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Research Article

Linear algebra and power method combined with method of deflation applied to Toeplitz sinc matrices: A new approach for the computation of eigenvalues and eigenvectors

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ABSTRACT. A new robust method for the computation of the eigenvalues and eigenvectors of real symmetric sinc $n{\times}n$-matrices $A(t) = A(t,n)$ with t in $0 < t < 1$ is developed. If the eigenvalues $\mu_j(t) = \mu_j(A(t)) = \mu_j(A(t,n))$, $j = 1,\dots,n$ are arranged in descending order, one has $\mu_1(t) \to n\ (t \to 0)$ and $\mu_j(t) \to 0\ (t \to 0)$, $j = 2,\dots,n$. The new approach consists in computing only a limited number i_q of the eigenvalues and to set the remaining very small eigenvalues equal to zero, of which the corresponding eigenvectors are reconstructed via the Gram-Schmidt process. This opens the way for conceiving a new method for the computation of the eigenvalues and associated eigenvectors of severely ill-conditioned real symmetric Toeplitz sinc matrices that delivers good results.

Keywords: Condition number, eigenvalues and eigenvectors, method of deflation, power method, Toeplitz sinc matrix.

2020 Mathematics Subject Classification: 15B05, 65F15, 65F22, 65F25.

1. INTRODUCTION

The purpose of this introduction is to deal with the following questions:

Question 1. *Why is this paper written?*

Question 2. *What is its relation to other literature on the subject?*

Question 3. *What is new in this paper, and perhaps what makes it unique?*

Question 4. *What is its content?*

These questions are interrelated, and the answers are partly interwoven. Problems arise in the computation of eigenvalues and associated eigenvectors of Toeplitz sinc $n \times n$-matrices $A(t) = A(t,n)$ that are ubiquitous in digital signal processing. These are made up of the entries $s(0) := 1$ and $s(jt) := \sin(j\pi t)/(j\pi t)$, $j = 1,\dots,n-1$ and are investigated for $0 < t < 1$. The eigenvalues $\mu_j(t) = \mu_j(A(t)) = \mu_j(A(t,n))$, $j = 1,\dots,n$ are positive and are assumed to be arranged in descending order. Since $\mu_1(t) \to n\ (t \to 0)$ and $\mu_j(t) \to 0\ (t \to 0)$, $j = 2,\dots,n$, see [4], for the condition number $\kappa_2(t)$, one obtains $\kappa_2(t) = \mu_1(t)/\mu_n(t) \to \infty\ (t \to 0)$. This leads to numerical problems in the determination of eigenvalues and associated eigenvectors for small values of t since on a computer the set of real numbers $\mathbb{R}$ is replaced by machine numbers with only a restricted number of digital places.

Special Toeplitz matrices, namely those that commute with tridiagonal matrices were discussed by Grünbaum in [1]. However, apart from its credentials, it has some shortcomings.

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For example, in [8], the authors proved that the tridiagonal matrices used there have distinct eigenvalues whereas Grünbaum had to assume this. A weak point of this paper is also that it does not present numerical results. In [4], we discussed also the paper of Hertz [2] titled "Simple Bounds on the Extreme Eigenvalues of Toeplitz Matrices", and we concluded that the obtained results have merits, but have only limited applicability.

This paper is written in order to do without the restriction in [1] that the Toeplitz sinc matrix should commute with a tridiagonal matrix, and we choose a totally different approach to determine the eigenvalues and associated eigenvectors of a Toeplitz sinc $n \times n$-matrix $A(t) = A(t, n)$.

This new approach starts from the following observation. Since $\mu_j(t) \to 0 \ (t \to 0), \ j = 2, \ldots, n$, for small values of t in $0 < t < 1$ we determine an index i_q such that $\mu_j(t)$ is less than a prescribed value for $j = i_q + 1, \ldots, n$ and set $\mu_j(t) = 0$, $j = i_q + 1, \ldots, n$. In other words, we replace the eigenvalues $\mu_j(t)$, $j = 1, \ldots, n$ by $\mu_j(t)$, $j = 1, \ldots, i_q$ and $\mu_j(t) = 0$, $j = i_q + 1, \ldots, n$. Then, the eigenvalues $\mu_j(t)$, $j = 1, \ldots, i_q$ and associated eigenvectors can be determined without difficulty by the power method combined with the method of deflation, and for the eigenvalues $\mu_j(t) = 0$, $j = i_q + 1, \ldots, n$, many systems of orthonormal eigenvectors can easily be given where we choose a very simple one. Application of the Gram-Schmidt orthonormalization process to the eigenvectors corresponding to the eigenvalues $\mu_j(t)$, $j = 1, \ldots, i_q$ and $\mu_j(t) = 0$, $j = i_q + 1, \ldots, n$ delivers for the whole set of eigenvalues $\mu_j(t)$, $j = 1, \ldots, i_q, i_q + 1, \ldots, n$ and the associated eigenvectors $w_j(t)$, $j = 1, \ldots, i_q, i_q + 1, \ldots, n$ good approximations.

The important point is that we avoid-for small values of t in $0 < t < 1$-the difficulty caused by $\mu_j(t) \to 0 \ (t \to 0), \ j = 2, \ldots, n$ by selecting a limited number i_q of eigenvalues that are computed whereas simple eigenvectors for the other eigenvalues $\mu_j(t) = 0$, $j = i_q + 1, \ldots, n$ can easily be given. This idea seems to be unique, so far. This was also the reason why we commented only shortly on other methods for the computation of eigenvalues and associated eigenvectors of Toeplitz sinc matrices.

Next, we come to the content of this paper; it is organized as follows. In Section 2, Linear-Algebra Tools are collected consisting in material on Projections and Invariance of Vector Spaces under Linear Transformations in Subsection 2.1, Real Symmetric Matrices in Subsection 2.2, Gram-Schmidt Orthonormalization in Subsection 2.3, and Toeplitz Sinc Matrices in Subsection 2.4. These results are known, but the presentation is much more detailed than can be found in literature. In Section 3, the Power Method for a Real Symmetric Matrix with Non-Negative Eigenvalues is described which is used to compute its largest positive eigenvalue. In Section 4, the Method of Deflation is discussed. In combination with the power method, it allows one to determine all eigenvalues of a positive definite matrix. In Section 5, the obtained results are applied to Toeplitz sinc matrices $A(t, n)$ for $t = 0.1$ with $n = 5$ in Subsection 5.1, with $n = 10$ in Subsection 5.2, and with $n = 15$ in Subsection 5.3, and Subsection 5.4 contains a scrutiny of the computed eigenvectors. Finally, in Section 6, conclusions are drawn.

The style of the paper is expository in order to address a large readership. Only cited literature is given in the References.

2. Collection of Linear-Algebra Tools

This section collects those tools from Linear Algebra that play an important role in this paper. It is organized in subsections and contains, for instance, material on projections and invariance properties of linear transformations, on eigenvalues and eigenvectors of real symmetric matrices, on the Gram-Schmidt orthonormalization process as well as on Toeplitz sinc matrices.

The underlying vector spaces over the field $\mathbb{F} = \mathbb{R}$ resp. $\mathbb{F} = \mathbb{C}$ are supposed to be finite dimensional, and the operators (also termed mappings or transformations) are represented by matrices.

2.1. **Projections and Invariance of Vector Spaces under Linear Operators.** The first subsection is taken from [3, Chapter I, §3, Sections 3. and 4., pp.20-23] with many verbatim passages.

In a (finite-dimensional) vector space, the projection operator is introduced. It is idempotent and is associated with the notions complementary space or the direct sum of subspaces. Further, linear operators that map a subspace into itself are considered. One says that they let the corresponding subspace invariant or that the subspace is invariant under the linear operators.
(i) Projections
Let M, N be complementary subspaces in the vector space X; by this, we mean that

$$X = M \oplus N,$$

i.e., each $u \in X$ can be uniquely expressed in the form $u = u' + u''$ with $u' \in M$ and $u'' \in N$, see [3, Chapter I, §3, Section 4.]. We also say that X is the direct sum of M and N. The element u' is called the projection of u on M along N. If $v = v' + v''$ in the same sense, $\alpha\, u + \beta\, v$ has the projection $\alpha\, u' + \beta\, v'$ on M along N. If we set $u' = P\,u$, it follows that P is a linear operator on X to itself. P is called projection operator (or simply projection) on M along N. We have $P\,u = u$ if and only if $u \in M$ and $Pu = 0$ if and only if $u \in N$. The range of P is M and the null space of P is N. Since $P\,u \in M$, we have $P(Pu) = Pu$, that is, P is idempotent:

$$P^2 = P.$$

Conversely, any idempotent operator P is a projection. In fact, set $M = R(P)$, where $R(P)$ is the range of P and $N = R(Q) = R(I - Q)$. Then, $u' \in M$ implies that $u' = Pu$ for some $u \in X$ and therefore $Pu' = P^2u = Pu = u'$. Similarly, $u'' \in N$ implies $Pu'' = 0$. Hence, $u \in M \cap N = \{0\}$. Each $u \in X$ has the expression $u = u' + u''$ with $u' = Pu \in M$ and $u'' = (I - P)u \in N$. This shows that P is the projection on M along N.
(ii) Invariance of Vector Spaces under Linear Operators
This part is taken from [3, Chapter I, §3, Section 5.]. A subspace M of a vector space X is said to be invariant under a linear operator $T \in L(X)$ if $TM \subset M$. In this case, T induces a linear operator T_M on M to M defined by $T_M u = Tu$ for $u \in M$. If there are two invariant subspaces M, $N \subset X$ such that $X = M \oplus N$, T is said to be decomposed (or reduced) by the pair M, N.

2.2. **Real Symmetric Matrices.** In this subsection, we investigate the eigenvalues and associated eigenvectors of real symmetric $n \times n$-matrices B in the space of n-tuples $H = \mathbb{R}^{n\times n}$ endowed with the scalar product $(\cdot,\cdot) = (\cdot,\cdot)_2$. We shall see that there is a basis of real orthonormal eigenvectors $w_j, j = 1,\dots,n$ associated with the eigenvalues $\lambda_j = \lambda_j(B)$, $j = 1,\dots,n$ and standard eigenvector expansions for u and Bu with arbitrary $u \in H$. In the special case when $\lambda_j \neq 0$, $j = 1,\dots,m$ with $m < n$, we derive what can be called displayed eigenvector expansions for u and Bu with arbitrary $u \in H$. For $M_m := [w_1,\dots,w_m]$, one can show that $BM_m \subset M_m$ and that B is invertible on M_m.
(i) Eigenvalues and Eigenvectors
We have the following theorem.

Theorem 2.1 (Real eigenvalues and real eigenvectors of real symmetric mappings)**.** *Let the matrix $B = (b_{jk})_{j,k=1,\dots,n}$ be a real symmetric mapping in the space of n-tuples $H = \mathbb{R}^n$ endowed with the scalar product $(\cdot,\cdot) = (\cdot,\cdot)_2$. Then, the eigenvalues $\lambda_1,\dots,\lambda_n$ of B are real, and there is a corresponding orthonormal system of real eigenvectors $w_1,\dots,w_n$ with*

$$Bw_j = \lambda_j w_j, \quad (w_j, w_k) = \delta_{jk}, \quad j,k = 1,\dots,n.$$

Proof. First, we consider B in the space of n-tuples $\mathbb{F}^n = \mathbb{C}^n$. Then, the proof follows from [5, (15), p.91], where the eigenvectors can be elements of $\mathbb{C}^n$. Now, $w_j = Re\, w_j + iIm\, w_j = w_j^{(r)} + iw_j^{(i)}$, $j = 1, \ldots, n$ with $w_j^{(r)} = Re\, w_j \quad and \quad w_j^{(i)} = Im\, w_j$, $j = 1, \ldots, n$. Since $w_j \neq 0$, one has $w_j^{(r)} \neq 0$ or $w_j^{(i)} \neq 0$ for $j = 1, \ldots, n$. If one replaces $w_j \in \mathbb{C}^n$ by $w_j^{(r)}$ if $w_j^{(r)} \neq 0$ or $w_j \in \mathbb{C}^n$ by $w_j^{(i)}$ if $w_j^{(i)} \neq 0$ for $j = 1, \ldots, n$, one obtains, after normalization, real normed eigenvectors w_j, $j = 1, \ldots, n$. They are orthonormal or can be made orthonormal. To see this, consider two eigenvalues $\lambda_k \neq \lambda_l$. Then, $\lambda_k(w_k, w_l) = (\lambda_k w_k, w_l) = (Bw_k, w_l) = (w_k, Bw_l) = (w_k, \lambda_l w_l) = \lambda_l(w_k, w_l)$ leading to $(\lambda_k - \lambda_l)(w_k, w_l) = 0$ or $(w_k, w_l) = 0$. For $\lambda_k = \lambda_l$, the Gram-Schmidt procedure delivers orthonormal, real eigenvectors w_k, w_l. □

We mention that we have included the proof of Theorem 2.1 since the last case is usually not treated explicitly, at least not in engineering literature. From Theorem 2.1, we get the following theorem.

Theorem 2.2 (Standard eigenvector expansions). *Let the matrix $B = (b_{jk})_{j,k=1,\ldots,n}$ be a real symmetric mapping in the space of n-tuples $H = \mathbb{R}^n$ endowed with the scalar product $(\cdot,\cdot) = (\cdot,\cdot)_2$. Let $\lambda_1, \ldots, \lambda_n$ be the real eigenvalues of B, and let $w_1, \ldots, w_n$ be a corresponding orthonormal system of real eigenvectors so that*

$$Bw_j = \lambda_j w_j, \quad (w_j, w_k) = \delta_{jk}, \quad j, k = 1, \ldots, n.$$

Then, the following eigenvector expansions hold:

$$u = \sum_{j=1}^{n} (u, w_j) w_j, \quad u \in H,$$

$$Bu = \sum_{j=1}^{n} \lambda_j (u, w_j) w_j, \; u \in H.$$

Proof. The proof follows immediately from Theorem 2.1. □

(ii) Special Case When $\lambda_j = 0$ for $j = m+1, \ldots, n$
Let $m \in \mathbb{N}$ with $1 \leq m < n$. Then, we consider the special case

$$\lambda_j \neq 0, \; j = 1, \ldots, m, \quad \lambda_j = 0, \; j = m+1, \ldots, n.$$

Set

$$M_m = [w_1, \ldots, w_m], \quad M_0 = [w_{m+1}, \ldots, w_n].$$

Then,

$$H = M_m \oplus M_0$$

with

$$M_0 = M_m^{\perp}.$$

This situation leads to the subsequent corollary.

Corollary 2.1 (Displayed eigenvector expansions in special case). *In the above special case, the following eigenvector expansions for u and Bu are valid:*

$$u = \sum_{j=1}^{m} (u, w_j) w_j + P_0 u, \quad u \in H$$

where $P_0 : H \mapsto N(B)$ is the projection of H on the null space $N(B)$ and where P_0 is defined by

$$P_0 u = \sum_{j=m+1}^{n} (u, w_j) w_j, \quad u \in H.$$

Further,

$$Bu = \sum_{j=1}^{m} \lambda_j (u, w_j) w_j, \quad u \in H.$$

Proof. We show that $P_0 : H \mapsto N(B)$ and that it is a projection. First,

$$\begin{aligned} B(P_0 u) =& B \left[\sum_{j=m+1}^{n} (u, w_j) w_j \right] = \sum_{j=m+1}^{n} (u, w_j) B w_j \\ =& \sum_{j=m+1}^{n} (u, w_j) 0 w_j = 0, \quad u \in H \end{aligned}$$

so that $B : H \mapsto N(B)$. Further, $u' := P_0 u$ is uniquely determined by the representation of $P_0 u$ and is therefore a projection according to Subsection 2.1. Next, we have

$$\begin{aligned} Bu =& B \left[\sum_{j=1}^{m} (u, w_j) w_j + \sum_{j=m+1}^{n} (u, w_j) w_j \right] \\ =& \sum_{j=1}^{m} (u, w_j) B w_j + \sum_{j=m+1}^{n} (u, w_j) B w_j \\ =& \sum_{j=1}^{m} (u, w_j) \lambda_j w_j + \sum_{j=m+1}^{n} (u, w_j) 0 w_j \\ =& \sum_{j=1}^{m} (u, w_j) \lambda_j w_j, \quad u \in H. \end{aligned}$$

□

By the way,

- we have $M_0 = M_m^{\perp}$ and thus $H = M_m \oplus M_0 = M_m \oplus M_m^{\perp}$,
- the standard eigenvector expansions in Theorem 2.2 remain valid, of course, but are less precise,
- the displayed eigenvector expansions for u and Bu in Corollary 2.1 were inspired by the first vector expansion in [6, Satz 7.3, p.34] as well as the first vector expansion in [6, Satz 7.2, p.32], as the case may be.

(iii) Invertibility of the Operator B on M_m

We have $BM_m \subset M_m$, that is, M_m is invariant under the linear operator B. This leads to the following theorem.

Theorem 2.3 (Eigenvector expansion of B^{-1} on M_m)**.** *In the above special case, the $n \times n$-matrix B restricted to M_m is invertible and has the eigenvector expansion*

$$B^{-1} u = \sum_{j=1}^{m} \frac{1}{\lambda_j} (u, w_j) w_j, \; u \in M_m.$$

Proof. To prove this, let $u \in M_m$. Then,

$$\begin{aligned} u =& \sum_{j=1}^{m}(u,w_j)w_j = \sum_{j=1}^{m}\lambda_j(u,w_j)\frac{1}{\lambda_j}w_j \\ =& \sum_{j=1}^{m}\lambda_j(u,w_j)B^{-1}w_j = B^{-1}\left[\sum_{j=1}^{m}\lambda_j(u,w_j)w_j\right] \\ =& B^{-1}(Bu) = (B^{-1}B)u,\ u \in M_m \end{aligned}$$

so that B has a left inverse on M_m. Similarly,

$$\begin{aligned} u =& \sum_{j=1}^{m}(u,w_j)w_j = \sum_{j=1}^{m}\frac{1}{\lambda_j}(u,w_j)\lambda_j w_j \\ =& \sum_{j=1}^{m}\frac{1}{\lambda_j}(u,w_j)Bw_j = B\left[\sum_{j=1}^{m}\frac{1}{\lambda_j}(u,w_j)w_j\right] \\ =& B(B^{-1}u) = (BB^{-1})u,\ u \in M_m \end{aligned}$$

so that B has a right inverse on M_m. On the whole, we have proven that B has an inverse on M_m that commutes with B. □

We mention that in the case $m = n$, one has $M_m = H$. Then, it follows that B is invertible on H.

2.3. **Gram-Schmidt Orthonormalization.** We have the following theorem.

Theorem 2.4 (Gram-Schmidt orthonormalization process)**.** *Let E be a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ endowed with the scalar product $(\cdot,\cdot)$. Further, let $u_1,\ldots,u_m$ be a system of linearly independent vectors in E. Then, from the formulae*

$$\begin{aligned} v_1 &= u_1, & w_1 &= \tfrac{1}{\|v_1\|}v_1 \\ v_j &= u_j - \sum_{k=1}^{j-1}(u,w_k)w_k, & w_j &= \tfrac{1}{\|v_j\|}v_j,\ j = 2,\ldots,m \end{aligned}$$

an orthogonal system $v_1,\ldots,v_m$ and an orthonormal system $w_1,\ldots,w_m$ are obtained and are such that

$$M_j = [u_1,\ldots,u_j] = [v_1,\ldots,v_j] = [w_1,\ldots,w_j],\ j = 1,\ldots,m.$$

Proof. See [5, Section 7.1(3), p.111]. □

2.4. **Toeplitz Sinc Matrices.** In this subsection, we collect the important properties of Toeplitz sinc matrices obtained in [4] and [7]. They serve, in Section 4, to test whether the power method combined with the method of deflation are appropriate in computing the eigenvalues and associated eigenvectors with a given precision. Many verbatim passages in this subsection are taken from [4].

A Toeplitz sinc matrix $A(t) = A_n(t) = A(t,n)$ is defined as

$$
(2.1)\quad A(t) = A(t,n) = \begin{bmatrix} s(0) & s(t) & s(2\,t) & \cdots & s((n-2)\,t) & s((n-1)\,t) \\ s(t) & s(0) & s(t) & \cdots & s((n-3)\,t) & s((n-2)\,t) \\ s(2\,t) & s(t) & s(0) & \cdots & & s((n-3)\,t) \\ & & & \vdots & & \\ s((n-1)\,t) & s((n-2)\,t) & & \cdots & s(t) & s(0) \end{bmatrix}
$$

where $0 < t < 1$ and $s(0) = 1$ as well as $s(t) = sinc(t) = \sin(\pi t)/(\pi t)$. From [7, Theorem 2.2], it follows that this matrix is positive definite by setting there $t_1 = 0, t_i = (i-1)\,t,\ i = 2,\dots,n$ and taking into account that $s(-t) = s(t)$ for $0 < t < 1$. As t gets smaller, the condition number of this matrix deteriorates quickly. As a rule, this leads to numerical problems, for instance, when one wants to compute the eigenvalues and associated eigenvectors. For the above Toeplitz sinc matrix, we state the following theorem.

Theorem 2.5. *Let the matrix $A(t) = A_n(t) = A(t,n)$ in (2.1) be given. Further, let the eigenvalues $\mu_j(t) = \mu_j(A(t)) = \mu_j(A_n(t)),\ j = 1,\dots,n$ be arranged according to*

$$
(2.2)\qquad \mu_1(t) \ge \mu_2(t) \ge \cdots \ge \mu_n(t).
$$

Then, the limits

$$
(2.3)\qquad A := \lim_{t\to 0} A(t)
$$

as well as

$$
(2.4)\qquad \mu_j = \mu_j(A) := \lim_{t\to 0} \mu_j(t) = \lim_{t\to 0} \mu_j(A(t)), \quad j = 1,\dots,n
$$

exist and

$$
(2.5)\qquad A = A_n = \lim_{t\to 0} A(t) = \lim_{t\to 0} A_n(t) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ & & & \vdots & & \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.
$$

Further, the limits in (2.4) are eigenvalues of A, and with appropriate enumeration of the eigenvalues $\mu_j := \mu_j(A),\ j = 1,\dots,n$, one has

$$
(2.6)\qquad \lim_{t\to 0} \mu_1(A(t)) = \lim_{t\to 0} \mu_1(A_n(t)) = \mu_1(A) = \mu_1 = n,
$$

$$
(2.7)\qquad \lim_{t\to 0} \mu_j(A(t)) = \lim_{t\to 0} \mu_j(A_n(t)) = \mu_j(A) = \mu_j = 0,\ j = 2,\dots,n.
$$

Proof. See [4, Theorem 2.1]. □

We continue with comments on these results. For the special limits $\lim_{t\to 0} s(t)$ and $\lim_{t\to 1} s(t)$, one obtains

$$
\lim_{t\to 0} s(t) = \lim_{t\to 0} \frac{\sin(\pi t)}{\pi t} = 1 \quad \textit{and} \quad \lim_{t\to 1} s(t) = \lim_{t\to 1} \frac{\sin(\pi t)}{\pi t} = 0.
$$

Further, the limit of the condition number $\kappa_2(t) = \mu_1(t)/\mu_n(t)$ is given by

$$
\lim_{t\to 0} \kappa_2(t) = \infty
$$

which indicates that problems might arise in numerical computations involving Toeplitz sinc matrices for small values of t in $0 < t < 1$. But, we have seen in Theorem 2.5 that the eigenvalues of the limit matrix $A = \lim_{t\to 0} A(t) = \lim_{t\to 0} A_n(t)$ are given by $\mu_1(A) = n$ and $\mu_j(A) = 0,\ j = 2, \ldots, n$.

Moreover, a set of eigenvectors $u_1, \ldots, u_n$ of A is given by

$$U = [u_1 \,|\, u_2, \ldots, u_n] = \left[\begin{array}{c|cccccc} 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 1 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & -1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & -1 \end{array}\right]$$

with the property $rank(U) = n$ as well as

$$(u_1, u_1) = n, \quad (u_1, u_j) = 0,\ j = 2, \ldots, n,$$

see [4, Section 8, for $n = 5$] where we used the corresponding normed vectors, instead. Here, u_1 is eigenvector associated with the largest eigenvalue $\mu_1 = \mu_1(A) = n$, and $u_j, j = 2, \ldots, n$ are eigenvectors associated with the eigenvalues $\mu_j = \mu_j(A) = 0, j = 2, \ldots, n$. Starting from these eigenvectors, applying the Gram-Schmidt process to $u_1, \ldots, u_n$, orthonormal eigenvectors $w_1, \ldots, w_n$ with $[w_1, \ldots, w_n] = [u_1, \ldots, u_n]$ can be found. This will be exploited in Section 5.

3. Power Method for Real Symmetric Matrices with Non-Negative Eigenvalues

This section discusses the power method for symmetric real $n \times n$-matrices B with non-negative eigenvalues $\lambda_j, j = 1, \ldots, n$ arranged in descending order. Since this method determines the largest positive eigenvalue, it is clear that among the eigenvalues must be at least a positive one. Otherwise, all eigenvalues were equal to zero, a particular case that is not further investigated in this paper since it does not seem to have interesting applications.

In the following, we give a detailed account of the convergence of the power method exhibiting that - under mild conditions - it computes the largest positive eigenvalue and a corresponding normed eigenvector.

All other positive eigenvalues and associated orthonormal eigenvectors can be obtained by the power method when it is combined with the method of deflation, the last one being described in the next section. We take over most passages from [5, Subsection 10.1.1] on the power method for symmetric matrices $B \in \mathbb{R}^{n\times n}$ with non-negative eigenvalues $\lambda_j,\ j = 1, \ldots, n$ arranged according to

$$\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n \geq 0. \tag{3.8}$$

However, the presentation is much more elaborate. The power method or von Mises method considers, first, for an arbitrarily chosen, but fixed initial vector $u^{(0)} \neq 0$ in $\mathbb{F}^n = \mathbb{R}^n$, the corresponding sequence of iterated vectors

$$u^{(\imath+1)} = Bu^{(\imath)},\ \imath = 0, 1, 2, \ldots .$$

Using the orthonormal basis of eigenvectors of B, one can express, with $u^{(0)} = u$, these iterated vectors as

$$u^{(\imath)} = B^{\imath} u = \sum_{k=1}^{n} \lambda_k^{\imath} (u, w_k) w_k, \;\; \imath = 0, 1, 2, \dots . \tag{3.9}$$

If the initial vector $u = u^{(0)}$ satisfies the hypothesis

$$u^{(1)} = Bu^{(0)} \neq 0 \quad \text{or equivalently} \quad \|u^{(1)}\|^2 = \|Bu\|^2 = \sum_{k=1}^{n} \lambda_k^2 |(u, w_k)|^2 > 0,$$

then there is a natural number m such that

$$\lambda_m (u, w_m) \neq 0, \;\; \lambda_k (u, w_k) = 0, \;\; k < m. \tag{3.10}$$

In the expression (3.9) for $u^{(\imath)}$, we assume that $\lambda = \lambda_m$ is the largest positive eigenvalue that occurs, and for sufficiently large values of $\imath$ the terms in $\lambda_k = \lambda$ dominate the other terms with $\lambda_k < \lambda$ as far as these occur in the summation. Accordingly, we split up the sum in (3.9), and writing

$$\lambda = \lambda_m > 0, \quad z = \sum_{\substack{k=m \\ \lambda_k = \lambda}}^{n} (u, w_k) w_k, \quad r^{(\imath)} = \sum_{\substack{k=m \\ \lambda_k < \lambda}}^{n} \left(\frac{\lambda_k}{\lambda} \right)^{\imath} (u, w_k) w_k, \tag{3.11}$$

we can express the iterated vectors $u^{(\imath)}$ as

$$u^{(\imath)} = \lambda^{\imath} (z + r^{(\imath)}), \;\; \imath = 1, 2, \dots . \tag{3.12}$$

From the representation in (3.11), we see that the vector z is orthogonal to $r^{(\imath)}$, and so the relation

$$\|u^{(\imath)}\| = \lambda^{\imath} (\|z\|^2 + \|r^{(\imath)}\|^2)^{\frac{1}{2}}, \;\; \imath = 1, 2, \dots \tag{3.13}$$

holds. The vector z is a linear combination of the orthonormal eigenvectors of B for the eigenvalue $\lambda = \lambda_m$, and by condition (3.10), $z \neq 0$, and so z is an eigenvector corresponding to the eigenvalue λ, i.e.,

$$Az = \lambda z, \;\; z \neq 0.$$

Let λ' be the largest positive eigenvalue of B with $\lambda' = \lambda_k < \lambda_m = \lambda$ for $k = m+1, \dots, n$ if there is such a one, and otherwise let $\lambda' = 0$. Then, the vectors $r^{(\imath)}$ in (3.11) converge to zero according to the estimate

$$0 \leq q = \frac{\lambda'}{\lambda} < 1, \quad \|r^{(\imath)}\| \leq q^{\imath} \|r^{(0)}\| \to 0 \; (\imath \to \infty). \tag{3.14}$$

As a result of these considerations, we therefore have the convergence of the normalized iterated vectors

$$v^{(\imath)} = \frac{u^{(\imath)}}{\|u^{(\imath)}\|} = \frac{z + r^{(\imath)}}{\|z + r^{(\imath)}\|} \to \frac{z}{\|z\|} = w \; (\imath \to \infty),$$

i.e., convergence to the normalized eigenvector $w = z/\|z\|$ of B corresponding to the eigenvalue λ. We determine the corresponding approximations to the eigenvalue by means of the Rayleigh quotients of B in the form

$$\varrho(u^{(\imath)}) = \frac{(Bu^{(\imath)}, u^{(\imath)})}{\|u^{(\imath)}\|^2} = (Bv^{(\imath)}, v^{(\imath)}) = \varrho(v^{(\imath)}), \quad \imath = 0, 1, 2, \dots ,$$

and we continue with a two-sided estimate on $(Bu^{(\imath)}, u^{(\imath)})$ contained in the following lemma.

Lemma 3.1 (Two-sided estimate on $(Bu^{(\imath)}, u^{(\imath)})$). *From the representation*

$$u^{(\imath)} = B^{\imath} u = \sum_{k=1}^{n} \lambda_k^{\imath}(u, w_k) w_k, \ \ \imath = 0, 1, 2, \dots,$$

we obtain the two-sided estimate

$$\lambda^{2\imath+1} \|z\|^2 \leq (Bu^{(\imath)}, u^{(\imath)}) \leq \sum_{k=1}^{n} \lambda_k^{2\imath+1} |(u, w_k)|^2 \leq \lambda \, \|u^{(\imath)}\|^2, \quad \imath = 0, 1, 2, \dots .$$

Proof. One has

$$Bu^{(\imath)} = B[\, \lambda^{\imath}(z + r^{(\imath)})\,] = \lambda^{\imath}[Bz + Br^{(\imath)}\,] = \lambda^{\imath}[\lambda z + Br^{(\imath)}\,] = \lambda^{\imath+1} z + \lambda^{\imath} Br^{(\imath)},$$

where $z \perp Br^{(\imath)}$, $\imath = 0, 1, 2, \dots$ due to the representation (3.11). Herewith, we can express the iterated vectors $u^{(\imath)}$ as

$$u^{(\imath)} = \lambda^{\imath}(z + r^{(\imath)}), \quad \imath = 0, 1, 2, \dots$$

leading to

$$\begin{aligned}
(Bu^{(\imath)}, u^{(\imath)}) =& (\lambda^{\imath+1} z + \lambda^{\imath} Br^{(\imath)}, \lambda^{\imath}(z + r^{(\imath)})) \\
=& \lambda^{2\imath+1}(z, z) + \lambda^{2\imath+1} \underbrace{(z, r^{(\imath)})}_{=0} + \lambda^{2\imath} \underbrace{(Br^{(\imath)}, z)}_{=0} + \lambda^{2\imath} \underbrace{(Br^{(\imath)}, r^{(\imath)})}_{\geq 0} \\
\geq& \lambda^{2\imath+1} \|z\|^2, \quad \imath = 0, 1, 2, \dots
\end{aligned}$$

so that we have the lower estimate

$$\lambda^{2\imath+1} \|z\|^2 \leq (Bu^{(\imath)}, u^{(\imath)}), \quad \imath = 0, 1, 2, \dots .$$

Further,

$$Bu^{(\imath)} = BB^{\imath} u = \sum_{k=1}^{n} \lambda_k^{\imath}(u, w_k) B w_k = \sum_{k=1}^{n} \lambda_k^{\imath+1}(u, w_k) w_k, \ \ \imath = 0, 1, 2, \dots$$

so that with $\lambda_k \leq \lambda$,

$$(Bu^{(\imath)}, u^{(\imath)}) = \sum_{k=1}^{n} \lambda_k^{2\imath+1} |(u, w_k)|^2 \leq \lambda \sum_{k=1}^{n} \lambda_k^{2\imath} |(u, w_k)|^2 = \lambda \, \|u^{(\imath)}\|^2, \quad \imath = 0, 1, 2, \dots .$$

Thus we get, indeed, the upper bound

$$(Bu^{(\imath)}, u^{(\imath)}) = \sum_{k=1}^{n} \lambda_k^{2\imath+1} |(u, w_k)|^2 \leq \lambda \, \|u^{(\imath)}\|^2, \quad \imath = 0, 1, 2, \dots .$$

□

From the upper bound in Lemma 3.1, we see that

$$0 \leq \lambda - \frac{(Bu^{(\imath)}, u^{(\imath)})}{\|u^{(\imath)}\|^2} = \lambda - \varrho(u^{(\imath)}) = \lambda - \varrho(v^{(\imath)}), \quad \imath = 0, 1, 2, \dots ,$$

i.e.,

$$0 \leq \lambda - \varrho(v^{(\imath)}), \quad \imath = 0, 1, 2, \dots .$$

In the next lemma, we estimate the right-hand side further from above.

Lemma 3.2 (Estimate from above on $0 \le \lambda - \varrho(v^{(\imath)}),\ \imath = 0,1,2,\dots$). *For* $0 \le \lambda - \varrho(v^{(\imath)}),\ \imath = 0,1,2,\dots$, *we obtain the upper estimate*

$$0 \le \lambda - \varrho(v^{(\imath)}) \le \lambda\, q^{2\imath} \frac{\|r^{(0)}\|^2}{\|z\|^2}, \quad \imath = 0,1,2,\dots. \tag{3.15}$$

Proof. We depart from the expression for $(Bu^{(\imath)}, u^{(\imath)})/\|u^{(\imath)}\|^2$ yielding with $u^{(\imath)} = \lambda^\imath(z + r^{(\imath)})$,

$$\begin{aligned}\frac{(Bu^{(\imath)}, u^{(\imath)})}{\|u^{(\imath)}\|^2} &= \frac{(B[\lambda^\imath(z + r^{(\imath)})], \lambda^\imath[z + r^{(\imath)}])}{(\lambda^\imath(z + r^{(\imath)}), \lambda^\imath(z + r^{(\imath)}))} = \frac{(B(z + r^{(\imath)}), z + r^{(\imath)})}{(z + r^{(\imath)}, z + r^{(\imath)})}\\ &= \frac{(\lambda z + Br^{(\imath)}, z + r^{(\imath)})}{(z + r^{(\imath)}, z + r^{(\imath)})}, \quad \imath = 0,1,2,\dots.\end{aligned}$$

Now, $z \perp r^{(\imath)}$ leading to

$$(z + r^{(\imath)}, z + r^{(\imath)}) = \|z + r^{(\imath)}\|^2 = \|z\|^2 + \|r^{(\imath)}\|^2 \ge \|z\|^2$$

and

$$(\lambda z + Br^{(\imath)}, z + r^{(\imath)}) = \lambda\,\|z\|^2 + (Br^{(\imath)}, r^{(\imath)}).$$

Hereby,

$$\frac{(Bu^{(\imath)}, u^{(\imath)})}{\|u^{(\imath)}\|^2} = \frac{\lambda\,\|z\|^2 + (Br^{(\imath)}, r^{(\imath)})}{\|z\|^2 + \|r^{(\imath)}\|^2}.$$

This yields

$$\begin{aligned}0 \le& \lambda - \varrho(v^{(\imath)}) = \lambda - \varrho(u^{(\imath)}) = \lambda - \frac{(Bu^{(\imath)}, u^{(\imath)})}{\|u^{(\imath)}\|^2} = \lambda - \frac{\lambda\,\|z\|^2 + (Br^{(\imath)}, r^{(\imath)})}{\|z\|^2 + \|r^{(\imath)}\|^2}\\ =& \frac{\lambda\,[\,\|z\|^2 + \|r^{(\imath)}\|^2\,] - [\,\lambda\,\|z\|^2 + (Br^{(\imath)}, r^{(\imath)})\,]}{\|z\|^2 + \|r^{(\imath)}\|^2} = \frac{\lambda\,\|r^{(\imath)}\|^2 - (Br^{(\imath)}, r^{(\imath)})}{\|z\|^2 + \|r^{(\imath)}\|^2}\\ \le& \frac{\lambda\,\|r^{(\imath)}\|^2}{\|z\|^2 + \|r^{(\imath)}\|^2} \le \lambda\,\frac{\|r^{(\imath)}\|^2}{\|z\|^2} \le \lambda\, q^{2\imath}\frac{\|r^{(0)}\|^2}{\|z\|^2}, \quad \imath = 0,1,2,\dots.\end{aligned}$$

□

Accordingly, in the power method or the von Mises method, for initial vectors $u^{(0)}$ such that $u^{(1)} = Bu^{(0)} \ne 0$, we determine the sequence of corresponding normalized iterated vectors. These are calculated from the recurrence relation

$$v^{(0)} = \frac{u^{(0)}}{\|u^{(0)}\|}$$

and

$$v^{(\imath+1)} = \frac{1}{\|Bv^{(\imath)}\|}\, Bv^{(\imath)}, \quad \imath = 0,1,2,\dots \tag{3.16}$$

that follows immediately from the equation

$$v^{(\imath+1)} = \frac{u^{(\imath+1)}}{\|u^{(\imath+1)}\|} = \frac{Bu^{(\imath)}}{\|Bu^{(\imath)}\|} = \frac{Bv^{(\imath)}}{\|Bv^{(\imath)}\|}, \quad \imath = 0,1,2,\dots$$

The denominators $\|Bv^{(\imath)}\|$ are always non-zero due to (3.13) and $z \ne 0$. From the convergence of the normalized iterated vectors to the eigenvector $z/\|z\|$ of B corresponding to the eigenvalue $\lambda > 0$, we get the convergence

$$\|Bv^{(\imath)}\| \to \lambda \quad (\imath \to \infty)$$

since $\|Bw\| = |\lambda| \, \|w\| = \lambda$, and so the numbers $\|Bv^{(\imath)}\|$ also determine approximations to the eigenvalue λ. From these derivations, we have the following theorem.

Theorem 3.6 (Convergence theorem with a priori error estimates). *Let the matrix $B = (b_{jk}) \in \mathbb{R}^{n\times n}$ be symmetric with non-negative eigenvalues, $H = \mathbb{R}^n$ be endowed with the ordinary scalar product $(\cdot,\cdot) = (\cdot,\cdot)_2$, the eigenvalues λ_j, $j = 1,\dots,n$ be arranged according to*

$$\lambda_1 \geq \lambda_2 \geq \dots, \geq \lambda_n \geq 0,$$

and let the associated mutually orthonormal eigenvectors be

$$w_1, w_2, \dots, w_n.$$

Then, for every initial vector $u = u^{(0)}$ such that $Bu^{(0)} \neq 0$, the corresponding sequence (3.16) of iterated vectors $v^{(\imath)}$ converges to the eigenvector $w = z/\|z\|$ of B corresponding to the largest eigenvalue $\lambda = \lambda_m > 0$ satisfying the condition (3.10). The following a priori error estimate holds:

$$\|v^{(\imath)} - w\| \leq \gamma q^{\imath}, \quad \imath = 0,1,2,\dots \tag{3.17}$$

where $0 \leq q \leq \dfrac{\lambda'}{\lambda} < 1$ and $\gamma = \|r^{(0)}\|/\|z\|$. The sequence of corresponding Rayleigh quotients $\varrho(v^{(\imath)})$ converges to the eigenvalue λ with the a priori error estimate

$$0 \leq \frac{1}{\lambda}(\lambda - \varrho(v^{(\imath)}) \leq \gamma^2 \, q^{2\imath}, \quad \imath = 0,1,2,\dots. \tag{3.18}$$

Proof. The relation (3.17) is obtained as follows. We have

$$\|v^{(\imath)} - w\|^2 = 2 - \frac{1}{\|z\|}\,[(v^{(\imath)}, z) + (z, v^{(\imath)})] = 2 - \frac{2}{\|z\|}\,(v^{(\imath)}, z) \quad \imath = 0,1,2,\dots.$$

Now, since $z \perp r^{(\imath)}$,

$$(v^{(\imath)}, z) = \frac{(u^{(\imath)}, z)}{\|u^{(\imath)}\|} = \frac{(z + r^{(\imath)}, z)}{\|z + r^{(\imath)}\|} = \frac{\|z\|^2}{\|z + r^{(\imath)}\|}, \quad \imath = 0,1,2,\dots.$$

This leads to

$$\begin{aligned}\|v^{(\imath)} - w\|^2 =& 2 - \frac{2}{\|z\|}\,\frac{\|z\|^2}{\|z + r^{(\imath)}\|} = 2 - \frac{2\,\|z\|}{\|z + r^{(\imath)}\|} = \frac{2\,\|z + r^{(\imath)}\| - 2\,\|z\|}{\|z + r^{(\imath)}\|} \\ \leq& \frac{2}{\|z\|}\,(\|z + r^{(\imath)}\| - \|z\|) = \frac{2}{\|z\|}\,\frac{\|z + r^{(\imath)}\|^2 - \|z\|^2}{\|z + r^{(\imath)}\| + \|z\|} \leq \frac{2}{\|z\|}\,\frac{\|r^{(\imath)}\|^2}{2\,\|z\|}\end{aligned}$$

for $\imath = 0,1,2,\dots$. Hence, by means of (3.14), the a priori inequality follows

$$\|v^{(\imath)} - w\|^2 \leq \frac{\|r^{(\imath)}\|^2}{\|z\|^2} \leq q^{2\,\imath}\,\frac{\|r^{(0)}\|^2}{\|z\|^2}, \quad \imath = 0,1,2,\dots.$$

This delivers (3.17). The estimate (3.18) follows directly from (3.15). □

Since $\|v^{(\imath)}\| = 1$, the squares of the eigenvalue approximations $\|Bv^{(\imath)}\|$ are the Rayleigh quotients of the matrix B^2,

$$\|Bv^{(\imath)}\|^2 = (Bv^{(\imath)}, Bv^{(\imath)}) = (B^2 v^{(\imath)}, v^{(\imath)}) = \frac{(B^2 v^{(\imath)}, v^{(\imath)})}{\|v^{(\imath)}\|^2}, \quad \imath = 0,1,2,\dots.$$

Here, it is of interest that these eigenvalue approximations form a monotone sequence as may be seen from this expression by using the Schwarz inequality

$$\begin{aligned}\|Bv^{(\imath)}\| =&\frac{\|Bv^{(\imath)}\|^2}{\|Bv^{(\imath)}\|} = \frac{(Bv^{(\imath)}, Bv^{(\imath)})}{\|Bv^{(\imath)}\|} = \frac{(B^2v^{(\imath)}, v^{(\imath)})}{\|Bv^{(\imath)}\|} = \frac{(Bv^{(\imath+1)}, v^{(\imath)})}{\|v^{(\imath+1)}\|}\\ =&(Bv^{(\imath+1)}, v^{(\imath)}) \le \|Bv^{(\imath+1)}\|, \quad \imath = 0, 1, 2, \ldots.\end{aligned} \tag{3.19}$$

In Section 5, we apply the power method to positive definite matrices B so that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n > 0$. Additionally, we choose $u^{(0)} = u$ such that $(u, w_1) \ne 0$ in (3.10) so that $\lambda_m = \lambda_1$. The further eigenvalues $\lambda_j > 0,\ j = 2, \ldots, n$ are determined by the power method combined with the method of deflation. The last one is discussed in the next section.

4. Method of Deflation

Let $B \in \mathbb{R}^{n\times n}$ be a positive definite matrix with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ arranged such that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n > 0$, and let $w_1, w_2, \ldots, w_n$ be an associated orthonormal system of real eigenvectors.

Most important in the method of deflation is the use of projection operators $P_j \in \mathbb{R}^{n\times n},\ j = 1, \ldots, n$ in the vector space $H = \mathbb{R}^n$ endowed with the ordinary scalar product $(\cdot, \cdot) = (\cdot, \cdot)_2$. For this, we depart from the eigenvector expansions

$$u = \sum_{j=1}^{n} (u, w_j) w_j,\ u \in H,$$

$$Bu = \sum_{j=1}^{n} \lambda_j (u, w_j) w_j,\ u \in H.$$

Next, we rewrite the expressions $(u, w_j)w_j,\ j = 1, \ldots, n$ in the form $(u, w_j)w_j = P_j u,\ j = 1, \ldots, n$ so that

$$u = \sum_{j=1}^{n} P_j u,\ u \in H,$$

$$Bu = \sum_{j=1}^{n} \lambda_j P_j u,\ u \in H,$$

and use the obtained projection operators (i.e. $n\times n$- projection matrices) $P_j \in \mathbb{R}^{n\times n}$ to construct from $B = B_0$ and with the obtained projections P_j the deflated matrices $B_j = B_{j-1} - \lambda_j P_j,\ j = 1, 2, \ldots, n$ where it will be shown that B_j has the eigenvalues $\lambda_k > 0,\ k = j + 1, \ldots, n$ and $\lambda_k = 0,\ k = 1, \ldots, j$. Now, the details follow.

Let $B = (B_{jk}) \in \mathbb{R}^{n\times n}$ be a positive definite matrix with eigenvalues $\lambda_j > 0,\ j = 1, \ldots, n$ in descending order and associated orthonormal eigenvectors that can be chosen real as we know from Theorem 2.1. Further, we have the above eigenvector expansions

$$u = \sum_{j=1}^{n} (u, w_j) w_j,\ u \in H,$$

$$Bu = \sum_{j=1}^{n} \lambda_j (u, w_j) w_j,\ u \in H.$$

Next, let w_j^T be the transpose corresponding to the column vector w_j for $j = 1, \ldots, n$. Herewith, we rewrite $(u, w_j)w_j$ in the form

$$(u, w_j)w_j = (w_j^T u)w_j = w_j(w_j^T u) = (w_j w_j^T)u, \; u \in H$$

for $j = 1, \ldots, n$. The matrices

$$\boxed{P_j = w_j w_j^T \in \mathbb{R}^{n \times n}, \; j = 1, \ldots, n}$$

are projection operators with the property

$$P_j P_k = \delta_{jk} P_k, \; j, k = 1, \ldots, n.$$

Further, we have

$$u = \sum_{j=1}^{n} (u, w_j)w_j = \sum_{j=1}^{n} (P_j u) = (\sum_{j=1}^{n} P_j)u, \quad u \in H$$

or

$$\boxed{I = \sum_{j=1}^{n} P_j}$$

as well as

$$Bu = \sum_{j=1}^{n} \lambda_j (u, w_j)w_j = \sum_{j=1}^{n} \lambda_j (P_j u) = (\sum_{j=1}^{n} \lambda_j P_j)u, \quad u \in H$$

or

$$\boxed{B = \sum_{j=1}^{n} \lambda_j P_j.}$$

This expression is the starting point for the method of deflation which we take essentially from [5, Section 10.1.2, last paragraph on p. 185]. But, we restrict ourselves to the ordinary scalar product $(\cdot, \cdot) = (\cdot, \cdot)_2$ as opposed to the weighted scalar product $(u, v) = (Cu, v)_2, \; u, v \in \mathbb{R}^n$ with positive definite $C \in \mathbb{R}^{n \times n}$ applied there.

The derivation of the method of deflation is as follows: We rewrite the projection expansion of B in the form

$$B - \lambda_1 P_1 = \sum_{j=2}^{n} P_j.$$

With the deflated matrix $B_1 = B - \lambda_1 P_1$, we obtain

$$B_1 = \sum_{j=2}^{n} P_j.$$

Then, we get

$$B_1 w_1 = (B - \lambda_1 P_1)w_1 = Bw_1 - \lambda_1 w_1 = 0.$$

This means that the deflated matrix B_1 has the eigenvalues $\lambda_j > 0, \, j = 2, \ldots, n, \; \lambda_1 = 0$. Further, for the null space $N(B_1)$, we obtain $N(B_1) = [w_1]$. In the next step, we form the deflated matrix B_2 defined by

$$B_2 = \sum_{j=3}^{n} \lambda_j P_j = \sum_{j=2}^{n} \lambda_j P_j - \lambda_2 P_2 = B_1 - \lambda_2 P_2.$$

Then,

$$B_2 w_k = \sum_{j=3}^{n} \lambda_j P_j w_k = \lambda_j \delta_{jk} w_k,\ j,k = 1,\dots,n$$

which means that the deflated matrix B_2 has the eigenvalues $\lambda_j > 0,\ j = 3,\dots,n,\ \lambda_1 = 0,\ \lambda_2 = 0$. Further, for the null space $N(B_2)$, we obtain $N(B_2) = [w_1, w_2]$.

Generalizing this process, we determine the deflated matrix B_m defined by

$$B_m = \sum_{j=m+1}^{n} \lambda_j P_j = \sum_{j=m}^{n} \lambda_j P_j - \lambda_m P_m = B_{m-1} - \lambda_m P_m.$$

Then,

$$B_m w_k = \sum_{j=m+1}^{n} \lambda_j P_j w_k = \lambda_j \delta_{jk} w_k,\ j,k = 1,\dots,n$$

which means that the deflated matrix B_m has the eigenvalues $\lambda_j > 0,\ j = m+1,\dots,n,\ \lambda_1 = 0,\dots,\lambda_m = 0$. Further, for the null space $N(B_m)$, we obtain $N(B_m) = [w_1,\dots,w_m]$. For $m = n-1$, $B_m = B_{n-1}$ has the eigenvalues $\lambda_n > 0,\ \lambda_1 = 0,\dots,\lambda_{n-1} = 0$. Further, $N(B_m) = B_{n-1} = [w_1,\dots,w_{n-1}]$.

In the special case $m = n$, all eigenvalues of B_n are equal to zero. Further, $N(B_n) = [w_1,\dots,w_n] = H = \mathbb{R}^n$.

5. Application to Toeplitz Sinc Matrices for $t = 0.1$

In Section 2, we considered symmetric matrices $B \in \mathbb{R}^{n\times n}$ with non-negative eigenvalues and an associated set of orthonormal eigenvectors. In Section 3, we studied symmetric matrices $B \in \mathbb{R}^{n\times n}$ with positive eigenvalues, i.e., real positive definite matrices.

In this section, we choose for $B \in \mathbb{R}^{n\times n}$ even more special matrices, namely Toeplitz sinc matrices $A_n(t) = A(t,n)$ that are real, positive definite and depend on the time variable t in $0 < t < 1$ as well as on $n \in \mathbb{N}$ with $n \geq 2$.

As we shall see later, the power method combined with the method of deflation will deliver the eigenvalues and associated eigenvectors, but not in descending order. However, in a subsequent step, the eigenvalues are sorted such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0,$$

i.e., in the desired descending order. At this point, we mention that the eigenvalues $\lambda_j > 0,\ j = 1,\dots,n$ determined by the applied numerical procedure turn out to be all distinct, that is, one obtains

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$$

for $t = 0.1$ and $n = 5$, $n = 10$, $n = 15$ where we used as numerical procedure the power method combined with the method of deflation as well as for comparison reasons the Matlab program $eig.m$.

Further, we note that the eigenvalues and corresponding eigenvectors will be computed with a prescribed precision. Those eigenvalues in descending order that meet the prescribed precision criterion are denoted by $\lambda_j,\ j = 1,\dots,i_q$, and the associated eigenvectors are also exact in the above sense. The inexact eigenvalues $\lambda_j,\ j = i_q+1,\dots,n$ are replaced by $\lambda_j = 0,\ j = i_q+1,\dots,n$, and associated eigenvectors can easily be provided.

By the prescribed precision, we mean that the computational results are exact in the 5 digits of the Matlab format shortE given by $x.xxxxe-006$. To obtain this, we form the quotient vector q of the Matlab floating-point relative accuracy $eps = 2.2204e-016$ and of the eigenvalue vector λ (in Matlab: $q = eps./\lambda$) whose components are sorted in descending order and are computed by the power method combined with the method of deflation. Then, we determine the index i_q as the number of the components $q(j)$ of q, such that $q(j-1) < 0.5e-0.006$ and $q(j) > 0.5e-0.006$ for $j = 2, \ldots, n$.

As a consequence, i_q is determined such that λ_j, $j = 1, \ldots, i_q$ in the Matlab format shortE $x.xxxxe{-}006$ with its 5 significant digits x (1 digit before the decimal point and 4 digits after the decimal point) are exact to the 5 displayed digits whereas the eigenvalues λ_j, $j = i_q+1, \ldots, n$ are less than λ_{i_q} and can thus be set equal to zero.

In order to verify this result, we compute the eigenvalues and eigenvectors also by the Matlab program $eig.m$. Thereby, we obtain identical results in the 5 significant digits for the first i_q eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_{i_q}$ and the components of the associated eigenvectors which confirms that the index i_q in correctly determined. The computations are restricted to Toeplitz sinc matrices $A(t,n)$ with $t = 0.1$ and $n = 5$, $n = 10$, $n = 15$. But, it should pose no problem to carry out corresponding computations for other pairs (t,n). When in an applied Matlab program the input argument tol is required, we have chosen $tol = eps$.

Note that in Subsection 2.2, we have assumed that $\lambda_j = 0$, $j = m+1, \ldots, n$, whereas in this section we set $\lambda_j = 0$, $j = m+1, \ldots, n$ with $m = i_q$ due to the exactness demand. Further, the equality sign in this section - depending on the context - may mean only equality up to the exactness criterion in the above-mentioned sense. The largest eigenvalue of the limit matrix $A = \lim\limits_{\substack{0<t<1 \\ t\to 0}} A(t) = \lim\limits_{\substack{0<t<1 \\ t\to 0}} A(t,n)$ is denoted by $\mu_1 = \mu_1(A)$ as well as its normed eigenvector by w, and we write out the projection matrices P_j, $j = 1, \ldots, n$ only for $n = 5$. The computed matrices $A(t,n)$ and the corresponding computed eigenvalues $\mu_j(t)$, $j = 1, \ldots, n$ are denoted by B and λ_j, $j = 1, \ldots, n$, as the case may be.

5.1. **The Case** $n = 5$**.** For $t = 0.1$ and $n = 5$, we obtain

$$
\begin{aligned}
A =& \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{5\times 5}, \\
e =& [1 \quad 1 \quad 1 \quad 1 \quad 1]^T \in \mathbb{R}^5, \\
\mu_1 =& \mu_1(A) = 5, \\
w =& (1/\sqrt{n})\, e,
\end{aligned}
$$

leading to

$$
w = \begin{bmatrix} 4.4721e-001 \\ 4.4721e-001 \\ 4.4721e-001 \\ 4.4721e-001 \\ 4.4721e-001 \end{bmatrix} \in \mathbb{R}^5,
$$

$$
B = \begin{bmatrix} 1.0000e+000 & 9.8363e-001 & 9.3549e-001 & 8.5839e-001 & 7.5683e-001 \\ 9.8363e-001 & 1.0000e+000 & 9.8363e-001 & 9.3549e-001 & 8.5839e-001 \\ 9.3549e-001 & 9.8363e-001 & 1.0000e+000 & 9.8363e-001 & 9.3549e-001 \\ 8.5839e-001 & 9.3549e-001 & 9.8363e-001 & 1.0000e+000 & 9.8363e-001 \\ 7.5683e-001 & 8.5839e-001 & 9.3549e-001 & 9.8363e-001 & 1.0000e+000 \end{bmatrix} \in \mathbb{R}^{5\times 5}.
$$

The vector λ of eigenvalues in descending order computed by the Power Method combined with the Method of Deflation is given by

$$\lambda = \quad [4.6893e+000, \quad 3.0768e-001, \quad 3.0357e-003, \quad 8.8969e-006, \quad 8.0081e-009] \quad \in \mathbb{R}^5,$$

and the corresponding vector λ computed by the Matlab routine $eig.m$ reads also

$$\lambda = \quad [4.6893e+000, \quad 3.0768e-001, \quad 3.0357e-003, \quad 8.8969e-006, \quad 8.0081e-009] \quad \in \mathbb{R}^5.$$

The power method combined with the method of deflation delivers further

$$W = \begin{bmatrix} 4.3259e-001 & -6.2863e-001 & 5.4578e-001 & -3.2377e-001 & 1.2244e-001 \\ 4.5424e-001 & -3.2377e-001 & -2.5245e-001 & 6.2863e-001 & -4.7952e-001 \\ 4.6160e-001 & \underline{-4.9285e-012} & -5.2610e-001 & \underline{3.1519e-013} & 7.1424e-001 \\ 4.5424e-001 & 3.2377e-001 & -2.5245e-001 & -6.2863e-001 & -4.7952e-001 \\ 4.3259e-001 & 6.2863e-001 & 5.4578e-001 & 3.2377e-001 & 1.2244e-001 \end{bmatrix} \in \mathbb{R}^{5\times 5}$$

with

$$W = [w_1,\, w_2,\, w_3,\, w_4,\, w_5]$$

where

$$w_j \in \mathbb{R}^5,\ j = 1, \ldots, n = 5$$

are the eigenvectors of B and where

$$P_j = w_j\, w_j^T,\ j = 1, \ldots, n = 5$$

are the associated matrix projections. The $eig.m$-routine delivers

$$W = \begin{bmatrix} 4.3259e-001 & -6.2863e-001 & 5.4578e-001 & 3.2377e-001 & -1.2244e-001 \\ 4.5424e-001 & -3.2377e-001 & -2.5245e-001 & -6.2863e-001 & 4.7952e-001 \\ 4.6160e-001 & \underline{1.2295e-015} & -5.2610e-001 & \underline{2.5060e-012} & -7.1424e-001 \\ 4.5424e-001 & 3.2377e-001 & -2.5245e-001 & 6.2863e-001 & 4.7952e-001 \\ 4.3259e-001 & 6.2863e-001 & 5.4578e-001 & -3.2377e-001 & -1.2244e-001 \end{bmatrix} \in \mathbb{R}^{5\times 5}.$$

Comparing the results, setting the underlined (negligible) entries equal to zero, some normed eigenvectors $w_j,\ j = 2, \ldots, 5$ differ only in the factor -1. Therefore, both computed vectors $w_j,\ j = 1, \ldots, (n = 5)$ are equal as eigenvectors. Written out, we obtain the projection operators

$$P_1 = \begin{bmatrix} 1.8713e-001 & 1.9650e-001 & 1.9968e-001 & 1.9650e-001 & 1.8713e-001 \\ 1.9650e-001 & 2.0633e-001 & 2.0968e-001 & 2.0633e-001 & 1.9650e-001 \\ 1.9968e-001 & 2.0968e-001 & 2.1307e-001 & 2.0968e-001 & 1.9968e-001 \\ 1.9650e-001 & 2.0633e-001 & 2.0968e-001 & 2.0633e-001 & 1.9650e-001 \\ 1.8713e-001 & 1.9650e-001 & 1.9968e-001 & 1.9650e-001 & 1.8713e-001 \end{bmatrix} \in \mathbb{R}^{5\times 5},$$

$$P_2 = \begin{bmatrix} 3.9518e-001 & 2.0353e-001 & 3.0982e-012 & -2.0353e-001 & -3.9518e-001 \\ 2.0353e-001 & 1.0482e-001 & 1.5957e-012 & -1.0482e-001 & -2.0353e-001 \\ 3.0982e-012 & 1.5957e-012 & 2.4290e-023 & -1.5957e-012 & -3.0982e-012 \\ -2.0353e-001 & -1.0482e-001 & -1.5957e-012 & 1.0482e-001 & 2.0353e-001 \\ -3.9518e-001 & -2.0353e-001 & -3.0982e-012 & 2.0353e-001 & 3.9518e-001 \end{bmatrix} \in \mathbb{R}^{5\times 5},$$

$$P_3 = \begin{bmatrix} 2.9788e-001 & -1.3778e-001 & -2.8714e-001 & -1.3778e-001 & 2.9788e-001 \\ -1.3778e-001 & 6.3732e-002 & 1.3282e-001 & 6.3732e-002 & 1.3778e-001 \\ -2.8714e-001 & 1.3282e-001 & 2.7678e-001 & 1.3282e-001 & -2.8714e-001 \\ -1.3778e-001 & 6.3732e-002 & 1.3282e-001 & 6.3732e-002 & -1.3778e-001 \\ 2.9788e-001 & -1.3778e-001 & -2.8714e-001 & -1.3778e-001 & 2.9788e-001 \end{bmatrix} \in \mathbb{R}^{5\times 5},$$

$$P_4 = \begin{bmatrix} 1.0482e-001 & -2.0353e-001 & -1.0205e-013 & 2.0353e-001 & -1.0482e-001 \\ -2.0353e-001 & 3.9518e-001 & 1.9814e-013 & -3.9518e-001 & 2.0353e-001 \\ -1.0205e-013 & 1.9814e-013 & 9.9344e-026 & -1.9814e-013 & 1.0205e-013 \\ 2.0353e-001 & -3.9518e-001 & -1.9814e-013 & 3.9518e-001 & -2.0353e-001 \\ -1.0482e-001 & 2.0353e-001 & 1.0205e-013 & -2.0353e-001 & 1.0482e-001 \end{bmatrix} \in \mathbb{R}^{5\times 5},$$

$$P_5 = \begin{bmatrix} 1.4993e-002 & -5.8714e-002 & 8.7455e-002 & -5.8714e-002 & 1.4993e-002 \\ -5.8714e-002 & 2.2994e-001 & -3.4249e-001 & 2.2994e-001 & -5.8714e-002 \\ 8.7455e-002 & -3.4249e-001 & 5.1014e-001 & -3.4249e-001 & 8.7455e-002 \\ -5.8714e-002 & 2.2994e-001 & -3.4249e-001 & 2.2994e-001 & -5.8714e-002 \\ 1.4993e-002 & -5.8714e-002 & 8.7455e-002 & -5.8714e-002 & 1.4993e-002 \end{bmatrix} \in \mathbb{R}^{5\times 5}.$$

This entails the standard eigenvector expansions for u and Bu, namely

$$\boxed{\begin{array}{l} u = \sum_{j=1}^{5} (u, w_j)\, w_j, \qquad u \in H = \mathbb{R}^5 \\ \text{and} \\ Bu = \sum_{j=1}^{5} \lambda_j\, (u, w_j)\, w_j, \quad u \in H = \mathbb{R}^5. \end{array}}$$

Correspondingly, using

$$\boxed{(u, w_j)\, w_j = (w_j^T\, u)\, w_j = w_j\, (w_j^T\, u) = (w_j\, w_j^T)\, u = P_j\, u,\ u \in H = \mathbb{R}^5, \quad j = 1, \ldots, n = 5,}$$

we obtain the standard projection-matrix expansions for the matrices I and B themselves, namely

$$\boxed{\begin{array}{l} I = \sum_{j=1}^{n=5} P_j \\ \text{and} \\ B = \sum_{j=1}^{n=5} \lambda_j\, P_j. \end{array}}$$

5.2. **The Case** $n = 10$**.** For $t = 0.1$ and $n = 10$, we obtain

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{10\times 10},$$

$$\begin{aligned} e =& [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^T \in \mathbb{R}^{10}, \\ \mu_1 =& \mu_1(A) = 10, \\ w =& (1/\sqrt{n})\, e, \end{aligned}$$

leading to

$$
\begin{aligned}
w =& \begin{bmatrix} 3.1623e-001 \\ 3.1623e-001 \\ 3.1623e-001 \\ 3.1623e-001 \\ 3.1623e-001 \\ 3.1623e-001 \\ 3.1623e-001 \\ 3.1623e-001 \\ 3.1623e-001 \\ 3.1623e-001 \end{bmatrix} \in \mathbb{R}^{10}, \\
B =& [B_l \mid B_r]
\end{aligned}
$$

where

$$
B_l = \begin{bmatrix}
1.0000e+000 & 9.8363e-001 & 9.3549e-001 & 8.5839e-001 & 7.5683e-001 \\
9.8363e-001 & 1.0000e+000 & 9.8363e-001 & 9.3549e-001 & 8.5839e-001 \\
9.3549e-001 & 9.8363e-001 & 1.0000e+000 & 9.8363e-001 & 9.3549e-001 \\
8.5839e-001 & 9.3549e-001 & 9.8363e-001 & 1.0000e+000 & 9.8363e-001 \\
7.5683e-001 & 8.5839e-001 & 9.3549e-001 & 9.8363e-001 & 1.0000e+000 \\
6.3662e-001 & 7.5683e-001 & 8.5839e-001 & 9.3549e-001 & 9.8363e-001 \\
5.0455e-001 & 6.3662e-001 & 7.5683e-001 & 8.5839e-001 & 9.3549e-001 \\
3.6788e-001 & 5.0455e-001 & 6.3662e-001 & 7.5683e-001 & 8.5839e-001 \\
2.3387e-001 & 3.6788e-001 & 5.0455e-001 & 6.3662e-001 & 7.5683e-001 \\
1.0929e-001 & 2.3387e-001 & 3.6788e-001 & 5.0455e-001 & 6.3662e-001
\end{bmatrix} \in \mathbb{R}^{10\times 5},
$$

$$
B_r = \begin{bmatrix}
6.3662e-001 & 5.0455e-001 & 3.6788e-001 & 2.3387e-001 & 1.0929e-001 \\
7.5683e-001 & 6.3662e-001 & 5.0455e-001 & 3.6788e-001 & 2.3387e-001 \\
8.5839e-001 & 7.5683e-001 & 6.3662e-001 & 5.0455e-001 & 3.6788e-001 \\
9.3549e-001 & 8.5839e-001 & 7.5683e-001 & 6.3662e-001 & 5.0455e-001 \\
9.8363e-001 & 9.3549e-001 & 8.5839e-001 & 7.5683e-001 & 6.3662e-001 \\
1.0000e+000 & 9.8363e-001 & 9.3549e-001 & 8.5839e-001 & 7.5683e-001 \\
9.8363e-001 & 1.0000e+000 & 9.8363e-001 & 9.3549e-001 & 8.5839e-001 \\
9.3549e-001 & 9.8363e-001 & 1.0000e+000 & 9.8363e-001 & 9.3549e-001 \\
8.5839e-001 & 9.3549e-001 & 9.8363e-001 & 1.0000e+000 & 9.8363e-001 \\
7.5683e-001 & 8.5839e-001 & 9.3549e-001 & 9.8363e-001 & 1.0000e+000
\end{bmatrix} \in \mathbb{R}^{10\times 5},
$$

$$
\begin{aligned}
\lambda =& [\lambda_l \mid \lambda_r], \\
\lambda_l =& \quad [\underline{7.8465e+000,\ 2.0423e+000,\ 1.0926e-001,\ 1.8886e-003,\ 1.5966e-005}] \quad \in \mathbb{R}^5, \\
\lambda_r =& \quad [\underline{7.5864e-008\ 2.1201e-010},\ 3.4535e-013,\ 1.1076e-015,\ 5.4224e-018] \quad \in \mathbb{R}^5.
\end{aligned}
$$

$\boxed{\text{Determination of } i_q = 7 \text{ in Matlab:}}$

Form the quotient vector q of the Matlab quantity $eps = 2.2204e-016$ and of the sorted vector λ computed by the power method combined with the method of deflation. Then, determine i_q as the number of the components of q such that $q(j-1) < 0.5e-006$ and $q(j) > 0.5e-006$ for $j = 2\ldots,(n=10)$. As a consequence, i_q is determined such that $\lambda_j,\ j = 1,\ldots,i_q$ in the Matlab format shortE, given by $x.xxxxe-006$ with its 5 significant digits (1 digit before the decimal point and 4 digits after the decimal point) are exact whereas the eigenvalues $\lambda_j,\ j = i_q+1,\ldots,(n=10)$ are less than λ_{iq} as well as are considered being inexact; they therefore can be set equal to zero. The eigenvalues for $j = 1,\ldots,i_q = 7$ being exact in this sense are underlined above.

Correspondingly, the associated eigenvectors for $\lambda_j,\ j = i_q+1,\ldots,(n=10) = 8,9,(n=10)$ are also considered being inexact. But, as we shall see below, sets of linearly independent eigenvectors associated with the eigenvalues $\lambda_j = 0,\ j = i_q + 1, \ldots, (n = 10) = 8, 9, (n = 10)$ can be given without difficulty. Thus,

$$i_q = 7.$$

For comparison reasons, we apply also the Matlab routine $eig.m$ giving

$$\lambda_l = \ \underline{[7.8465e+000,\ 2.0423e+000,\ 1.0926e-001,\ 1.8886e-003,\ 1.5966e-005]} \ \in \mathbb{R}^5,$$

$$\lambda_r = \ [\underline{7.5864e-008,\ 2.1201e-010},\ 3.4521e-013,\ 2.7815e-016,\ 1.2737e-016] \ \in \mathbb{R}^5.$$

Now, the columns of the following matrix U form a basis of eigenvectors of the limit matrix A (the first column is eigenvector associated with the largest eigenvalue $\mu_1 = \mu_1(A) = n = 10$, the other columns are eigenvectors associated with the eigenvalues $\mu_j = \mu_j(A) = 0,\ j = 2, \ldots, (n = 10)$:

$$U = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \in \mathbb{R}^{10\times 10}.$$

Next, we replace the first $i_q = 7$ columns by the eigenvectors computed by the power method combined with the method of deflation. This leads to

$$U = [U_l \,|\, U_r]$$

where

$$U_l = \begin{bmatrix} 2.5812e-001 & 4.6675e-001 & 5.4006e-001 & -4.7703e-001 & 3.5511e-001 \\ 2.9405e-001 & 3.9338e-001 & 2.2361e-001 & 1.1321e-001 & -3.9522e-001 \\ 3.2291e-001 & 2.9807e-001 & -3.9039e-002 & 3.6088e-001 & -3.2820e-001 \\ 3.4308e-001 & 1.8592e-001 & -2.2680e-001 & 3.3467e-001 & 4.1481e-002 \\ 3.5346e-001 & 6.3176e-002 & -3.2461e-001 & 1.3187e-001 & 3.2904e-001 \\ 3.5346e-001 & -6.3176e-002 & -3.2461e-001 & -1.3187e-001 & 3.2904e-001 \\ 3.4308e-001 & -1.8592e-001 & -2.2680e-001 & -3.3467e-001 & 4.1481e-002 \\ 3.2291e-001 & -2.9807e-001 & -3.9039e-002 & -3.6088e-001 & -3.2820e-001 \\ 2.9405e-001 & -3.9338e-001 & 2.2361e-001 & -1.1321e-001 & -3.9522e-001 \\ 2.5812e-001 & -4.6675e-001 & 5.4006e-001 & 4.7703e-001 & 3.5511e-001 \end{bmatrix} \in \mathbb{R}^{10\times 5},$$

$$U_r = \begin{bmatrix} -2.2686e-001 & 1.2336e-001 & 0 & 0 & 0 \\ 5.0123e-001 & -4.3495e-001 & 0 & 0 & 0 \\ -1.8605e-002 & 3.7956e-001 & 0 & 0 & 0 \\ -3.8786e-001 & 2.3917e-001 & 0 & 0 & 0 \\ -2.1569e-001 & -3.0713e-001 & 0 & 0 & 0 \\ 2.1569e-001 & -3.0713e-001 & 0 & 0 & 0 \\ 3.8786e-001 & 2.3917e-001 & 1.0000e+000 & 0 & 0 \\ 1.8605e-002 & 3.7956e-001 & -1.0000e+000 & 1.0000e+000 & 0 \\ -5.0123e-001 & -4.3495e-001 & 0 & -1.0000e+000 & 1.0000e+000 \\ 2.2686e-001 & 1.2336e-001 & 0 & 0 & -1.0000e+000 \end{bmatrix} \in \mathbb{R}^{10\times 5}.$$

Gram-Schmidt orthonormalization with the last U as input and W as output delivers

$$W = [W_l \,|\, W_r]$$

where

$$W_l = \begin{bmatrix} 2.5812e-001 & 4.6675e-001 & 5.4006e-001 & -4.7703e-001 & 3.5511e-001 \\ 2.9405e-001 & 3.9338e-001 & 2.2361e-001 & 1.1321e-001 & -3.9522e-001 \\ 3.2291e-001 & 2.9807e-001 & -3.9039e-002 & 3.6088e-001 & -3.2820e-001 \\ 3.4308e-001 & 1.8592e-001 & -2.2680e-001 & 3.3467e-001 & 4.1481e-002 \\ 3.5346e-001 & 6.3176e-002 & -3.2461e-001 & 1.3187e-001 & 3.2904e-001 \\ 3.5346e-001 & -6.3176e-002 & -3.2461e-001 & -1.3187e-001 & 3.2904e-001 \\ 3.4308e-001 & -1.8592e-001 & -2.2680e-001 & -3.3467e-001 & 4.1481e-002 \\ 3.2291e-001 & -2.9807e-001 & -3.9039e-002 & -3.6088e-001 & -3.2820e-001 \\ 2.9405e-001 & -3.9338e-001 & 2.2361e-001 & -1.1321e-001 & -3.9522e-001 \\ 2.5812e-001 & -4.6675e-001 & 5.4006e-001 & 4.7703e-001 & 3.5511e-001 \end{bmatrix} \in \mathbb{R}^{10\times 5},$$

$$W_r = \left[\begin{array}{cc|ccc} -2.2686e-001 & 1.2336e-001 & 2.0320e-002 & -4.8138e-002 & 2.7991e-002 \\ 5.0123e-001 & -4.3495e-001 & -8.6248e-002 & 2.4622e-001 & -1.7692e-001 \\ -1.8605e-002 & 3.7956e-001 & 9.6871e-002 & -4.3938e-001 & 4.6964e-001 \\ -3.8786e-001 & 2.3917e-001 & 6.3933e-002 & 1.6843e-001 & -6.6518e-001 \\ -2.1569e-001 & -3.0713e-001 & -1.2715e-001 & 4.6341e-001 & 5.1796e-001 \\ 2.1569e-001 & -3.0713e-001 & -2.3447e-001 & -6.3856e-001 & -1.9093e-001 \\ 3.8786e-001 & 2.3917e-001 & 6.6404e-001 & 1.8419e-001 & 6.8279e-003 \\ 1.8605e-002 & 3.7956e-001 & -6.2373e-001 & 1.8419e-001 & 6.8279e-003 \\ -5.0123e-001 & -4.3495e-001 & 2.7432e-001 & -1.5512e-001 & 6.8279e-003 \\ 2.2686e-001 & 1.2336e-001 & -4.7907e-002 & 3.4776e-002 & -3.0424e-003 \end{array}\right] \in \mathbb{R}^{10\times 5}$$

with

$$W = [w_1,\, w_2,\, w_3,\, w_4,\, w_5,\, w_6,\, w_7|,\, w_8,\, w_9,\, w_{10}]$$

where

$$w_j \in \mathbb{R}^{10},\ j = 1, \dots n = 10$$

and

$$P_j = w_j\, w_j^T,\ j = 1, \dots, n = 10.$$

The $eig.m$-routine delivers

$$W = [W_l \,|\, W_r]$$

where

$$W_l = \begin{bmatrix} 2.5812e-001 & -4.6675e-001 & 5.4006e-001 & -4.7703e-001 & -3.5511e-001 \\ 2.9405e-001 & -3.9338e-001 & 2.2361e-001 & 1.1321e-001 & 3.9522e-001 \\ 3.2291e-001 & -2.9807e-001 & -3.9039e-002 & 3.6088e-001 & 3.2820e-001 \\ 3.4308e-001 & -1.8592e-001 & -2.2680e-001 & 3.3467e-001 & -4.1481e-002 \\ 3.5346e-001 & -6.3176e-002 & -3.2461e-001 & 1.3187e-001 & -3.2904e-001 \\ 3.5346e-001 & 6.3176e-002 & -3.2461e-001 & -1.3187e-001 & -3.2904e-001 \\ 3.4308e-001 & 1.8592e-001 & -2.2680e-001 & -3.3467e-001 & -4.1481e-002 \\ 3.2291e-001 & 2.9807e-001 & -3.9039e-002 & -3.6088e-001 & 3.2820e-001 \\ 2.9405e-001 & 3.9338e-001 & 2.2361e-001 & -1.1321e-001 & 3.9522e-001 \\ 2.5812e-001 & 4.6675e-001 & 5.4006e-001 & 4.7703e-001 & -3.5511e-001 \end{bmatrix} \in \mathbb{R}^{10\times 5},$$

$$W_r = \left[\begin{array}{cc|ccc} -2.2686e-001 & -1.2336e-001 & -5.5684e-002 & -2.0045e-002 & -3.3536e-003 \\ 5.0123e-001 & 4.3495e-001 & 2.8187e-001 & 1.3735e-001 & 3.2321e-002 \\ -1.8605e-002 & -3.7956e-001 & -5.0320e-001 & -3.8824e-001 & -1.3807e-001 \\ -3.8786e-001 & -2.3917e-001 & 2.3886e-001 & 5.4752e-001 & 3.4364e-001 \\ -2.1569e-001 & 3.0713e-001 & 3.2752e-001 & -2.9934e-001 & -5.4984e-001 \\ 2.1569e-001 & 3.0713e-001 & -3.2760e-001 & -2.1738e-001 & 5.8704e-001 \\ 3.8786e-001 & -2.3917e-001 & -2.3869e-001 & 4.9272e-001 & -4.1854e-001 \\ 1.8605e-002 & -3.7956e-001 & 5.0308e-001 & -3.6461e-001 & 1.9227e-001 \\ -5.0123e-001 & 4.3495e-001 & -2.8183e-001 & 1.3140e-001 & -5.1671e-002 \\ 2.2686e-001 & -1.2336e-001 & 5.5677e-002 & -1.9375e-002 & 6.1920e-003 \end{array}\right] \in \mathbb{R}^{10\times 5}.$$

Comparing the results, some normed eigenvectors $w_j,\ j = 2, \dots, (i_q = 7)$ differ only in the factor -1. Therefore, both computed vectors $w_j,\ j = 1, \dots, (i_q = 7)$ are equal as eigenvectors.

The eigenvectors $w_j,\ j = 1,\ldots,10$ computed by the power method combined with the method of deflation are exact in the 5 significant digital places whereas the eigenvectors $w_j,\ j = 8,9,10$ computed by the Matlab program $eig.m$ are inexact in these 5 digital places. Using the first method, this entails what could be called displayed eigenvector expansions for u and Bu, namely

$$u = \sum_{j=1}^{n=7} (u, w_j)\, w_j + P_{0,10} u, \quad u \in H = \mathbb{R}^{10}$$

where

$$P_{0,10} u = \sum_{j=8}^{n=10} (u, w_j) w_j, \qquad u \in H = \mathbb{R}^{10} \mapsto N(B).$$

Further,

$$Bu = \sum_{j=1}^{n=7} \lambda_j\, (u, w_j) w_j, \qquad u \in H = \mathbb{R}^{10}.$$

Correspondingly, using

$$\boxed{(u, w_j)\, w_j = (w_j^T\, u)\, w_j = w_j\, (w_j^T\, u) = (w_j\, w_j^T)\, u = P_j\, u,\ u \in H = \mathbb{R}^{10}, \quad j = 1,\ldots,n = 10,}$$

we obtain what could be called displayed projection-matrix expansions for the matrices I and B themselves, namely

$$I = \sum_{j=1}^{n=7} P_j + P_{0,10}$$

where

$$P_{0,10} = \sum_{j=8}^{n=10} P_j.$$

Further,

$$B = \sum_{j=1}^{n=7} \lambda_j\, P_j.$$

At this point, we want to mention that still the standard eigenvector expansions for u and Bu hold, namely

$$u = \sum_{j=1}^{n=10} (u, w_j) w_j, \qquad u \in H = \mathbb{R}^{10}$$

and

$$Bu = \sum_{j=1}^{n=10} \lambda_j\, (u, w_j) w_j, \quad u \in H = \mathbb{R}^{10}$$

as well as the standard matrix-projection expansions for the matrices I and B themselves, namely

$$\boxed{\begin{aligned} &I = \sum_{j=1}^{n=10} P_j \\ &\text{and} \\ &B = \sum_{j=1}^{n=10} \lambda_j \, P_j, \end{aligned}}$$

but they hide the interesting details.

5.3. **The Case $n = 15$.** For $t = 0.1$ and $n = 15$, we obtain

$$\begin{aligned}
A =& \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{15\times 15}, \\
e =& [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^T \in \mathbb{R}^{15}, \\
\mu_1 =& \mu_1(A) = 15, \\
w =& (1/\sqrt{n})\, e,
\end{aligned}$$

leading to

$$w = \begin{bmatrix} 2.5820e-001 \\ 2.5820e-001 \\ 2.5820e-001 \\ 2.5820e-001 \\ 2.5820e-001 \\ 2.5820e-001 \\ 2.5820e-001 \\ 2.5820e-001 \\ 2.5820e-001 \\ 2.5820e-001 \\ 2.5820e-001 \\ 2.5820e-001 \\ 2.5820e-001 \\ 2.5820e-001 \\ 2.5820e-001 \end{bmatrix} \in \mathbb{R}^{15},$$

$$B = [B_l \,|\, B_m \,|\, B_r\,]$$

where

$$B_l = \begin{bmatrix}
1.0000e+000 & 9.8363e-001 & 9.3549e-001 & 8.5839e-001 & 7.5683e-001 \\
9.8363e-001 & 1.0000e+000 & 9.8363e-001 & 9.3549e-001 & 8.5839e-001 \\
9.3549e-001 & 9.8363e-001 & 1.0000e+000 & 9.8363e-001 & 9.3549e-001 \\
8.5839e-001 & 9.3549e-001 & 9.8363e-001 & 1.0000e+000 & 9.8363e-001 \\
7.5683e-001 & 8.5839e-001 & 9.3549e-001 & 9.8363e-001 & 1.0000e+000 \\
6.3662e-001 & 7.5683e-001 & 8.5839e-001 & 9.3549e-001 & 9.8363e-001 \\
5.0455e-001 & 6.3662e-001 & 7.5683e-001 & 8.5839e-001 & 9.3549e-001 \\
3.6788e-001 & 5.0455e-001 & 6.3662e-001 & 7.5683e-001 & 8.5839e-001 \\
2.3387e-001 & 3.6788e-001 & 5.0455e-001 & 6.3662e-001 & 7.5683e-001 \\
1.0929e-001 & 2.3387e-001 & 3.6788e-001 & 5.0455e-001 & 6.3662e-001 \\
3.8982e-017 & 1.0929e-001 & 2.3387e-001 & 3.6788e-001 & 5.0455e-001 \\
-8.9421e-002 & 3.8982e-017 & 1.0929e-001 & 2.3387e-001 & 3.6788e-001 \\
-1.5591e-001 & -8.9421e-002 & 3.8982e-017 & 1.0929e-001 & 2.3387e-001 \\
-1.9809e-001 & -1.5591e-001 & -8.9421e-002 & 3.8982e-017 & 1.0929e-001 \\
-2.1624e-001 & -1.9809e-001 & -1.5591e-001 & -8.9421e-002 & 3.8982e-017
\end{bmatrix} \in \mathbb{R}^{15\times 5},$$

$$B_m = \begin{bmatrix}
6.3662e-001 & 5.0455e-001 & 3.6788e-001 & 2.3387e-001 & 1.0929e-001 \\
7.5683e-001 & 6.3662e-001 & 5.0455e-001 & 3.6788e-001 & 2.3387e-001 \\
8.5839e-001 & 7.5683e-001 & 6.3662e-001 & 5.0455e-001 & 3.6788e-001 \\
9.3549e-001 & 8.5839e-001 & 7.5683e-001 & 6.3662e-001 & 5.0455e-001 \\
9.8363e-001 & 9.3549e-001 & 8.5839e-001 & 7.5683e-001 & 6.3662e-001 \\
1.0000e+000 & 9.8363e-001 & 9.3549e-001 & 8.5839e-001 & 7.5683e-001 \\
9.8363e-001 & 1.0000e+000 & 9.8363e-001 & 9.3549e-001 & 8.5839e-001 \\
9.3549e-001 & 9.8363e-001 & 1.0000e+000 & 9.8363e-001 & 9.3549e-001 \\
8.5839e-001 & 9.3549e-001 & 9.8363e-001 & 1.0000e+000 & 9.8363e-001 \\
7.5683e-001 & 8.5839e-001 & 9.3549e-001 & 9.8363e-001 & 1.0000e+000 \\
6.3662e-001 & 7.5683e-001 & 8.5839e-001 & 9.3549e-001 & 9.8363e-001 \\
5.0455e-001 & 6.3662e-001 & 7.5683e-001 & 8.5839e-001 & 9.3549e-001 \\
3.6788e-001 & 5.0455e-001 & 6.3662e-001 & 7.5683e-001 & 8.5839e-001 \\
2.3387e-001 & 3.6788e-001 & 5.0455e-001 & 6.3662e-001 & 7.5683e-001 \\
1.0929e-001 & 2.3387e-001 & 3.6788e-001 & 5.0455e-001 & 6.3662e-001
\end{bmatrix} \in \mathbb{R}^{15\times 5},$$

$$B_r = \begin{bmatrix}
3.8982e-017 & -8.9421e-002 & -1.5591e-001 & -1.9809e-001 & -2.1624e-001 \\
1.0929e-001 & 3.8982e-017 & -8.9421e-002 & -1.5591e-001 & -1.9809e-001 \\
2.3387e-001 & 1.0929e-001 & 3.8982e-017 & -8.9421e-002 & -1.5591e-001 \\
3.6788e-001 & 2.3387e-001 & 1.0929e-001 & 3.8982e-017 & -8.9421e-002 \\
5.0455e-001 & 3.6788e-001 & 2.3387e-001 & 1.0929e-001 & 3.8982e-017 \\
6.3662e-001 & 5.0455e-001 & 3.6788e-001 & 2.3387e-001 & 1.0929e-001 \\
7.5683e-001 & 6.3662e-001 & 5.0455e-001 & 3.6788e-001 & 2.3387e-001 \\
8.5839e-001 & 7.5683e-001 & 6.3662e-001 & 5.0455e-001 & 3.6788e-001 \\
9.3549e-001 & 8.5839e-001 & 7.5683e-001 & 6.3662e-001 & 5.0455e-001 \\
9.8363e-001 & 9.3549e-001 & 8.5839e-001 & 7.5683e-001 & 6.3662e-001 \\
1.0000e+000 & 9.8363e-001 & 9.3549e-001 & 8.5839e-001 & 7.5683e-001 \\
9.8363e-001 & 1.0000e+000 & 9.8363e-001 & 9.3549e-001 & 8.5839e-001 \\
9.3549e-001 & 9.8363e-001 & 1.0000e+000 & 9.8363e-001 & 9.3549e-001 \\
8.5839e-001 & 9.3549e-001 & 9.8363e-001 & 1.0000e+000 & 9.8363e-001 \\
7.5683e-001 & 8.5839e-001 & 9.3549e-001 & 9.8363e-001 & 1.0000e+000
\end{bmatrix} \in \mathbb{R}^{15\times 5},$$

$$\begin{aligned}
\lambda =& [\lambda_l \,|\, \lambda_m \,|\, \lambda_r], \\
\lambda_l =& \; [\underline{9.3119e+000,\ 4.9075e+000,\ 7.4593e-001,\ 3.3948e-002,\ 7.2882e-004}] \quad \in \mathbb{R}^5, \\
\lambda_m =& \; [\underline{9.2853e-006,\ 7.6801e-008,\ 4.3229e-010},\ 1.6988e-012,\ 4.1672e-015] \quad \in \mathbb{R}^5, \\
\lambda_r =& \; [3.0175e-017,\ 2.9844e-017,\ 2.9517e-017,\ 2.9194e-017,\ 2.8874e-017] \quad \in \mathbb{R}^5.
\end{aligned}$$

$\boxed{\text{Determination of } i_q = 8 \text{ in Matlab:}}$

Form the quotient vector q of the Matlab quantity $eps = 2.2204e-016$ and of the sorted vector λ computed by the power method combined with the method of deflation. Then, determine i_q as the number of the components of q such that $q(j-1) < 0.5e-006$ and $q(j) > 0.5e-006$ for $j = 2\ldots,(n=15)$. As a consequence, i_q is determined such that $\lambda_j,\ j = 1,\ldots,i_q$ in the Matlab format shortE $x.xxxxe-006$ with its 5 significant digits (1 digit before the decimal point and 4 digits after the decimal point) are exact whereas the eigenvalues $\lambda_j,\ j = i_q+1,\ldots,(n=15)$ are less than λ_{iq} as well as are considered being inexact; they therefore can be set equal to zero. The eigenvalues for $j = 1,\ldots,i_q = 8$ being exact in this sense are underlined above.

Correspondingly, the associated eigenvectors for $\lambda_j,\ j = i_q+1,\ldots,(n=15) = 9,\ldots,(n=15)$ are also considered being inexact. But, as we shall see below, sets of linearly independent eigenvectors associated with the eigenvalues $\lambda_j = 0,\ j = i_q+1,\ldots,(n=15) = 9,\ldots,(n=15)$ can be given without difficulty. Thus,

$$i_q = 8.$$

For comparison reasons, we apply also the Matlab routine $eig.m$ giving

$$\begin{aligned}
\lambda_l &= \ [\underline{9.3119e+000,\ 4.9075e+000,\ 7.4593e-001,\ 3.3948e-002,\ 7.2882e-004}] \ \in \mathbb{R}^5,\\
\lambda_m &= \ [\underline{9.2853e-006,\ 7.6801e-008,\ 4.3229e-010},\ 1.6991e-012,\ 4.4048e-015] \ \in \mathbb{R}^5,\\
\lambda_r &= \ [5.6015e-016,\ 3.4319e-018,\ -7.6322e-017,\ -1.2156e-016,\ -2.0392e-016] \ \in \mathbb{R}^5.
\end{aligned}$$

Now, the columns of the following matrix U form a basis of eigenvectors of the limit matrix A (the first column is eigenvector associated with the largest eigenvalue $\mu_1 = \mu_1(A) = n = 15$, the other columns are eigenvectors associated with the eigenvalues $\mu_j = \mu_j(A) = 0,\ j = 2,\ldots,(n=15)$:

$$U = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix} \in \mathbb{R}^{15\times 15}.$$

Next, we replace the first $i_q = 8$ columns by the eigenvectors computed by the power method combined with the method of deflation. This leads to

$$U = [U_l \,|\, U_m \,|\, U_r]$$

where

$$U_l = \begin{bmatrix} 1.5416e-001 & -3.4966e-001 & 4.7804e-001 & 4.8946e-001 & 4.2678e-001 \\ 1.9085e-001 & -3.4298e-001 & 3.3329e-001 & 1.4066e-001 & -1.1587e-001 \\ 2.2573e-001 & -3.1907e-001 & 1.8468e-001 & -1.0801e-001 & -3.2307e-001 \\ 2.5709e-001 & -2.7861e-001 & 4.3562e-002 & -2.5063e-001 & -2.9188e-001 \\ 2.8334e-001 & -2.2336e-001 & -7.9298e-002 & -2.9208e-001 & -1.2684e-001 \\ 3.0314e-001 & -1.5605e-001 & -1.7448e-001 & -2.4741e-001 & 7.3476e-002 \\ 3.1545e-001 & -8.0229e-002 & -2.3468e-001 & -1.4015e-001 & 2.2874e-001 \\ 3.1963e-001 & -9.2175e-014 & -2.5528e-001 & 1.1263e-009 & 2.8656e-001 \\ 3.1545e-001 & 8.0229e-002 & -2.3468e-001 & 1.4015e-001 & 2.2874e-001 \\ 3.0314e-001 & 1.5605e-001 & -1.7448e-001 & 2.4741e-001 & 7.3476e-002 \\ 2.8334e-001 & 2.2336e-001 & -7.9298e-002 & 2.9208e-001 & -1.2684e-001 \\ 2.5709e-001 & 2.7861e-001 & 4.3562e-002 & 2.5063e-001 & -2.9188e-001 \\ 2.2573e-001 & 3.1907e-001 & 1.8468e-001 & 1.0801e-001 & -3.2307e-001 \\ 1.9085e-001 & 3.4298e-001 & 3.3329e-001 & -1.4066e-001 & -1.1587e-001 \\ 1.5416e-001 & 3.4966e-001 & 4.7804e-001 & -4.8946e-001 & 4.2678e-001 \end{bmatrix} \in \mathbb{R}^{15\times 5},$$

$$U_m = \begin{bmatrix} 3.3454e-001 & 2.3836e-001 & 1.5449e-001 & 0 & \\ -3.2607e-001 & -4.3382e-001 & -4.3364e-001 & 0 & 0 \\ -3.1912e-001 & -1.1578e-001 & 1.5983e-001 & 0 & 0 \\ -4.8217e-002 & 2.4844e-001 & 3.4139e-001 & 0 & 0 \\ 2.0726e-001 & 3.0528e-001 & 5.1350e-002 & 0 & 0 \\ 3.0126e-001 & 9.0343e-002 & -2.6264e-001 & 0 & 0 \\ 2.0949e-001 & -1.8209e-001 & -2.7273e-001 & 0 & 0 \\ 4.8494e-013 & -3.0115e-001 & 1.2309e-007 & 1.0000e+000 & 0 \\ -2.0949e-001 & -1.8209e-001 & 2.7273e-001 & -1.0000e+000 & 1.0000e+000 \\ -3.0126e-001 & 9.0343e-002 & 2.6264e-001 & 0 & -1.0000e+000 \\ -2.0726e-001 & 3.0528e-001 & -5.1350e-002 & 0 & 0 \\ 4.8217e-002 & 2.4844e-001 & -3.4139e-001 & 0 & 0 \\ 3.1912e-001 & -1.1578e-001 & -1.5983e-001 & 0 & 0 \\ 3.2607e-001 & -4.3382e-001 & 4.3364e-001 & 0 & 0 \\ -3.3454e-001 & 2.3836e-001 & -1.5449e-001 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{15\times 5},$$

$$U_r = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1.0000e+000 & 0 & 0 & 0 & 0 \\ -1.0000e+000 & 1.0000e+000 & 0 & 0 & 0 \\ 0 & -1.0000e+000 & 1.0000e+000 & 0 & 0 \\ 0 & 0 & -1.0000e+000 & 1.0000e+000 & 0 \\ 0 & 0 & 0 & -1.0000e+000 & 1.0000e+000 \\ 0 & 0 & 0 & 0 & -1.0000e+000 \end{bmatrix} \in \mathbb{R}^{15\times 5}.$$

Gram-Schmidt orthonormalization with the last U as input and W as output delivers

$$W = [W_l \,|\, W_m \,|\, W_r]$$

where

$$W_l = \begin{bmatrix} 1.5416e-001 & -3.4966e-001 & 4.7804e-001 & 4.8946e-001 & 4.2678e-001 \\ 1.9085e-001 & -3.4298e-001 & 3.3329e-001 & 1.4066e-001 & -1.1587e-001 \\ 2.2573e-001 & -3.1907e-001 & 1.8468e-001 & -1.0801e-001 & -3.2307e-001 \\ 2.5709e-001 & -2.7861e-001 & 4.3562e-002 & -2.5063e-001 & 2.9188e-001 \\ 2.8334e-001 & -2.2336e-001 & -7.9298e-002 & -2.9208e-001 & -1.2684e-001 \\ 3.0314e-001 & -1.5605e-001 & -1.7448e-001 & -2.4741e-001 & 7.3476e-002 \\ 3.1545e-001 & -8.0229e-002 & -2.3468e-001 & -1.4015e-001 & 2.2874e-001 \\ 3.1963e-001 & -9.2166e-014 & -2.5528e-001 & 1.1263e-009 & 2.8656e-001 \\ 3.1545e-001 & 8.0229e-002 & -2.3468e-001 & 1.4015e-001 & 2.2874e-001 \\ 3.0314e-001 & 1.5605e-001 & -1.7448e-001 & 2.4741e-001 & 7.3476e-002 \\ 2.8334e-001 & 2.2336e-001 & -7.9298e-002 & 2.9208e-001 & -1.2684e-001 \\ 2.5709e-001 & 2.7861e-001 & 4.3562e-002 & 2.5063e-001 & -2.9188e-001 \\ 2.2573e-001 & 3.1907e-001 & 1.8468e-001 & 1.0801e-001 & -3.2307e-001 \\ 1.9085e-001 & 3.4298e-001 & 3.3329e-001 & -1.4066e-001 & -1.1587e-001 \\ 1.5416e-001 & 3.4966e-001 & 4.7804e-001 & -4.8946e-001 & 4.2678e-001 \end{bmatrix} \in \mathbb{R}^{15\times 5},$$

$$W_m = \left[\begin{array}{ccc|cc} 3.3454e-001 & 2.3836e-001 & 1.5449e-001 & 1.8810e-002 & 3.0811e-002 \\ -3.2607e-001 & -4.3382e-001 & -4.3364e-001 & -7.1290e-002 & -1.0422e-001 \\ -3.1912e-001 & -1.1578e-001 & 1.5983e-001 & 5.7138e-002 & 5.7890e-002 \\ -4.8217e-002 & 2.4844e-001 & 3.4139e-001 & 6.7868e-002 & 1.0746e-001 \\ 2.0726e-001 & 3.0528e-001 & 5.1350e-002 & -3.4973e-002 & -9.6212e-004 \\ 3.0126e-001 & 9.0343e-002 & -2.6264e-001 & -1.3299e-001 & -1.5319e-001 \\ 2.0949e-001 & -1.8209e-001 & -2.7273e-001 & -1.3676e-001 & -2.2452e-001 \\ 2.1841e-013 & -3.0115e-001 & -4.8001e-008 & 6.9414e-001 & 3.6622e-001 \\ -2.0949e-001 & -1.8209e-001 & 2.7273e-001 & -6.6148e-001 & 3.6622e-001 \\ -3.0126e-001 & 9.0343e-002 & 2.6264e-001 & 1.3542e-001 & -7.4602e-001 \\ -2.0726e-001 & 3.0528e-001 & -5.1350e-002 & 9.5257e-002 & 2.1427e-001 \\ 4.8217e-002 & 2.4844e-001 & -3.4139e-001 & 4.0423e-004 & 1.4368e-001 \\ 3.1912e-001 & -1.1578e-001 & -1.5983e-001 & -4.5697e-002 & -1.2195e-003 \\ 3.2607e-001 & -4.3382e-001 & 4.3364e-001 & 1.3928e-002 & -8.5197e-002 \\ -3.3454e-001 & 2.3836e-001 & -1.5449e-001 & 2.2594e-004 & 2.8778e-002 \end{array}\right] \in \mathbb{R}^{15\times 5},$$

$$W_r = \begin{bmatrix} 1.9648e-002 & -1.4242e-002 & -7.9476e-002 & 5.0415e-002 & -1.9917e-002 \\ 5.4351e-002 & 7.3854e-002 & 3.2376e-001 & -2.6729e-001 & 1.3593e-001 \\ 2.6320e-004 & -1.1387e-001 & -3.4982e-001 & 5.0804e-001 & -3.9528e-001 \\ 7.4893e-002 & -1.5865e-002 & -1.7421e-001 & -2.7900e-001 & 6.2895e-001 \\ 5.7608e-002 & 1.4555e-001 & 3.5567e-001 & -3.6645e-001 & -5.8037e-001 \\ 7.5675e-002 & 6.6402e-002 & 3.1861e-001 & 6.1479e-001 & 2.9670e-001 \\ 2.4707e-001 & -2.9156e-001 & -5.6428e-001 & -2.7855e-001 & -6.6574e-002 \\ 1.8374e-001 & 8.0959e-002 & 5.9911e-002 & 4.1446e-003 & 9.1637e-005 \\ 1.8374e-001 & 8.0959e-002 & 5.9911e-002 & 4.1446e-003 & 9.1637e-005 \\ 1.8374e-001 & 8.0959e-002 & 5.9911e-002 & 4.1446e-003 & 9.1637e-005 \\ 7.4713e-001 & 8.0959e-002 & 5.9911e-002 & 4.1446e-003 & 9.1637e-005 \\ 4.1956e-001 & -5.7053e-001 & 5.9911e-002 & 4.1446e-003 & 9.1637e-005 \\ 1.7197e-001 & 6.4891e-001 & -3.2177e-001 & 4.1446e-003 & 9.1637e-005 \\ 2.2900e-001 & -3.0769e-001 & 2.5379e-001 & -1.0729e-002 & 9.1637e-005 \\ 5.8597e-002 & 5.5201e-002 & -6.1827e-002 & 3.8970e-003 & -8.6567e-005 \end{bmatrix} \in \mathbb{R}^{15\times 5}$$

with

$$W = [w_1,\, w_2,\, w_3,\, w_4,\, w_5,\, w_6,\, w_7,\, w_8|,\, w_9,\, w_{10},\, w_{11},\, w_{12},\, w_{13},\, w_{14},\, w_{15}]$$

where

$$w_j \in \mathbb{R}^{15},\ j = 1, \dots n = 15$$

and

$$P_j = w_j\, w_j^T,\ j = 1, \dots, n = 15.$$

The $eig.m$-routine delivers

$$W_l = \begin{bmatrix}
1.5416e-001 & -3.4966e-001 & 4.7804e-001 & 4.8946e-001 & 4.2678e-001 \\
1.9085e-001 & -3.4298e-001 & 3.3329e-001 & 1.4066e-001 & -1.1587e-001 \\
2.2573e-001 & -3.1907e-001 & 1.8468e-001 & -1.0801e-001 & -3.2307e-001 \\
2.5709e-001 & -2.7861e-001 & 4.3562e-002 & -2.5063e-001 & -2.9188e-001 \\
2.8334e-001 & -2.2336e-001 & -7.9298e-002 & -2.9208e-001 & -1.2684e-001 \\
3.0314e-001 & -1.5605e-001 & -1.7448e-001 & -2.4741e-001 & 7.3476e-002 \\
3.1545e-001 & -8.0229e-002 & -2.3468e-001 & -1.4015e-001 & 2.2874e-001 \\
3.1963e-001 & -7.2587e-017 & -2.5528e-001 & -1.5330e-015 & 2.8656e-001 \\
3.1545e-001 & 8.0229e-002 & -2.3468e-001 & 1.4015e-001 & 2.2874e-001 \\
3.0314e-001 & 1.5605e-001 & -1.7448e-001 & 2.4741e-001 & 7.3476e-002 \\
2.8334e-001 & 2.2336e-001 & -7.9298e-002 & 2.9208e-001 & -1.2684e-001 \\
2.5709e-001 & 2.7861e-001 & 4.3562e-002 & 2.5063e-001 & -2.9188e-001 \\
2.2573e-001 & 3.1907e-001 & 1.8468e-001 & 1.0801e-001 & -3.2307e-001 \\
1.9085e-001 & 3.4298e-001 & 3.3329e-001 & -1.4066e-001 & -1.1587e-001 \\
1.5416e-001 & 3.4966e-001 & 4.7804e-001 & -4.8946e-001 & 4.2678e-001
\end{bmatrix} \in \mathbb{R}^{15\times 5},$$

$$W_m = \left[\begin{array}{ccc|cc}
3.3454e-001 & 2.3836e-001 & 1.5449e-001 & -9.0766e-002 & 4.7379e-002 \\
-3.2607e-001 & -4.3382e-001 & -4.3364e-001 & 3.5720e-001 & -2.4523e-001 \\
-3.1912e-001 & -1.1578e-001 & 1.5982e-001 & -3.6835e-001 & 4.2183e-001 \\
-4.8217e-002 & 2.4844e-001 & 3.4139e-001 & -1.6126e-001 & -1.2176e-001 \\
2.0726e-001 & 3.0528e-001 & 5.1350e-002 & 2.7310e-001 & -3.5820e-001 \\
3.0126e-001 & 9.0343e-002 & -2.6264e-001 & 2.5678e-001 & 1.4611e-001 \\
2.0949e-001 & -1.8209e-001 & -2.7273e-001 & -1.0810e-001 & 2.9274e-001 \\
-6.6509e-012 & -3.0115e-001 & 1.8672e-007 & -3.1726e-001 & 2.5096e-002 \\
-2.0949e-001 & -1.8209e-001 & 2.7273e-001 & -1.0790e-001 & -3.3278e-001 \\
-3.0126e-001 & 9.0343e-002 & 2.6264e-001 & 2.5660e-001 & -1.2442e-001 \\
-2.0726e-001 & 3.0528e-001 & -5.1350e-002 & 2.7313e-001 & 3.4682e-001 \\
4.8217e-002 & 2.4844e-001 & -3.4139e-001 & -1.6122e-001 & 1.2992e-001 \\
3.1912e-001 & -1.1578e-001 & -1.5983e-001 & -3.6839e-001 & -4.2686e-001 \\
3.2607e-001 & -4.3382e-001 & 4.3364e-001 & 3.5721e-001 & 2.4699e-001 \\
-3.3454e-001 & 2.3836e-001 & -1.5449e-001 & -9.0767e-002 & -4.7635e-002
\end{array}\right] \in \mathbb{R}^{15\times 5},$$

$$W_r = \begin{bmatrix}
1.8963e-002 & 2.2311e-003 & 9.7443e-003 & 1.4281e-002 & -2.1643e-004 \\
-1.2517e-001 & -1.6545e-002 & -5.8688e-002 & -1.0933e-001 & 4.7213e-003 \\
3.2478e-001 & 5.6758e-002 & 1.1489e-001 & 3.4515e-001 & -4.0587e-002 \\
-3.8792e-001 & -1.2254e-001 & 5.2930e-003 & -5.4036e-001 & 1.6563e-001 \\
1.7161e-001 & 1.8206e-001 & -3.4482e-001 & 3.1992e-001 & -3.6960e-001 \\
-5.9896e-002 & -1.5719e-001 & 4.5565e-001 & 2.5513e-001 & 4.8012e-001 \\
3.0644e-001 & -4.1660e-002 & -7.5345e-003 & -5.2326e-001 & -3.8562e-001 \\
-3.6822e-001 & 3.7146e-001 & -4.2660e-001 & 1.8385e-001 & 2.6312e-001 \\
-4.0145e-002 & -6.0083e-001 & 1.7940e-001 & 2.1152e-001 & -2.3167e-001 \\
2.3354e-001 & 5.5225e-001 & 3.7100e-001 & -1.8263e-001 & 8.3404e-002 \\
1.3959e-001 & -3.2100e-001 & -4.9758e-001 & -4.6602e-002 & 2.3702e-001 \\
-4.7573e-001 & 1.1882e-001 & 2.3722e-001 & 1.2718e-001 & -4.1235e-001 \\
3.7984e-001 & -2.7410e-002 & -2.7237e-002 & -7.0361e-002 & 2.9609e-001 \\
-1.3731e-001 & 3.9426e-003 & -1.4867e-002 & 1.7044e-002 & -1.0539e-001 \\
1.9615e-002 & -3.4920e-004 & 4.1238e-003 & -1.5145e-003 & 1.5326e-002
\end{bmatrix} \in \mathbb{R}^{15\times 5}.$$

Comparing the results, both computed eigenvectors $w_j,\ j = 1, \ldots, (i_q = 8)$ are equal. The eigenvectors $w_j,\ j = 1, \ldots, 15$ computed by the power method combined with the method of deflation are exact in the 5 significant digital places whereas the eigenvectors $w_j,\ j = 9, \ldots, 15$ computed by the Matlab program $eig.m$ are inexact in these 5 digital places.

Using the first method, this entails what could be called displayed eigenvector expansions for u and Bu, namely

$$\boxed{\begin{array}{l} u = \displaystyle\sum_{j=1}^{n=8} (u, w_j)\, w_j + P_{0,15} u, \quad u \in H = \mathbb{R}^{15} \\ \text{where} \\ P_{0,15} u = \displaystyle\sum_{j=9}^{n=15} (u, w_j) w_j, \qquad u \in H = \mathbb{R}^{15} \mapsto N(B). \\ \text{Further,} \\ Bu = \displaystyle\sum_{j=1}^{n=8} \lambda_j (u, w_j) w_j, \qquad u \in H = \mathbb{R}^{15}. \end{array}}$$

Correspondingly, using

$$\boxed{(u, w_j)\, w_j = (w_j^T\, u)\, w_j = w_j\, (w_j^T\, u) = (w_j\, w_j^T)\, u = P_j\, u,\ u \in H = \mathbb{R}^{15}, \quad j = 1, \ldots, n = 15,}$$

we obtain what could be called displayed projection-matrix expansions for the matrices I and B themselves, namely

$$\boxed{\begin{array}{l} I = \displaystyle\sum_{j=1}^{n=8} P_j + P_{0,15} \\ \text{where} \\ P_{0,15} = \displaystyle\sum_{j=9}^{n=15} P_j. \\ \text{Further,} \\ B = \displaystyle\sum_{j=1}^{n=8} \lambda_j\, P_j. \end{array}}$$

At this point, we want to mention again that still the standard eigenvector expansions for u and Bu hold, namely

$$\boxed{\begin{array}{l} u = \displaystyle\sum_{j=1}^{n=15} (u, w_j) w_j, \qquad u \in H = \mathbb{R}^{15} \\ \text{and} \\ Bu = \displaystyle\sum_{j=1}^{n=15} \lambda_j (u, w_j) w_j, \quad u \in H = \mathbb{R}^{15} \end{array}}$$

as well as the standard matrix-projection expansions for the matrices I and B themselves, namely

$$\boxed{\begin{array}{l} I = \sum_{j=1}^{n=15} P_j \\ \text{and} \\ B = \sum_{j=1}^{n=15} \lambda_j \, P_j, \end{array}}$$

but, again, they hide the interesting details.

5.4. **Scrutiny of the Computed Eigenvectors and of Other Results.** Even when an operator $T(\kappa)$ in a finite-dimensional vector space is continuous in κ, the eigenvectors or eigenspaces are not necessarily continuous. That the eigenspaces can behave quite singularly even for a very smooth function $T(\kappa)$ is seen from an example due to Rellich. For these statements, see [3, Chapter II, §5.3, pp. 110-111]. Because of these remarks, we have to scrutinize as to whether the Power Method combined with the Method of Deflation for the Toeplitz sinc matrices $A(t) = A_n(t) = A(t,n)$ can be applied for the determination of the eigenvectors $w_j,\ j = 1, \ldots, n$. We remind the reader in this context that the eigenvalues $\lambda_j,\ j = i_q + 1, \ldots, n$ are set equal to zero, that we have chosen $t = 0.1$, and that the computed orthonormal eigenvectors are denoted by $w_j,\ j = 1, \ldots, n$.

As abbreviation, we set $Dvec = A(t,n)W - W\Lambda$ which is equivalent to $A(t,n)w_j - \lambda_j w_j,\ j = 1, \ldots, n$ where the columns of W contain the computed eigenvectors and where Λ is a diagonal matrix containing in its diagonal the sorted computed eigenvalues $\lambda_1 > \lambda_1 > \cdots > \lambda_{i_q}$ and $\lambda_{i_q+1} = \ldots = \lambda_n = 0$. As further abbreviation, we set $mavec$ for the vector containing the maxima of the absolute values of the components of the columns of $Dvec$.

(i) Testing Results for $n = 5$

We obtain

$$Dvec = \begin{bmatrix} -1.1990e-014 & -1.5575e-012 & 1.7844e-012 & -1.8024e-017 & 1.1505e-016 \\ 5.7732e-015 & 7.2047e-013 & 9.1375e-013 & -4.0365e-017 & 2.4174e-017 \\ 1.1990e-014 & 1.5014e-012 & -1.0246e-014 & -2.8042e-018 & 6.2335e-017 \\ 5.3291e-015 & 7.2045e-013 & -9.3418e-013 & -1.1376e-017 & 6.5803e-017 \\ -1.1990e-014 & -1.5575e-012 & -1.8036e-012 & 7.2568e-017 & 1.2893e-016 \end{bmatrix} \in \mathbb{R}^{5\times 5},$$

and

$$mavec = [1.1990e-014,\ 1.5575e-012,\ 1.8036e-012,\ 7.2568e-017,\ 1.2893e-016] \in \mathbb{R}^5.$$

(ii) Testing Results for $n = 10$

We obtain

$$Dvec = [Dvec_l \,|\, Dvec_r]$$

where

$$Dvec_l = \begin{bmatrix} -8.8818e-016 & -5.4539e-012 & -4.7138e-012 & -1.6187e-014 & 2.1664e-014 \\ 0 & -2.2582e-012 & -3.9732e-012 & 1.7792e-014 & -4.9823e-015 \\ -4.4409e-016 & 3.9424e-013 & -3.0103e-012 & 1.4656e-014 & -1.6283e-014 \\ 0 & 2.2904e-012 & -1.8777e-012 & -1.9610e-015 & -1.5173e-014 \\ 4.4409e-016 & 3.2782e-012 & -6.3800e-013 & -1.5069e-014 & -5.9199e-015 \\ 0 & 3.2782e-012 & 6.3786e-013 & -1.5103e-014 & 6.0224e-015 \\ 0 & 2.2902e-012 & 1.8774e-012 & -1.9497e-015 & 1.5333e-014 \\ -4.4409e-016 & 3.9391e-013 & 3.0100e-012 & 1.4958e-014 & 1.6563e-014 \\ 0 & 2.2583e-012 & 3.9726e-012 & 1.7836e-014 & 5.3725e-015 \\ 0 & -5.4545e-012 & 4.7136e-012 & -1.6250e-014 & -2.1510e-014 \end{bmatrix} \in \mathbb{R}^{10\times 5},$$

$$Dvec_r = \begin{bmatrix} 1.7545e-016 & -2.2594e-016 & 1.1152e-014 & -1.5222e-014 & 1.0723e-014 \\ 1.4024e-016 & -8.8844e-017 & -5.6025e-014 & 7.6742e-014 & 1.3726e-014 \\ 1.6845e-016 & -1.0975e-016 & 1.0001e-013 & -1.3711e-013 & 4.5160e-014 \\ 2.0179e-016 & -1.8193e-016 & -4.7320e-014 & 6.5010e-014 & -9.7843e-015 \\ 2.6614e-016 & -2.0088e-016 & -6.5184e-014 & 8.9147e-014 & -1.6072e-014 \\ 2.1866e-016 & -2.7277e-016 & 6.5156e-014 & -8.9373e-014 & 3.2181e-014 \\ 2.5998e-016 & -9.2108e-017 & 4.7705e-014 & -6.5090e-014 & 2.5208e-014 \\ 1.1626e-016 & -9.5703e-017 & -1.0007e-013 & 1.3688e-013 & -3.0069e-014 \\ 1.6013e-016 & -3.7740e-017 & 5.6191e-014 & -7.6869e-014 & 2.7214e-014 \\ 1.0046e-016 & 7.4239e-017 & -1.0922e-014 & 1.5148e-014 & 1.5283e-015 \end{bmatrix} \in \mathbb{R}^{10\times 5},$$

and

$$mavec = [mavec_l \,|\, mavec_r]$$

where

$$mavec_l = \; [8.8818e-016,\, 5.4545e-012,\, 4.7138e-012,\, 1.7836e-014,\, 2.1664e-014] \;\in \mathbb{R}^5,$$
$$mavec_r = \; [2.6614e-016,\, 2.7277e-016,\, 1.0007e-013,\, 1.3711e-013,\, 4.5160e-014] \;\in \mathbb{R}^5.$$

(iii) Testing Results for $n = 15$
We obtain

$$Dvec = [Dvec_l \,|\, Dvec_m \,|\, Dvec_r\,]$$

where

$$Dvec_l = \begin{bmatrix} -5.1070e-015 & -7.1854e-013 & 5.2358e-013 & -5.5722e-011 & -6.3905e-011 \\ -3.3307e-015 & -5.0160e-013 & 5.1312e-013 & 1.5129e-011 & -1.8365e-011 \\ -1.7764e-015 & -2.7800e-013 & 4.7687e-013 & 4.2181e-011 & 1.4103e-011 \\ 4.4409e-016 & -6.5947e-014 & 4.1565e-013 & 3.8108e-011 & 3.2723e-011 \\ 8.8818e-016 & 1.1924e-013 & 3.3241e-013 & 1.6561e-011 & 3.8135e-011 \\ 1.7764e-015 & 2.6168e-013 & 2.3104e-013 & -9.5931e-012 & 3.2303e-011 \\ 2.6645e-015 & 3.5238e-013 & 1.1718e-013 & -2.9865e-011 & 1.8299e-011 \\ 3.9968e-015 & 3.8313e-013 & -3.7470e-015 & -3.7414e-011 & -2.2524e-017 \\ 3.1086e-015 & 3.5250e-013 & -1.2390e-013 & -2.9865e-011 & -1.8299e-011 \\ 2.6645e-015 & 2.6223e-013 & -2.3784e-013 & -9.5933e-012 & -3.2303e-011 \\ 1.3323e-015 & 1.1924e-013 & -3.3883e-013 & 1.6561e-011 & -3.8136e-011 \\ 0 & -6.5059e-014 & -4.2163e-013 & 3.8108e-011 & -3.2723e-011 \\ -1.3323e-015 & -2.7711e-013 & -4.8167e-013 & 4.2181e-011 & -1.4103e-011 \\ -3.3307e-015 & -5.0049e-013 & -5.1731e-013 & 1.5129e-011 & 1.8365e-011 \\ -5.1070e-015 & -7.1765e-013 & -5.2680e-013 & -5.5722e-011 & 6.3905e-011 \end{bmatrix} \in \mathbb{R}^{15\times 5},$$

$$Fvec_m = \begin{bmatrix} 8.7083e-017 & -8.3483e-017 & 6.7980e-017 & 2.3852e-014 & 6.7613e-014 \\ -9.2601e-017 & -2.7581e-017 & -1.5402e-016 & -9.4089e-014 & -2.6609e-013 \\ -3.8362e-017 & -1.6132e-016 & 7.2962e-017 & 9.7183e-014 & 2.7430e-013 \\ -4.3969e-017 & 7.3537e-017 & 6.0734e-017 & 4.2218e-014 & 1.2030e-013 \\ -1.5728e-016 & -1.2473e-017 & -3.7211e-017 & -7.2109e-014 & -2.0344e-013 \\ -2.2853e-017 & 4.0571e-017 & -5.8071e-017 & -6.7177e-014 & -1.9122e-013 \\ -1.3398e-016 & 6.2083e-018 & -8.7916e-017 & 2.8681e-014 & 8.0496e-014 \\ -1.4081e-016 & 3.1602e-017 & -5.5577e-017 & 8.3283e-014 & 2.3645e-013 \\ -1.5282e-016 & -2.2218e-017 & -3.9310e-017 & 2.7934e-014 & 8.0505e-014 \\ -4.8842e-017 & -9.0342e-018 & 4.3674e-017 & -6.7428e-014 & -1.9117e-013 \\ -1.3168e-016 & 4.4989e-018 & -4.7067e-017 & -7.1451e-014 & -2.0350e-013 \\ -1.4477e-016 & -1.9973e-018 & -3.2573e-017 & 4.2429e-014 & 1.2021e-013 \\ -7.7979e-017 & -1.1565e-016 & 1.1062e-016 & 9.6018e-014 & 2.7447e-013 \\ -1.9811e-017 & -4.4871e-017 & 4.0796e-017 & -9.3506e-014 & -2.6616e-013 \\ -8.4387e-017 & -7.5298e-018 & 5.5178e-017 & 2.3768e-014 & 6.7703e-014 \end{bmatrix} \in \mathbb{R}^{15\times 5},$$

$$Dvec_r = \begin{bmatrix} 7.1290e-014 & 1.4221e-014 & -1.1300e-013 & 1.2664e-014 & -1.9473e-013 \\ -2.8071e-013 & -5.5565e-014 & 4.4363e-013 & -1.1547e-013 & -3.7196e-013 \\ 2.8887e-013 & 5.6894e-014 & -4.5845e-013 & 7.7955e-014 & -5.8111e-013 \\ 1.2738e-013 & 2.5923e-014 & -2.0083e-013 & 1.5469e-014 & -7.3921e-013 \\ -2.1427e-013 & -4.2103e-014 & 3.3878e-013 & -1.0785e-013 & -8.5831e-013 \\ -2.0239e-013 & -4.0816e-014 & 3.1769e-013 & -1.0637e-013 & -9.5018e-013 \\ 8.4299e-014 & 1.5973e-014 & -1.3563e-013 & -9.2369e-015 & -1.0035e-012 \\ 2.4998e-013 & 5.0085e-014 & -3.9545e-013 & 4.5670e-014 & -9.9696e-013 \\ 8.5917e-014 & 1.8079e-014 & -1.3499e-013 & -1.2350e-014 & -9.2147e-013 \\ -2.0172e-013 & -4.0183e-014 & 3.1761e-013 & -1.0986e-013 & -7.8879e-013 \\ -2.1570e-013 & -4.4263e-014 & 3.3751e-013 & -1.1107e-013 & -6.2147e-013 \\ 1.2677e-013 & 2.4952e-014 & -2.0178e-013 & 1.0368e-014 & -4.3230e-013 \\ 2.9114e-013 & 5.9550e-014 & -4.5806e-013 & 7.0337e-014 & -2.1132e-013 \\ -2.8162e-013 & -5.7121e-014 & 4.4202e-013 & -1.2048e-013 & 5.0226e-014 \\ 7.1755e-014 & 1.4516e-014 & -1.1374e-013 & 6.5880e-015 & 2.7011e-013 \end{bmatrix} \in \mathbb{R}^{15\times 5},$$

and

$$mavec = [mavec_l \,|\, mavec_m \,|\, mavec_r]$$

where

$$\begin{aligned} mavec_l &= [5.1070e-015,\ 7.1854e-013,\ 5.2680e-013,\ 5.5722e-011,\ 6.3905e-011] &\in \mathbb{R}^5, \\ mavec_m &= [1.5728e-016,\ 1.6132e-016,\ 1.5402e-016,\ 9.7183e-014,\ 2.7447e-013] &\in \mathbb{R}^5, \\ mavec_r &= [2.9114e-013,\ 5.9550e-014,\ 4.5845e-013,\ 1.2048e-013,\ 1.0035e-012] &\in \mathbb{R}^5. \end{aligned}$$

(iv) Evaluation of the Above Testing Results
The above testing results show that the computed eigenvectors are indeed good eigenvectors of the considered Toeplitz sinc matrices.
(v) Verification of the Eigenvector Expansions for u and Bu
The expansion formulas for u and Bu in Subsection 5.2 were tested with some random vectors u in Matlab. The results verified the respective computed expansions. The expansion formula for u means that $\mathbb{R}^{10}$ is the direct sum of the subspace spanned by $w_1, \ldots, w_7$ and that spanned by w_8, w_9, w_{10}.

Corresponding verifications for the expansion formulas in Subsections 5.1 and 5.3 are left to the reader.
(vi) Further Tests
The computed orthonormal vectors w_j, $j = i_q + 1, \ldots, n$ span the same orthogonal complement of the subspace spanned by the eigenvectors w_j, $j = 1, \ldots, i_q$ as the one spanned by the eigenvectors of the original matrix corresponding to those cutoff eigenvalues close to zero for $j = i_q+1, \ldots, n$. This is because the orthogonal complement of a subspace is unique. Therefore, we suspect that the computed orthonormal vectors might eventually span the same eigenspace as the original ones (or very closely). Some experiments were done, and the numerical results tend to agree. What was done is to construct the eigenvector matrix V with the first i_q columns from the i_q eigenvectors of the original Toeplitz sinc matrix and the last $n-i_q$ columns from the computed orthonormal vectors w_j, $j = i_q + 1, \ldots, n$. The diagonal matrix D has on the main diagonal the first i_q eigenvalues from the Toeplitz sinc matrix and then zeros for the rest. When $VDV^T - A$ was computed, the error is very small, the 2-norm of this residual is at $1e-14$. Therefore, the Gram-Schmidt process can help us reconstruct the eigenvectors corresponding to the eigenvectors close to zero.

6. Conclusions

A new approach for the computation of the eigenvalues and eigenvectors for a Toeplitz sinc matrix $A(t,n)$ is developed that circumvents the difficulties usually encountered for small values of t in $0 < t < 1$. This is important since Toeplitz sinc matrices are ubiquitous also outside the field of mathematics, namely in digital signal processing. The new approach essentially consists in computing only a limited number i_q of eigenvalues and eigenvectors by a conventional method as well as in setting the remaining very small eigenvalues equal to zero and in choosing from the many possible associated eigenvectors very simple ones. At this point we want to mention that this is different from truncating the expansions after i_q terms since then the expansion for $u \in H$ would be incorrect. The new method was scrutinized in several ways showing good agreement between theory and numerical results. The applied strategy leads to a new method for the computation of the eigenvalues and associated eigenvectors of real symmetric Toeplitz sinc matrices that delivers good results. For comparison reasons, we have computed the eigenvalues and eigenvectors also with the Matlab routine $eig.m$ showing only agreement in the first i_q results within the prescribed precision. We think that the power method combined with the method of deflation is also superior to the Matlab routine $eig.m$ in the following aspect: since the last one computes only the whole set of eigenvalues and eigenvectors, the first one is more flexible because it can compute any number of the n eigenvalues and eigenvectors, especially a small number of these quantities. It seems likely that the new method can be extended to other Toeplitz matrices or even to other classes of matrices. One can also imagine that the applied power method combined with the method of deflation may be replaced by different ones. It seems to be even possible that the new approach can be carried over to other sorts of problems.

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