On the other hand, the Wright function naturally appears in the fundamental solutions of time-fractional and space-fractional diffusion equations and is closely connected with stable probability distributions. Both functions are extensively used in the modeling of memory-dependent processes, viscoelastic materials, and non-local diffusion phenomena. In approximation theory, kernels expressed in terms of the Prabhakar and Wright functions enable the construction of positive linear operators with adjustable parameters, leading to enhanced flexibility and improved convergence behavior. Consequently, the availability of accurate asymptotic expansions for $E_{\alpha,\beta}^{\gamma}$ and $\phi_{\alpha,\beta}$ is essential for rigorous theoretical analysis and efficient numerical implementation, particularly in large-parameter regimes, stability investigations, and the derivation of precise rate of convergence results. Such expansions also facilitate closed-form representations of moments and provide deeper insight into the asymptotic behavior of the associated operators as $n \to \infty$, which is crucial in approximation theory and applied mathematics.

For a real number $A \geq 0$, let $E_A[0,\infty)$ be the class of all functions $f : [0,\infty) \to \mathbb{R}$ which satisfy, for some positive constant $K$, the growth condition $|f(t)| \leq K \exp(At)$, for $t \geq 0$. Let $E[0,\infty) = \bigcup_{A\geq 0} E_A[0,\infty)$ be the space of all functions of (at most) exponential growth. For $\beta > 0$ and $\gamma \geq 1$, we define the operators $W_n^{\beta,\gamma} : E[0,\infty) \to C[0,\infty)$ by

$$\text{(1.2)} \qquad \left(W_n^{\beta,\gamma} f\right)(x) = \left(E_{1,\beta}^{\gamma}(nx)\right)^{-1} \sum_{\nu=0}^{\infty} \binom{\gamma+\nu-1}{\nu} \frac{(nx)^{\nu}}{\nu!\Gamma(\nu+\beta)} \int_0^{\infty} e^{-t} t^{\nu} f\left(\frac{t}{n}\right) dt.$$

It is obvious that, the operators $W_n^{\beta,\gamma}$ are linear and positive. We observe that, for sufficiently large $n$, the operators $W_n^{\beta,\gamma}$ are well defined on the space $E[0,\infty)$ since $f \in E_A[0,\infty)$, implies that, for $n > A$,

$$\left|\int_0^\infty e^{-t} t^\nu f\left(\frac{t}{n}\right) dt\right| \le K \int_0^\infty e^{-t} t^\nu \exp\left(At/n\right) dt$$
$$= K \int_0^\infty t^\nu \exp\left(-t\left(1 - A/n\right)\right) dt = \frac{K\nu!}{\left(1 - A/n\right)^{\nu+1}}$$

and

$$\left|\left(W_n^{\beta,\gamma} f\right)(x)\right| \le \frac{Kn}{n - A}\left(E_{1,\beta}^{\gamma}\left(nx\right)\right)^{-1} E_{1,\beta}^{\gamma}\left(\frac{n^2 x}{n - A}\right).$$

At the point $x = 0$, the operators (1.2) approximate the function $f$, provided that it is continuous from the right at $x = 0$. In this case, we have $\left(W_n^{\beta,\gamma} f\right)(0) = \int_0^\infty e^{-t} f\left(\frac{t}{n}\right) dt \to f(0)$ as $n \to \infty$.

In the special case, $\gamma = 1$, the operator $W_n^{\beta,\gamma}$ was discussed by Patel [5], another variant was discussed in [7]. Further, the special case $\beta = \gamma \ge 1$, the operator $W_n^{\beta,\gamma}$ reduces to

$$\left(W_n^{\beta,\beta} f\right)(x) = e^{-nx} \int_0^\infty e^{-t} I_0\left(2\sqrt{nxt}\right) f\left(\frac{t}{n}\right) dt,$$

where $I_0$ denotes the modified Bessel function of the first kind given by

$$I_0\left(z\right) = \sum_{\nu=0}^{\infty} \frac{z^{2\nu}}{4^\nu \left(\nu!\right)^2}.$$

Our focus is to study the rate of convergence for the operators $W_n^{\beta,\gamma}$. Also, we derive a (pointwise) complete asymptotic expansion for the sequence $\left(\left(W_n^{\beta,\gamma} f\right)(x)\right)_{n=1}^\infty$ in the form

$$\left(W_n^{\beta,\gamma} f\right)(x) \sim f(x) + \sum_{k=1}^{\infty} a_k^{\beta,\gamma}\left(f, x\right) n^{-k} \qquad (n \to \infty), \tag{1.3}$$

provided that $f$ admits derivatives of sufficiently high order at $x > 0$. Formula (1.3) means that, for all $q = 1, 2, \ldots$, it holds

$$\left(W_n^{\beta,\gamma} f\right)(x) = f(x) + \sum_{k=1}^{q} a_k^{\beta,\gamma}\left(f, x\right)\ n^{-k} + o(n^{-q}) \qquad (n \to \infty).$$

The coefficients $a_k^{\beta,\gamma}\left(f, x\right)$, which are independent of $n$, will be given in terms of coefficients of a certain power series. As a particular case $\gamma = 1$, we obtain the complete asymptotic expansion for the sequence of Patel's operators established in [5]. In the special case $q = 1$, we complement his results on $W_n^{\beta,1}$ by a Voronovskaja-type theorem.

## 2. Main Results

The following theorem presents as our main result the (pointwise) complete asymptotic expansion for the integral generalization of the Mittag-Leffler operators $W_n^{\beta,\gamma}$.

**Theorem 2.1.** *Let $q \in \mathbb{N}$ and $x \in (0,\infty)$. For each function $f \in E[0,\infty)$ which is $2q$ times differentiable at $x$, the operators $W_n^{\beta,\gamma}$ possess the asymptotic expansion*

$$\left(W_n^{\beta,\gamma} f\right)(x) = f(x) + \sum_{k=1}^{q} a_k^{\beta,\gamma}\left(f, x\right) n^{-k} + o(n^{-q}) \qquad (n \to \infty) \tag{2.4}$$

*with the coefficients*

$$a_k^{\beta,\gamma}(f,x)=\sum_{s=1}^{2k}\frac{f^{(s)}(x)}{s!}x^{s-k}T_{k,s}(\gamma,\beta)\qquad(k=1,2,\ldots),\tag{2.5}$$

*where the numbers* $T_{k,s}(\alpha,\gamma,\beta)$ *and* $H_k(a,b,j)$ *are defined by*

$$T_{k,s}(\gamma,\beta):=\sum_{j=0}^{k}\sum_{r=j}^{s}(-1)^{s-r}\binom{s}{r}\binom{r}{j}r^{\underline{j}}H_{k-j}(\gamma,\beta,r-j)\tag{2.6}$$

*and by equation* (3.10)*, respectively.*

Before proceeding to the proof of the above theorems, we first establish several fundamental properties of the operators $W_n^{\beta,\gamma}$. In particular, we demonstrate their connections with other special functions, notably the confluent hypergeometric function. Throughout the paper, $z^{\underline{0}}=1$, $z^{\underline{j}}=z(z-1)\cdots(z-j+1)$, $j\in\mathbb{N}$, denote the falling factorials.

**Remark 2.1.** *If* $f\in\bigcap_{q=1}^{\infty}K[q;x]$*, the operators* $W_n^{\beta,\gamma}$ *possess the complete asymptotic expansion*

$$\left(W_n^{\beta,\gamma}f\right)(x)=f(x)+\sum_{k=1}^{\infty}a_k^{\beta,\gamma}(f,x)\ n^{-k}\qquad(n\to\infty),$$

*where the coefficients* $a_k^{\beta,\gamma}(f,x)$ *are defined in* (2.5)*.*

In the case $q=1$, an immediate consequence of Theorem 2.1 is the following Voronovskaja-type formula.

**Corollary 2.1.** *Let* $x\in(0,\infty)$*. For each function* $f\in K[2;x]$*, the operators* $W_n^{\beta,\gamma}$ *satisfy*

$$\lim_{n\to\infty}n\left(\left(W_n^{\beta,\gamma}f\right)(x)-f(x)\right)=(1+\gamma-\beta)f'(x)+xf''(x).$$

**Remark 2.2.** *For the convenience of the reader, we list the explicit expressions for the initial coefficients* $a_k^{\beta,\gamma}(f,x)$*:*

$$\begin{aligned}
a_1^{\beta,\gamma}(f,x)&=(1+\gamma-\beta)f'(x)+xf''(x)\\
a_2^{\beta,\gamma}(f,x)&=\frac{(\beta-\gamma)(\gamma-1)}{x}f'(x)\\
&+\frac{(\gamma-\beta+1)(\gamma-\beta+2)}{2}f''(x)+\frac{\gamma-\beta+2}{6}xf^{(3)}(x)+\frac{x^2}{2}f^{(4)}(x)\\
a_3^{\beta,\gamma}(f,x)&=\frac{(\gamma-1)(\gamma-\beta)(2\gamma-\beta-2)}{x^2}f'(x)+\frac{(1-\gamma)(\gamma-\beta)(\gamma-\beta+1)}{x}f''(x)\\
&+\frac{6(\gamma-\beta)^3+(17-6\beta)(\gamma-\beta)^4}{6}f^{(3)}(x)+\frac{(\gamma-\beta+2)(\gamma-\beta+3)}{2}xf^{(4)}(x)\\
&+\frac{\gamma-\beta+3}{2}x^2f^{(5)}(x)+\frac{x^3}{6}f^{(6)}(x).
\end{aligned}$$

We emphasize the fact that $a_k^{\beta,\gamma}(f,x)$ contains only derivatives $f^{(s)}(x)$ of order $s\le 2k$. By equation (2.5), $a_k^{\beta,\gamma}(f,x)$, $k\in\mathbb{N}$, is a linear combination of the terms $x^{s-k}f^{(s)}(x)$, for $s=1,\ldots,2k$. In the special case $\beta=\gamma$, we obtain the following corollary.

**Corollary 2.2.** *Let $q \in \mathbb{N}$ and $x \in (0,\infty)$. For each function $f \in E[0,\infty)$ which is $2q$ times differentiable at $x$, the operators $W_n^{\beta,\beta}$ possess the asymptotic expansion*

$$\left(W_n^{\beta,\beta} f\right)(x) = f(x) + \sum_{k=1}^{q} a_k^{\beta,\beta}(f,x)\, n^{-k} + o(n^{-q}) \qquad (n \to \infty)$$

*with the coefficients*

$$a_k^{\beta,\beta}(f,x) = \sum_{s=0}^{k} \binom{k}{s} \frac{x^s}{s!} f^{(k+s)}(x) \qquad (k = 1,2,\ldots).$$

## 3. Moments for the Operators $W_n^{\beta,\gamma}$

Throughout the paper, $e_r$ denote the monomials $e_r(t) = t^r$, $(r = 0,1,2,\ldots)$ and for each real $x$, we put $\psi_x = e_1 - xe_0$.

To demonstrate our main findings on the operators $W_n^{\beta,\gamma}$, we will need the moments of the integral operators (1.2). The proof of Theorem 2.1 relies on many lemmas, which are compiled in this section.

**Lemma 3.1.** *For $r = 0,1,2,\ldots$, the moments of the operators $W_n^{\beta,\gamma}$ have the representation*

$$\left(W_n^{\beta,\gamma} e_r\right)(x) = n^{-r} \sum_{j=0}^{r} \binom{r}{j} r^{\underline{r-j}} (nx)^j \frac{\gamma^{\overline{j}}}{\beta^{\overline{j}}} \frac{{}_1F_1(\gamma+j;\beta+j;nx)}{{}_1F_1(\gamma;\beta;nx)}. \tag{3.7}$$

*With the abbreviation*

$$R_{n,j}^{\beta,\gamma}(x) = \frac{\gamma^{\overline{j}}}{\beta^{\overline{j}}} \frac{{}_1F_1(\gamma+j;\beta+j;nx)}{{}_1F_1(\gamma;\beta;nx)},$$

*the first instances are given by*

$$\begin{aligned}
W_n^{\beta,\gamma} e_0 &= e_0, \\
W_n^{\beta,\gamma} e_1 &= R_{n,1}^{\beta,\gamma} e_1, \\
W_n^{\beta,\gamma} e_2 &= R_{n,2}^{\beta,\gamma} e_2 + \frac{2}{n} R_{n,1}^{\beta,\gamma} e_1, \\
W_n^{\beta,\gamma} e_3 &= R_{n,3}^{\beta,\gamma} e_3 + \frac{9}{n} R_{n,2}^{\beta,\gamma} e_2 + \frac{27}{n^2} R_{n,1}^{\beta,\gamma} e_1, \\
W_n^{\beta,\gamma} e_4 &= R_{n,4}^{\beta,\gamma} e_4 + \frac{16}{n} R_{n,3}^{\beta,\gamma} e_3 + \frac{96}{n^2} R_{n,2}^{\beta,\gamma} e_2 + \frac{256}{n^3} R_{n,1}^{\beta,\gamma} e_1.
\end{aligned}$$

*Proof.* Since $\int_0^\infty e^{-t} t^\nu e_r\left(\frac{t}{n}\right) dt = n^{-r}(\nu+r)!$, we obtain

$$\left(W_n^{\beta,\gamma} e_r\right)(x) = n^{-r} \left(E_{1,\beta}^{\gamma}(nx)\right)^{-1} \sum_{\nu=0}^{\infty} \binom{\gamma+\nu-1}{\nu} \frac{(nx)^\nu (\nu+r)!}{\nu!\,\Gamma(\nu+\beta)}.$$

By Vandermonde convolution, we have

$$\frac{(\nu+r)!}{\nu!} = \sum_{j=0}^{r} \binom{r}{j} \nu^{\underline{j}} r^{\underline{r-j}}$$

which leads to

$$\left(W_n^{\beta,\gamma} e_r\right)(x) = n^{-r} \left(E_{1,\beta}^{\gamma}(nx)\right)^{-1} \sum_{j=0}^{r} \binom{r}{j} r^{\underline{r-j}} \sum_{\nu=j}^{\infty} \binom{\gamma+\nu-1}{\nu} \frac{(nx)^\nu}{\Gamma(\nu+\beta)} \nu^{\underline{j}}.$$

Since

$$\nu^{\underline{j}}\binom{\gamma+\nu-1}{\nu}=\gamma^{\overline{j}}\binom{\gamma+\nu-1}{\nu-j}\qquad(\nu\geq j),$$

we obtain

$$\left(W_n^{\beta,\gamma}e_r\right)(x)=n^{-r}\left(E_{1,\beta}^{\gamma}(nx)\right)^{-1}\sum_{j=0}^{r}\binom{r}{j}r^{\underline{r-j}}\gamma^{\overline{j}}(nx)^j E_{1,\beta+j}^{\gamma+j}(nx).$$

Taking advantage of (3.8) leads to

$$\left(W_n^{\beta,\gamma}e_r\right)(x)=n^{-r}\sum_{j=0}^{r}\binom{r}{j}r^{\underline{r-j}}\gamma^{\overline{j}}(nx)^j\,\frac{\Gamma(\beta)}{\Gamma(\beta+j)}\,\frac{{}_1F_1(\gamma+j;\beta+j;nx)}{{}_1F_1(\gamma;\beta;nx)},$$

which is the desired formula, since $\Gamma(\beta+j)/\Gamma(\beta)=\beta^{\overline{j}}$. □

For the subsequent discussion, we require the connection between the confluent hypergeometric function and the three-parameters Mittag-Leffler function (1.1), as stated in [2, Equation 5.1.39] as

(3.8)
$$E_{1,\beta}^{\gamma}(z)=\frac{1}{\Gamma(\beta)}\sum_{\nu=0}^{\infty}\frac{\gamma^{\overline{\nu}}}{\beta^{\overline{\nu}}}\frac{z^\nu}{\nu!}=\frac{1}{\Gamma(\beta)}\,{}_1F_1(\gamma;\beta;z),$$

where the confluent hypergeometric function is given by

$${}_1F_1(a;b;z)=\sum_{\nu=0}^{\infty}\frac{a^{\overline{\nu}}}{b^{\overline{\nu}}}\frac{z^\nu}{\nu!}.$$

It is well known that ${}_1F_1$ satisfies the asymptotic relation (see [3, Formula 13.7.1])

(3.9)
$${}_1F_1(a;b;z)\sim\frac{e^z z^{a-b}\Gamma(b)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{(b-a)^{\overline{k}}(1-a)^{\overline{k}}}{k!z^k}\qquad(z\to+\infty).$$

As in [1], we define the numbers $H_k(a,b,j)$ by the relation

(3.10)
$$\frac{1+\sum_{k=1}^{\infty}(b-a)^{\overline{k}}(1-a-j)^{\overline{k}}\frac{w^k}{k!}}{1+\sum_{k=1}^{\infty}(b-a)^{\overline{k}}(1-a)^{\overline{k}}\frac{w^k}{k!}}=\sum_{k=0}^{\infty}H_k(a,b,j)\,w^k,$$

where all series are interpreted as formal power series. The following consequence will become useful related to $H_k(a,b,j)$

(3.11)
$$\frac{{}_1F_1(a+j;b+j;z)}{{}_1F_1(a;b;z)}\sim\frac{b^{\overline{j}}}{a^{\overline{j}}}\sum_{k=0}^{\infty}\frac{H_k(a,b,j)}{z^k}\qquad(z\to+\infty).$$

For the convenience of the reader we list some explicit expressions, which were also mentioned in [1]:

$$\begin{aligned}H_0(a,b,j)&=1,\\ H_1(a,b,j)&=(a-b)\,j,\\ H_2(a,b,j)&=\frac{(a-b)\,j}{2}(3-3a+b-j+aj-bj),\end{aligned}$$

$$\begin{aligned} H_3(a,b,j) = & \frac{(a-b)\,j}{6}\left(22-42a+20a^2+18b-16ab+2b^2\right. \\ & -3(a-b-1)(3a-b-4)\,j \\ & \left.+(a-b-2)(a-b-1)\,j^2\right). \end{aligned}$$

**Lemma 3.2.** *For $r = 0, 1, 2, \ldots$, the moments of the operators $W_n^{\beta,\gamma}$ satisfy the asymptotic relation*

$$\left(W_n^{\beta,\gamma} e_r\right)(x) \sim \sum_{k=0}^{\infty} \frac{1}{n^k} \sum_{j=0}^{\min\{k,r\}} \binom{r}{j} r^{\underline{j}} H_{k-j}(\gamma, \beta, r-j)\, x^{r-k} \qquad (n \to \infty),$$

*where $H_k(a, b, j)$ is defined in (3.10).*

*Proof.* The starting point is the Equation (3.7) in Lemma 3.1, i.e.,

$$\left(W_n^{\beta,\gamma} e_r\right)(x) = n^{-r} \sum_{j=0}^{r} \binom{r}{j} r^{\underline{r-j}} (nx)^j \frac{\gamma^{\overline{j}}}{\beta^{\overline{j}}} \frac{{}_1F_1(\gamma + j; \beta + j; nx)}{{}_1F_1(\gamma; \beta; nx)}.$$

Hence, by (3.11),

$$\left(W_n^{\beta,\gamma} e_r\right)(x) \sim n^{-r} \sum_{j=0}^{r} \binom{r}{j} r^{\underline{r-j}} (nx)^j \sum_{\ell=0}^{\infty} \frac{H_\ell(\gamma, \beta, j)}{(nx)^\ell} \qquad (n \to \infty).$$

Finally,

$$\left(W_n^{\beta,\gamma} e_r\right)(x) \sim \sum_{j=0}^{r} \binom{r}{j} r^{\underline{j}} n^{-j} x^{r-j} \sum_{\ell=0}^{\infty} \frac{H_\ell(\gamma, \beta, r-j)}{(nx)^\ell} \qquad (n \to \infty)$$

and collecting all terms with $j + \ell = k$, we obtain the desired formula. □

**Lemma 3.3.** *For $s = 0, 1, 2, \ldots$, the central moments of the operators $W_n^{\beta,\gamma}$ have the representation*

$$\left(W_n^{\beta,\gamma} \psi_x^s\right)(x) \sim \sum_{k=\lfloor (s+1)/2 \rfloor}^{\infty} \frac{x^{s-k}}{n^k} T_{k,s}(\gamma, \beta) \qquad (n \to \infty), \tag{3.12}$$

*where $T_{k,s}(\gamma, \beta)$ is defined by the Equation (2.6).*

A direct consequence of Lemma 3.3 is the asymptotic relation

$$\left(W_n^{\beta,\gamma} \psi_x^s\right)(x) = O\left(n^{-\lfloor (s+1)/2 \rfloor}\right) \qquad (n \to \infty). \tag{3.13}$$

*Proof.* By the binomial formula, we have

$$\left(W_n^{\beta,\gamma} \psi_x^s\right)(x) = \sum_{r=0}^{s} (-x)^{s-r} \binom{s}{r} \left(W_n^{\beta,\gamma} e_r\right)(x).$$

Application of Lemma 3.2 yields

$$\begin{aligned} \left(W_n^{\beta,\gamma} \psi_x^s\right)(x) & \sim \sum_{k=0}^{\infty} \frac{x^{s-k}}{n^k} \sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} \sum_{j=0}^{\min\{k,r\}} \binom{r}{j} r^{\underline{j}} H_{k-j}(\gamma, \beta, r-j) \\ & = \sum_{k=0}^{\infty} \frac{x^{s-k}}{n^k} T_{k,s}(\gamma, \beta) \qquad (n \to \infty), \end{aligned}$$

where $T_{k,s}(\gamma,\beta)$ is defined in equation (2.6). It remains to prove that $T_{k,s}(\gamma,\beta)=0$ if $2k<s$. Using the binomial identity

$$\binom{s}{r}\binom{r}{j}=\binom{s}{j}\binom{s-j}{r-j},$$

we obtain

$$\begin{aligned}T_{k,s}(\gamma,\beta)&=\sum_{j=0}^{k}\binom{s}{j}\sum_{r=j}^{s}(-1)^{s-r}\binom{s-j}{r-j}r^{\underline{j}}H_{k-j}(\gamma,\beta,r-j)\\&=\sum_{j=0}^{k}\binom{s}{j}\sum_{r=0}^{s-j}(-1)^{s-j-r}\binom{s-j}{r}(r+j)^{\underline{j}}H_{k-j}(\gamma,\beta,r).\end{aligned}$$

Equation (3.10) reveals that $H_k(a,b,z)$ is a polynomial in the variable $z$ of degree at most $k$. This implies that

$$\sum_{r=0}^{s-j}(-1)^{s-j-r}\binom{s-j}{r}(r+j)^{\underline{j}}H_{k-j}(\gamma,\beta,r)=0$$

if $k<s-j$, i.e., if $k+j<s$. Noting that $0\le j\le k$, we conclude that $T_{k,s}(\gamma,\beta)=0$ if $2k<s$. This completes the proof. □

## 4. Proof of Theorem 2.1

In order to derive Theorem 2.1, a general approximation theorem due to Sikkema [9, Theorem 3] will be applied. To this end, we need some notation. Let $I$ be a real interval and $x\in I$. For $s\in\mathbb{N}$, let $H^{(s)}(x)$ denote the class of all locally bounded real functions $f:I\to\mathbb{R}$, which are $s$ times differentiable at $x$. In the case that $I$ is an infinite interval, $f$ has to satisfy the additional condition $f(t)=O(|t|^s)$ as $t\to+\infty$ or $t\to-\infty$, respectively, or both if $I=\mathbb{R}$. An inspection of the proof of Sikkema's result reveals that it can be stated in the following form which is more appropriate for our purposes.

**Lemma 4.4.** *Let $q\in\mathbb{N}$ and let $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators, $L_n:H^{(2q)}(x)\to C[c,d]$, $x\in[c,d]$. Suppose that the operators $L_n$ apply to $\psi_x^{2q+1}$ and to $\psi_x^{2q+2}$. Then the condition*

$$(L_n\psi_x^s)(x)=O\left(n^{-\lfloor (s+1)/2\rfloor}\right)\qquad(n\to\infty),\qquad \text{for } s=0,1,\ldots,2q+2,$$

*implies, for each function $f\in H^{(2q)}(x)$, the asymptotic relation*

$$(L_nf)(x)=\sum_{s=0}^{2q}\frac{f^{(s)}(x)}{s!}(L_n\psi_x^s)(x)+o\left(n^{-q}\right)\qquad(n\to\infty).$$

Since Lemma 4.4 deals with functions of polynomial growth, we cannot apply it directly to functions $f\in E[0,\infty)$. First, we note that the definition of $E_A[0,\infty)$ implies that the functions of this class are bounded on each finite interval $[0,r]$, $r>0$. In order to prove Theorem 2.1 for functions of (at most) exponential growth, we need a localization result (Lemma 4.6) for the operators $W_n^{\beta,\gamma}$. The next result will be applied in its proof.

For real numbers $A$, define $\exp_A(t)=e^{At}$.

**Lemma 4.5.** *Let $A$ be a real number and $x>0$. Then, $\left(W_n^{\beta,\gamma}\exp_A\right)(x)$ is well defined, for all positive integers $n>A$, with*

$$\left(W_n^{\beta,\gamma}\exp_A\right)(x)=\frac{n}{n-A}\,\frac{{}_1F_1\left(\gamma;\beta;\dfrac{n}{n-A}nx\right)}{{}_1F_1(\gamma;\beta;nx)}\tag{4.14}$$

*and has* $\lim_{n\to\infty}\left(W_n^{\beta,\gamma}\exp_A\right)(x)=\exp_A(x)$.

*Proof.* Let $x>0$. An elementary calculation and application of (3.8) shows that, for $n>A$,

$$\left(W_n^{\beta,\gamma}\exp_A\right)(x)=\left(E_{1,\beta}^{\gamma}(nx)\right)^{-1}\sum_{\nu=0}^{\infty}\binom{\gamma+\nu-1}{\nu}\frac{(nx)^{\nu}}{\nu!\Gamma(\nu+\beta)}\int_0^{\infty}e^{-t}t^{\nu}e^{\frac{At}{n}}dt$$
$$=\frac{n}{n-A}\frac{E_{1,\beta}^{\gamma}\left(\frac{n^2x}{n-A}\right)}{E_{1,\beta}^{\gamma}(nx)}$$
$$=\frac{n}{n-A}\frac{{}_1F_1\left(\gamma;\beta;\frac{n^2x}{n-A}\right)}{{}_1F_1(\gamma;\beta;nx)}.$$

By (3.9), we conclude that

$$\left(W_n^{\beta,\gamma}\exp_A\right)(x)\sim\frac{n}{n-A}\frac{\frac{e^{z_1}z_1^{\gamma-\beta}\Gamma(\beta)}{\Gamma(\gamma)}\left(1+O\left(z_1^{-1}\right)\right)}{\frac{e^{z_2}z_2^{\gamma-\beta}\Gamma(\beta)}{\Gamma(\gamma)}\left(1+O\left(z_2^{-1}\right)\right)}$$

$$=\frac{n}{n-A}e^{z_1-z_2}\left(\frac{z_1}{z_2}\right)^{\gamma-\beta}\left(1+O\left(z_1^{-1}\right)-O\left(z_2^{-1}\right)\right),$$

where

$$z_1=\frac{n^2x}{n-A}=nx\frac{n}{n-A},\qquad z_2=nx.$$

As $n$ tends to infinity, we have

$$\frac{n}{n-A}=1+\frac{A}{n}+O\left(n^{-2}\right),\qquad z_1=nx+Ax+O\left(n^{-1}\right).$$

Hence

$$z_1-z_2=Ax+O\left(n^{-1}\right),\qquad e^{z_1-z_2}=e^{Ax}\left(1+O\left(n^{-1}\right)\right).$$

Similarly,

$$\left(\frac{z_1}{z_2}\right)^{\gamma-\beta}=\left(\frac{n}{n-A}\right)^{\gamma-\beta}=1+(\gamma-\beta)\frac{A}{n}+O\left(n^{-2}\right).$$

Therefore

$$\left(W_n^{\beta,\gamma}\exp_A\right)(x)\sim e^{Ax}\left(1+\frac{A}{n}\right)\left(1+(\gamma-\beta)\frac{A}{n}\right)\left(1+O\left(n^{-1}\right)\right).$$

Hence, we have $\lim_{n\to\infty}\left(W_n^{\beta,\gamma}\exp_A\right)(x)=\exp_A(x)$. □

**Lemma 4.6** (Localization Theorem). *Let* $\delta>0$ *and fix* $x>0$. *If a function* $f\in E_A[0,\infty)$ *vanishes on the interval* $(x-\delta,x+\delta)\cap[0,\infty)$, *then* $\left(W_n^{\beta,\gamma}f\right)(x)=O\left(n^{-m}\right)$ *as* $n\to\infty$, *for arbitrarily large* $m>0$.

*Proof.* Let $m \in \mathbb{N}$. For a certain positive constant $K$, we have

$$\left(E_{1,\beta}^{\gamma}\left(nx\right)\right)\left|\left(W_n^{\beta,\gamma}f\right)(x)\right| \leq K \sum_{\substack{\nu\geq 0,\\ |\nu/n-x|\geq\delta}} \binom{\gamma+\nu-1}{\nu}\frac{1}{\Gamma(\nu+\beta)}\left(\frac{n^2x}{n-A}\right)^{\nu}$$
$$\leq K\delta^{-2m}\sum_{\nu=0}^{\infty}\binom{\gamma+\nu-1}{\nu}\frac{1}{\Gamma(\nu+\beta)}\left(\frac{n^2x}{n-A}\right)^{\nu}\left(\frac{\nu}{n}-x\right)^{2m}.$$

Application of the Schwarz inequality yields

$$\left|\left(W_n^{\beta,\gamma}f\right)(x)\right| \leq K\delta^{-2m}\sqrt{\left(W_n^{\beta,\gamma}\exp_{2A}\right)(x)}\sqrt{\left(W_n^{\beta,\gamma}\psi_x^{4m}\right)(x)}\,.$$

By Lemma 4.5, the first square root has the finite limit $\exp_A(x)$ as $n\to\infty$. By relation (3.13), the second root satisfies $\sqrt{\left(W_n^{\beta,\gamma}\psi_x^{4m}\right)(x)} = O\left(n^{-m}\right)$ as $n\to\infty$. This completes the proof. □

Now we are in position to prove the asymptotic expansion.

*Proof of Theorem 2.1.* Let $x>0$ and suppose that $f^{(2q)}(x)$ exists. Firstly, we assume that $f\in H^{(2q)}(x)$. By the Equation (3.13), we have $\left(W_n^{\beta,\gamma}\psi_x^{2s}\right)(x) = O\left(n^{-s}\right)$ as $n\to\infty$. Lemma 4.4 implies that

$$\left(W_n^{\beta,\gamma}f\right)(x) = \sum_{s=0}^{2q}\frac{f^{(s)}(x)}{s!}\left(W_n^{\beta,\gamma}\psi_x^{s}\right)(x) + o\left(n^{-q}\right) \qquad (n\to\infty)\,.$$

Now, we suppose that $f\in E\left[0,\infty\right)$ and $f^{(2q)}(x)$ exists. Note that the above asymptotic expansion depends only on the behaviour of $f$ in an arbitrary small neighborhood of $x$. Therefore, by the localization theorem, we can assume, without loss of generality, that $f\in H^{(2q)}(x)$. By Lemma 3.3 and noting that $W_n^{\beta,\gamma}e_0 = e_0$, we infer that

$$\sum_{s=0}^{2q}\frac{f^{(s)}(x)}{s!}\left(W_n^{\beta,\gamma}\psi_x^{s}\right)(x) \sim f(x) + \sum_{k=1}^{\infty}\frac{1}{n^k}\sum_{s=1}^{2k}\frac{f^{(s)}(x)}{s!}x^{s-k}T_{k,s}(\gamma,\beta)$$

as $n\to\infty$. Finally, we obtain

$$\left(W_n^{\beta,\gamma}f\right)(x) = f(x) + \sum_{k=1}^{q}\frac{1}{n^k}\sum_{s=1}^{2k}\frac{f^{(s)}(x)}{s!}x^{s-k}T_{k,s}(\gamma,\beta) + o\left(n^{-q}\right)$$

as $n\to\infty$. This is the desired expansion. □

*Proof of Corollary 2.2.* Let $x>0$ and suppose that $f^{(2q)}(x)$ exists. Since ${}_1F_1(a;a;z) = e^z$, Equation (3.10) reveals that $H_0(a,a,j) = 1$ and $H_k(a,a,j) = 0$ $(k\in\mathbb{N})$. By (2.6), we obtain

$$T_{k,s}(\beta,\beta) = \sum_{r=k}^{s}(-1)^{s-r}\binom{s}{r}\binom{r}{k}r^{\underline{k}}$$

and Equation (2.5) yields

$$a_k^{\beta,\beta}(f,x) = \sum_{s=1}^{2k} \frac{f^{(s)}(x)}{s!} x^{s-k} \sum_{r=k}^{s} (-1)^{s-r} \binom{s}{r}\binom{r}{k} r^{\underline{k}}$$
$$= \sum_{s=1}^{2k} \binom{s}{k} \frac{f^{(s)}(x)}{s!} x^{s-k} \sum_{r=k}^{s} (-1)^{s-r} \binom{s-k}{r-k} r^{\underline{k}} \qquad (k=1,2,\ldots).$$

Observing that

$$\sum_{r=k}^{s} (-1)^{s-r} \binom{s-k}{r-k} r^{\underline{k}} = \sum_{r=0}^{s-k} (-1)^{s-k-r} \binom{s-k}{r} (r+k)^{\underline{k}}$$
$$= \sum_{r=0}^{s-k} (-1)^{s-k-r} \binom{s-k}{r} \left(\left(\frac{d}{dy}\right)^k y^{r+k}\right)\Bigg|_{y=1}$$
$$= \left(\left(\frac{d}{dy}\right)^k y^k (y-1)^{s-k}\right)\Bigg|_{y=1}$$
$$= \binom{k}{s-k} \frac{k!}{(s-k)!} (s-k)! = k! \binom{k}{s-k}$$

(note that $\binom{k}{s-k} = 0$ if $2 \le 2k < s$), we conclude that

$$a_k^{\beta,\beta}(f,x) = \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{(s-k)!} \binom{k}{s-k} x^{s-k} = \sum_{s=0}^{k} \binom{k}{s} \frac{x^s}{s!} f^{(k+s)}(x).$$

This completes the proof of the corollary. □

## References


[1] U. Abel and P. G. Patel: *An asymptotic expansion for the Mittag-Leffler operators*, Indian J. Pure Appl. Math., (2025). https://doi.org/10.1007/s13226-025-00908-0

[2] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogozin: *Mittag-Leffler functions, related topics and applications*, Springer, Berlin (2020).

[3] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark: *NIST handbook of mathematical functions*, Cambridge Univ. Press, New York (2010).

[4] J. Paneva-Konovska: *Inequalities and asymptotic formulae for the three parametric Mittag-Leffler functions*, Math. Balk., **26** (2012), 203–210.

[5] P. G. Patel: *On positive linear operators linking gamma, Mittag-Leffler and Wright functions*, Int. J. Appl. Comput. Math., **10** (2024), Article ID:152.

[6] P. G. Patel: *Some properties of Wright operators*, arXiv:2404.04651, (2024).

[7] P. G. Patel: *Generalized gamma-Wright integral operators: Approximation properties*, Proc. Int. Conf. Math. Adv. Appl., **2** (2025), 40–48.

[8] T. R. Prabhakar: *A singular integral equation with a generalized Mittag Leffler function in the kernel*, Yokohama Math. J., **19** (1) (1971), 7–15.

[9] P. C. Sikkema: On some linear positive operators, Indag. Math., **73** (1970), 327–337.

[10] E. M. Wright: *On the coefficients of power series having exponential singularities*, J. Lond. Math. Soc., **8** (1) (1933), 71–79.



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