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Research Article

On the limit operators of q-King type sequences of operators

Dedicated to Professor Ioan Raşa, on the occasion of his 75th birthday

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ABSTRACT. We determine the limit operators for a sequence of generalized q-King operators and the sequence of q-Aldaz-Kounchev-Render operators. For each sequence of operators we prove the uniform convergence to the corresponding limit operators, for which we provide quantitative estimates.

Keywords: Bernstein operators, King operators, q-Bernstein operators, q-King type operators, modulus of continuity.

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1. INTRODUCTION

The connection between regular summability matrices and convergent positive linear operators leads to the following operators introduced by King [13, p. 204, (2.1)]:

$$(V_n f)(x) \equiv V_n(f;x) = \sum_{k=0}^{n} p_{n,k}(r_n(x)) f\left(\frac{k}{n}\right),$$

where $f \in C[0,1]$, $\{r_n\}$ is a sequence of continuous functions defined on $[0,1]$ with $0 \le r_n(x) \le 1$ and $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0,1]$. For the special case $r_n(x) = x$, $x \in [0,1]$, we receive the Bernstein operators

$$(B_n f)(x) \equiv B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right). \tag{1.1}$$

We consider the monomial functions $e_i(x) = x^i$, where $x \in [0,1]$ and $i = 0,1,2,\ldots$ The King operators which preserve e_0 and e_2 are given by

$$(V_n^* f)(x) \equiv V_n^*(f;x) = \sum_{k=0}^{n} p_{n,k}(r_n^*(x)) f\left(\frac{k}{n}\right), \tag{1.2}$$

where

$$r_n^*(x) = \begin{cases} x^2, & \text{if} \quad n = 1 \\ \frac{1}{2(n-1)}\left(-1 + \sqrt{4n(n-1)x^2 + 1}\right), & \text{if} \quad n = 2,3,\ldots \end{cases}$$

(see [13, p. 205]). Thus $(V_n^* e_0)(x) = 1$, $(V_n^* e_1)(x) = r_n^*(x)$ and $(V_n^* e_2)(x) = x^2$, $x \in [0,1]$, in contrast with Bernstein operators: $(B_n e_0)(x) = 1$, $(B_n e_1)(x) = x$ and $(B_n e_2)(x) = x^2 + \frac{x(1-x)}{n}$,

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$x \in [0,1]$. The following quantitative estimation is obtained in [13, p. 206, Theorem 3.1]:

$$|(V_n^* f)(x) - f(x)| \leq 2\,\omega(f; \sqrt{2x(x - r_n^*(x))}\,), \quad x \in [0,1],$$

where the usual modulus of continuity of $f \in C[0,1]$ is defined by

$$\omega(f;\delta) = \sup\{|f(x) - f(y)| : x, y \in [0,1], |x - y| < \delta\}, \quad \delta > 0. \tag{1.3}$$

A systematic study of the operators V_n^* $(n = 1,2,3,\dots)$ is due to Gonska and Pițul [11], who determine new estimates for the rate of convergence in terms of the first and second modulus of continuity. The existence of a sequence of linear positive bounded *polynomial* operators on $C[0,1]$, possessing e_0 and e_2 as fixed points, was proved in [6]. Furthermore, in [10], we proved the unique existence of the functions r_n $(n = 1,2,3,\dots)$ on $[0,1]$ such that the corresponding sequence of King operators approximate each continuous function on $[0,1]$ and preserve e_0 and e_j, where $j \in \{2,3,\dots\}$ is fixed. Main results concerning certain King type modifications of the Bernstein operators and the Szász-Mirakyan operators were presented in the survey paper [1].

Replacing $f\left(\frac{k}{n}\right)$ in (1.1) with $f\left(\sqrt[j]{\frac{k(k-1)\dots(k-j+1)}{n(n-1)\dots(n-j+1)}}\right)$, $n \geq j \geq 2$, Aldaz, Kounchev and Render [4, p. 12, Proposition 11] defined a new King type operator

$$(U_n f)(x) \equiv U_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\sqrt[j]{\frac{k(k-1)\dots(k-j+1)}{n(n-1)\dots(n-j+1)}}\right), \tag{1.4}$$

where $f \in C[0,1]$ and $x \in [0,1]$. These operators reproduce e_0 and e_j, where $j \in \{2,3,\dots\}$ is fixed, and $U_n f \to f$ uniformly for all $f \in C[0,1]$. In [7], we proved that there exist infinitely many sequences of Bernstein type operators, which have similar properties to (1.4). Further properties of the Bernstein type operators of Aldaz, Kounchev and Render were obtained in the papers [2] and [5].

The q-Bernstein operators introduced by Phillips [16] are a new generalization of (1.1). Let $0 < q \leq 1$ and denote the q-integers by $[n]_q = 1 + q + \dots + q^{n-1}$ for $n = 1,2,\dots$ and $[0]_q = 0$. Besides, let $[n]_q! = [1]_q[2]_q \dots [n]_q$ for $n = 1,2,\dots$ and $[0]_q! = 1$. Then the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad 0 \leq k \leq n.$$

The Bernstein operators based on q-integers are given by

$$(B_{n,q} f)(x) \equiv B_{n,q}(f;x) = \sum_{k=0}^{n} p_{n,k}(q;x) f\left(\frac{[k]_q}{[n]_q}\right), \tag{1.5}$$

where $f \in C[0,1]$ and

$$p_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)(1-xq)\dots(1-xq^{n-k-1}), \quad x \in [0,1]$$

(an empty product in (1.5) is taken to equal 1). For $q = 1$, (1.5) reduces to (1.1). Il'inskii and Ostrovska [12] proved the existence of the limit $\lim\limits_{n\to\infty} B_{n,q}$ for $q \in (0,1)$ given, obtaining the so-called limit q-Bernstein operator defined by

$$(B_{\infty,q} f)(x) \equiv B_{\infty,q}(f;x) = \begin{cases} \sum\limits_{k=0}^{\infty} p_{\infty,k}(q;x) f(1-q^k), & \text{if} \quad x \in [0,1) \\ f(1), & \text{if} \quad x = 1, \end{cases} \tag{1.6}$$

where $p_{\infty,k}(q;x) = \frac{x^k}{(1-q)^k[k]_q!}\prod_{s=0}^{\infty}(1-xq^s)$. They proved for $q\in(0,1)$ fixed that $B_{n,q}f \to B_{\infty,q}f$ uniformly for each $f \in C[0,1]$, and $B_{\infty,q}f \to f$ uniformly for any $f \in C[0,1]$ and $q \uparrow 1$ (see [12, p. 104, Theorem 2 and Theorem 1]). Moreover, Wang and Meng showed in [20, p. 153, Theorem 1] that if $q \in (0,1)$ and $f \in C[0,1]$, then

$$|(B_{n,q}f)(x) - (B_{\infty,q}f)(x)| \le \left(2 + \frac{4\ln\frac{1}{1-q}}{q(1-q)}\right)\omega(f;q^n), \quad x\in[0,1]. \tag{1.7}$$

Videnskii obtained in [19, p. 221, Theorem 8.1] that if $f \in C[0,1]$, then

$$|(B_{\infty,q}f)(x) - f(x)| \le 2\,\omega\left(f;\frac{1}{2}\sqrt{1-q}\right), \quad x\in[0,1]. \tag{1.8}$$

The King type operators based on q-integers were introduced in [3, p. 7] as follows (see (1.2) and (1.5)):

$$(V^*_{n,q}f)(x) \equiv V^*_{n,q}(f;x) = \sum_{k=0}^{n} p_{n,k}(q;r^*_{n,q}(x))f\left(\frac{[k]_q}{[n]_q}\right), \tag{1.9}$$

where $0<q\le 1$, $f\in C[0,1]$, $n\ge 2$ and

$$r^*_{n,q}(x) = \frac{1}{2([n]_q-1)}\left(-1+\sqrt{4[n]_q[n-1]_qx^2+1}\right), \quad x\in[0,1].$$

We mention that $V^*_{n,q}$ preserves the functions e_0 and e_2. The generalized q-King operators were given in [8, p. 346] as follows:

$$(V_{n,q}f)(x) \equiv V_{n,q}(f;x) = \sum_{k=0}^{n} p_{n,k}(q;r_{n,q}(x))f\left(\frac{[k]_q}{[n]_q}\right), \tag{1.10}$$

where $0<q\le 1$, the continuous functions $r_{n,q}$ $(n=1,2,3,\ldots)$ are defined on $[0,1]$ with $0\le r_{n,q}(x)\le 1$, $f\in C[0,1]$ and $p_{n,k}(q;x)$ are the polynomials considered in (1.5). We proved [8, p. 349, Theorem 1] that there exists a unique sequence of continuous functions $\{r_{n,q}\}$ such that $V_{n,q}$ reproduces e_0 and e_j, where $j\in\{2,3,\ldots\}$ is fixed, and $V_{n,q_n}f \to f$ uniformly for each $f\in C[0,1]$ and $q_n\in(0,1)$, $q_n\to 1$; the rate of convergence is estimated with the aid of the modulus of continuity (1.3). For $n\ge j$ and $q\in(0,1)$, the function $r_{n,q}$ will be the unique solution of the equation

$$(B_{n,q}e_j)(y) - x^j \equiv \sum_{k=0}^{n} p_{n,k}(q;y)\left(\frac{[k]_q}{[n]_q}\right)^j - x^j = 0, \quad x\in[0,1] \tag{1.11}$$

(see [8, p. 349, Theorem 1] and Lemma 2.2 below). For $j=2$, $V_{n,q}$ is identical with $V^*_{n,q}$.

The q-Aldaz-Kounchev-Render operators were defined in [9, p. 756, (3.1)]: for $0<q\le 1$, $n\ge j$, $f\in C[0,1]$ and $x\in[0,1]$, we set

$$(U_{n,q}f)(x) \equiv U_{n,q}(f;x) = \sum_{k=0}^{n} p_{n,k}(q;x)f\left(\sqrt[j]{\frac{[k]_q[k-1]_q\ldots[k-j+1]_q}{[n]_q[n-1]_q\ldots[n-j+1]_q}}\right), \tag{1.12}$$

where $j\in\{2,3,\ldots\}$ is fixed and $p_{n,k}(q;x)$ is given in (1.5). We mention that $[k]_q[k-1]_q\ldots[k-j+1]_q = [0]_q = 0$ for $k=0,1,\ldots,j-1$, and the operator $U_{n,1}$ is identical with U_n defined by (1.4). Furthermore, we have $U_{n,q}e_0 = e_0$ and $U_{n,q}e_j = e_j$. Among others, a new Korovkin type theorem and its converse theorem are established in [9]. As applications we obtained quantitative estimates for q-Bernstein type operators which preserve e_0 and e_j.

The main goal of the paper is to determine the limit operators of the sequences of operators given by (1.10) and (1.12). We establish the essential properties of the functions $r_{n,q}$ defined by (1.11) that guarantee the uniform convergence of $V_{n,q}f$ to its limit operator for each $f \in C[0,1]$ and $q \in (0,1)$ given. We also give an estimate similar to (1.8) for the limit operator of the sequence of operators $\{V_{n,q}\}$. Furthermore, we establish quantitative estimates similar to (1.7) and (1.8) for the limit operator of the sequence of operators defined by (1.12).

2. The Case of Generalized q-King Operators

In what follows we need some lemmas.

Lemma 2.1. *Let $j \in \{2,3,\ldots\}$ be given, $n \geq j$ and $0<q<1$. Then*

$$(B_{n,q}e_j)(y) = \sum_{k=0}^{n} p_{n,k}(q;y)\left(\frac{[k]_q}{[n]_q}\right)^j, \quad y \in [0,1]$$

is a polynomial in y of degree $\leq j$. Moreover, $(B_{n,q}e_j)(y) = a_0y^j + a_1y^{j-1} + \ldots + a_{j-1}y$, $y \in [0,1]$, where $a_0, a_1, \ldots, a_{j-1}$ depend on n and q, and $a_0 = \left(1-\frac{[1]_q}{[n]_q}\right)\left(1-\frac{[2]_q}{[n]_q}\right)\ldots\left(1-\frac{[j-1]_q}{[n]_q}\right)$, a_0, a_1, $\ldots$, $a_{j-1} > 0$, $a_0 + a_1 + \ldots + a_{j-1} = 1$.

For the proof see [8, p. 346, Lemma 1].

Lemma 2.2. *Let $j \in \{2,3,\ldots\}$ be given, $n \geq j$ and $0<q<1$. Then, the equation*

$$(B_{n,q}e_j)(y) - x^j = 0, \quad x \in [0,1]$$

has a unique solution $r_{n,q}(x)$, $x \in [0,1]$ such that $r_{n,q}(0)=0$, $r_{n,q}(1)=1$ and $0<r_{n,q}(x)<1$ for $x \in (0,1)$.

Proof. By [15, p. 236, (6)-(7)], we have

$$(B_{n,q}f)(y) = \sum_{k=0}^{n} \lambda_{k,n}\, f\left[0, \frac{1}{[n]_q}, \ldots, \frac{[k]_q}{[n]_q}\right] y^k,$$

where $\lambda_{0,n} = \lambda_{1,n} = 1$, $\lambda_{k,n} = \prod_{i=1}^{k-1}\left(1-\frac{[i]_q}{[n]_q}\right)$, $k = 2,3,\ldots,n$ and $f[y_0,y_1,\ldots,y_k]$ is the k-th order divided difference of f with distinct nodes $y_0,y_1,\ldots,y_k$:

$$f[y_0] = f(y_0) \quad f[y_0,y_1,\ldots,y_k] = \frac{f[y_1,\ldots,y_k] - f[y_0,\ldots,y_{k-1}]}{y_k - y_0}.$$

Hence

$$(B_{n,q}e_j)(y) = \sum_{k=0}^{n} \lambda_{k,n}\, e_j\left[0, \frac{1}{[n]_q}, \ldots, \frac{[k]_q}{[n]_q}\right] y^k. \tag{2.13}$$

In view of [17, p. 10, (1.33)], there exists $\xi_k \in \left(0, \frac{[k]_q}{[n]_q}\right)$ such that

$$\begin{aligned} e_j\left[0, \frac{1}{[n]_q}, \ldots, \frac{[k]_q}{[n]_q}\right] &= \frac{1}{k!}e_j^{(k)}(\xi_k) \\ &= \begin{cases} 0, & \text{if} \quad k=0 \\ \frac{1}{k!}j(j-1)\ldots(j-k+1)(\xi_k)^{j-k}, & \text{if} \quad 1 \leq k \leq j \\ 0, & \text{if} \quad j+1 \leq k \leq n. \end{cases} \end{aligned} \tag{2.14}$$

By (2.13) and (2.14), we get

$$(B_{n,q}e_j)'(y) = \sum_{k=1}^{n} \lambda_{k,n}\, e_j \left[0, \frac{1}{[n]_q}, \ldots, \frac{[k]_q}{[n]_q}\right] ky^{k-1} > 0 \tag{2.15}$$

for all $y \in [0,1]$, thus the function $B_{n,q}e_j$ is strictly increasing on $[0,1]$. Because $(B_{n,q}e_j)(0) = 0$, $(B_{n,q}e_j)(1) = 1$ and $B_{n,q}e_j \in C[0,1]$, therefore the equation $(B_{n,q}e_j)(y) - x^j = 0$, $x \in [0,1]$ has a unique solution noted by $y = r_{n,q}(x)$, $x \in [0,1]$ such that $r_{n,q}(0) = 0$, $r_{n,q}(1) = 1$ and $0 < r_{n,q}(x) < 1$ for $x \in (0,1)$. □

The essential properties of $r_{n,q}$ $(n = j, j+1, j+2, \ldots)$ from our point of view are summarized by the following lemma.

Lemma 2.3. *Let $j \in \{2,3,\ldots\}$ be given, $n \geq j$ and $0 < q < 1$. The functions $r_{n,q}$ $(n \geq j)$ determined in Lemma 2.2 satisfy the following properties:*

a) *$r_{n,q}$ is differentiable on $[0,1]$;*
b) *$r_{n,q}(x) \leq r_{n+1,q}(x) \leq x$ for all $x \in [0,1]$;*
c) *there exists a function $r_q : [0,1] \to \mathbb{R}$ continuous on the $[0,1]$ such that $\lim\limits_{n\to\infty} r_{n,q}(x) = r_q(x)$ uniformly for $x \in [0,1]$;*
d) *$r_{n,q}(x) \leq r_q(x) \leq x$ for all $x \in [0,1]$.*

Proof. $a)$ Because the function $B_{n,q}e_j$ is continuous and strictly increasing on $[0,1]$, there exists the continuous and strictly increasing inverse function $(B_{n,q}e_j)^{-1}$ on $[0,1]$, due to the following continuous inverse theorem: If $\varphi : [a,b] \to \mathbb{R}$ is a strictly increasing and continuous function, then the inverse mapping $\varphi^{-1} : [\varphi(a), \varphi(b)] \to [a,b]$ exists and is strictly increasing and continuous function on $[\varphi(a), \varphi(b)]$.

On the other hand, by Lemma 2.2, we have $(B_{n,q}e_j)(r_{n,q}(x)) - x^j = 0$, $x \in [0,1]$, which implies that $r_{n,q}(x) = ((B_{n,q}e_j)^{-1} \circ e_j)(x)$, $x \in [0,1]$. Thus $r_{n,q}$ is differentiable on $[0,1]$ and

$$r'_{n,q}(x) = ((B_{n,q}e_j)^{-1})'(x^j) \cdot e'_j(x) = \frac{jx^{j-1}}{(B_{n,q}e_j)'(r_{n,q}(x))}, \quad x \in [0,1], \tag{2.16}$$

because of (2.15) and $(B_{n,q}e_j)(r_{n,q}(x)) = x^j$, $x \in [0,1]$.

$b)$ By [17, p. 268, (7.56)], we have $B_{n,q}(e_1; x) = x$, $x \in [0,1]$. Applying Lemma 2.2 and Jensen's inequality to the convex function e_j on $[0,1]$, we get

$$\begin{aligned} x^j &= B_{n,q}(e_j; r_{n,q}(x)) = \sum_{k=0}^{n} p_{n,k}(q; r_{n,q}(x)) \left(\frac{[k]_q}{[n]_q}\right)^j \geq \left(\sum_{k=0}^{n} p_{n,k}(q; r_{n,q}(x)) \left(\frac{[k]_q}{[n]_q}\right)\right)^j \\ &= (B_{n,q}(e_1; r_{n,q}(x)))^j = (r_{n,q}(x))^j. \end{aligned}$$

Hence $r_{n,q}(x) \leq x$, $x \in [0,1]$. Furthermore, by [17, p. 270, Theorem 7.3.4], $(B_{n,q}e_j)(x) \geq (B_{n+1,q}e_j)(x)$, $x \in [0,1]$. Because $(B_{n,q}e_j)(0) = 0 = (B_{n+1,q}e_j)(0)$ and $(B_{n,q}e_j)(1) = 1 = (B_{n+1,q}e_j)(1)$, by continuous inverse theorem (see also [10, p. 91, Lemma 2.1]), we find that $(B_{n,q}e_j)^{-1}(y) \leq (B_{n+1,q}e_j)^{-1}(y)$, $y \in [0,1]$. Then $r_{n,q}(x) = (B_{n,q}e_j)^{-1}(x^j) \leq (B_{n+1,q}e_j)^{-1}(x^j) = r_{n+1,q}(x)$, $x \in [0,1]$.

$c)$ Due to (2.16) and (2.15), we have $r'_{n,q}(x) \geq 0$ for each $x \in [0,1]$. Thus the functions $r_{n,q}$ $(n = j, j+1, j+2, \ldots)$ are increasing on $[0,1]$.

In view of $b)$, the sequence $\{r_{n,q}(x)\}_{n\geq j}$ is increasing and bounded above for all $x \in [0,1]$, therefore it is convergent. We set $r_q(x) = \lim\limits_{n\to\infty} r_{n,q}(x)$, $x \in [0,1]$.

Let $x_0, x \in [0,1]$. By Lemma 2.1, we have

$$\begin{aligned}
&(B_{n,q}e_j)(r_{n,q}(x)) - (B_{n,q}e_j)(r_{n,q}(x_0)) \\
&= a_0\left\{(r_{n,q}(x))^j - (r_{n,q}(x_0))^j\right\} + a_1\left\{(r_{n,q}(x))^{j-1} - (r_{n,q}(x_0))^{j-1}\right\} + \ldots \\
&+ a_{j-1}\left\{r_{n,q}(x) - r_{n,q}(x_0)\right\} \\
&= \left\{r_{n,q}(x) - r_{n,q}(x_0)\right\}\left\{a_0\left((r_{n,q}(x))^{j-1} + \ldots + (r_{n,q}(x_0))^{j-1}\right)\right. \\
&\left.+ a_1\left((r_{n,q}(x))^{j-2} + \ldots + (r_{n,q}(x_0))^{j-2}\right) + \ldots + a_{j-1}\right\}.
\end{aligned}$$

Hence, by Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}
&|(B_{n,q}e_j)(r_{n,q}(x)) - (B_{n,q}e_j)(r_{n,q}(x_0))| \\
&\geq |r_{n,q}(x) - r_{n,q}(x_0)|\, a_0\left\{(r_{n,q}(x))^{j-1} + \ldots + (r_{n,q}(x_0))^{j-1}\right\} \\
&\geq |r_{n,q}(x) - r_{n,q}(x_0)|\left(1 - \frac{[1]_q}{[n]_q}\right)\left(1 - \frac{[2]_q}{[n]_q}\right)\ldots\left(1 - \frac{[j-1]_q}{[n]_q}\right)(r_{n,q}(x_0))^{j-1}.
\end{aligned}$$

Due to Lemma 2.2,

$$\begin{aligned}
&|r_{n,q}(x) - r_{n,q}(x_0)|\left(1 - \frac{[1]_q}{[n]_q}\right)\left(1 - \frac{[2]_q}{[n]_q}\right)\ldots\left(1 - \frac{[j-1]_q}{[n]_q}\right)(r_{n,q}(x_0))^{j-1} \\
&\leq |x^j - x_0^j| = |x - x_0||x^{j-1} + \ldots + x_0^{j-1}| \leq j|x - x_0|.
\end{aligned}$$

By passing to the limit in terms of n, we get

$$|r_q(x) - r_q(x_0)|\, q\, q^2 \ldots q^{j-1}\,(r_q(x_0))^{j-1} \leq j|x - x_0|.$$

If $r_q(x_0) \neq 0$, then we obtain the continuity of r_q at x_0. Furthermore, in view of Lemma 2.2, we have $r_{n,q}(0) = 0$ and $0 < r_{n,q}(x) < 1$ for $x \in (0,1)$. At the same time, the sequence $\{r_{n,q}(x)\}_{n \geq j}$ is increasing for all $x \in (0,1)$, therefore if $r_q(x_0) = 0$, then $x_0 = 0$. Because $0 \leq r_{n,q}(x) \leq x$, $x \in [0,1]$ (see b)), we have $|r_{n,q}(x) - r_q(x_0)| \leq x$, $x \in [0,1]$. By passing to the limit in terms of n, we find that $|r_q(x) - r_q(x_0)| \leq x$, $x \in [0,1]$, therefore r_q is continuous at $x_0 = 0$.

Based on the properties established for the functions r_q and $r_{n,q}$ $(n = 1, 2, 3, \ldots)$, we obtain the uniform convergence $\lim_{n\to\infty} r_{n,q}(x) = r_q(x)$ on $[0,1]$ by Pólya's theorem of uniform convergence (see [18, p. 81, Problem 127]).

d) Due to *b*), we have $r_{n,q}(x) \leq r_{m,q}(x) \leq x$ for each $x \in [0,1]$ and $n \leq m$. By passing to the limit in terms of m, in view of *c*), we get $r_{n,q}(x) \leq r_q(x) \leq x$, $x \in [0,1]$, which completes the proof of the lemma. □

Taking into account (1.10), (1.5) and (1.6), we have $V_{n,q}(f;x) = B_{n,q}(f; r_{n,q}(x))$, therefore the corresponding limit operator is defined by

$$(V_{\infty,q}f)(x) \equiv V_{\infty,q}(f;x) = B_{\infty,q}(f; r_q(x)) = \begin{cases} \sum_{k=0}^{\infty} p_{\infty,k}(q; r_q(x)) f(1 - q^k), & \text{if} \quad x \in [0,1) \\ f(1), & \text{if} \quad x = 1. \end{cases}$$

We have the following result.

Theorem 2.1. *Let $q \in (0,1)$ be given. Then for each $f \in C[0,1]$ we have*

$$\lim_{n\to\infty} (V_{n,q}f)(x) = (V_{\infty,q}f)(x)$$

uniformly for $x \in [0,1]$.

Proof. For all $x \in [0,1]$, we have

$$\begin{aligned} &|(V_{n,q}f)(x) - (V_{\infty,q}f)(x)| \\ &\le |B_{n,q}(f; r_{n,q}(x)) - B_{\infty,q}(f; r_{n,q}(x))| + |B_{\infty,q}(f; r_{n,q}(x)) - B_{\infty,q}(f; r_q(x))|. \end{aligned} \tag{2.17}$$

Because $\omega(f; q^n) \to 0$ as $n \to \infty$, by (1.7), for each $\varepsilon > 0$ there exists $n'_\varepsilon \in \mathbb{N}$ such that for all $n > n'_\varepsilon$ and $x \in [0,1]$, we have

$$|B_{n,q}(f; r_{n,q}(x)) - B_{\infty,q}(f; r_{n,q}(x))| < \frac{\varepsilon}{2}. \tag{2.18}$$

On the other hand, [12, p. 104, Theorem 2] implies that $\lim_{n\to\infty} (B_{n,q}f)(x) = (B_{\infty,q}f)(x)$ uniformly on $[0,1]$. Thus $B_{\infty,q}f \in C[0,1]$, therefore $B_{\infty,q}f$ is uniformly continuous on $[0,1]$ for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $u, v \in [0,1]$, $|u - v| < \delta$, we have

$$|B_{\infty,q}(f; u) - B_{\infty,q}(f; v)| < \frac{\varepsilon}{2}. \tag{2.19}$$

In view of Lemma 2.3, c), we have $\lim_{n\to\infty} r_{n,q}(x) = r_q(x)$ uniformly on $[0,1]$. Then there exists $n''_\varepsilon \in \mathbb{N}$ such that for each $n > n''_\varepsilon$ and $x \in [0,1]$, we have

$$|r_{n,q}(x) - r_q(x)| < \delta. \tag{2.20}$$

Combining (2.19) and (2.20), we obtain for all $n > n''_\varepsilon$ and $x \in [0,1]$ that

$$|B_{\infty,q}(f; r_{n,q}(x)) - B_{\infty,q}(f; r_q(x))| < \frac{\varepsilon}{2}. \tag{2.21}$$

In conclusion, by (2.17), (2.18) and (2.21), we get that for each $\varepsilon > 0$ there exists $n_\varepsilon = \max(n'_\varepsilon, n''_\varepsilon)$ such that for all $n > n_\varepsilon$ and $x \in [0,1]$, we have $|(V_{n,q}f)(x) - (V_{\infty,q}f)(x)| < \varepsilon$, which means that $\lim_{n\to\infty} (V_{n,q}f)(x) = (V_{\infty,q}f)(x)$ uniformly for $x \in [0,1]$. □

Theorem 2.2. *Let $j \in \{2,3,\ldots\}$ be given and $q \in (0,1)$. If $f \in C[0,1]$ and $x \in [0,1]$, then*

$$|(V_{\infty,q}f)(x) - f(x)| \le \left(1 + \sqrt{\frac{1}{2} + 2\sqrt[j]{\frac{1}{4} j^2 (j-1)^2}}\right) \omega(f; \sqrt[j]{1-q}). \tag{2.22}$$

Proof. Because $\sum_{k=0}^{\infty} p_{\infty,k}(q; x) = 1$, $x \in [0,1)$ (see [12, p. 103, (13)]), and applying the proof of general Popoviciu's theorem [14, p. 20, Theorem 1.6.1], we obtain

$$|(V_{\infty,q}f)(x) - f(x)| \le \omega(f; \delta) \left\{ 1 + \delta^{-1} \left(\sum_{k=0}^{\infty} p_{\infty,k}(q; r_q(x))(1 - q^k - x)^2 \right)^{1/2} \right\}, \tag{2.23}$$

where $\delta > 0$ and $x \in [0,1)$. Because of $(a+b)^2 \le 2(a^2 + b^2)$, $a, b \in \mathbb{R}$ and $B_{\infty,q}((e_1 - xe_0)^2; x) = (1-q)x(1-x)$ (see [19, p. 221, (7.12)]), we get

$$\begin{aligned} &\sum_{k=0}^{\infty} p_{\infty,k}(q; r_q(x))(1 - q^k - x)^2 \\ &\le 2\sum_{k=0}^{\infty} p_{\infty,k}(q; r_q(x))(1 - q^k - r_q(x))^2 + 2\sum_{k=0}^{\infty} p_{\infty,k}(q; r_q x)(x - r_q(x))^2 \\ &= 2B_{\infty,q}((e_1 - r_q(x)e_0)^2; r_q(x)) + 2(x - r_q(x))^2 \\ &= 2(1-q)r_q(x)(1 - r_q(x)) + 2(x - r_q(x))^2. \end{aligned} \tag{2.24}$$

Due to the estimation $|x - r_{n,q}(x)| \leq ([1]_q + [2]_q + \ldots + [j-1]_q)^{1/j} [n]_q^{-1/j}$ given in [8, p. 350, (10)], we find

$$(x - r_{n,q}(x))^2 \leq ([1]_q + [2]_q + \ldots + [j-1]_q)^{2/j} [n]_q^{-2/j} \leq (1 + 2 + \ldots + j - 1)^{2/j} [n]_q^{-2/j}.$$

By passing to the limit in terms of n and using Lemma 2.3, c), we obtain

$$(x - r_q(x))^2 \leq \left(\frac{1}{2} j(j-1)\right)^{2/j} (1-q)^{2/j}, \quad x \in [0,1).$$

Hence, by (2.24),

$$\sum_{k=0}^{\infty} p_{\infty,k}(q; r_q(x))(1 - q^k - r_q(x))^2$$
$$\leq \frac{1}{2}(1-q) + 2\left(\frac{1}{2} j(j-1)\right)^{2/j} (1-q)^{2/j} \leq \left(\frac{1}{2} + 2\sqrt[j]{\frac{1}{4} j^2 (j-1)^2}\right)(1-q)^{2/j}.$$

Now, (2.23) implies that

$$|(V_{\infty,q} f)(x) - f(x)| \leq \omega(f;\delta) \left\{1 + \delta^{-1} \sqrt{\frac{1}{2} + 2\sqrt[j]{\frac{1}{4} j^2 (j-1)^2}}\, (1-q)^{1/j}\right\}.$$

Taking $\delta = (1-q)^{1/j}$, we obtain (2.22) for $x \in [0,1)$. Finally, if $x = 1$, then $(V_{\infty,q} f)(1) = (B_{\infty,q} f)(r_q(1)) = (B_{\infty,q} f)(1) = f(1)$, because $1 = r_{n,q}(1) = \lim_{n\to\infty} r_{n,q}(1) = r_q(1)$ in view of Lemma 2.2 and Lemma 2.3, c). Thus (2.22) is true for all $x \in [0,1]$. □

Remark 2.1. *The limit operator of the sequence of operators given by (1.9) is the following:*

$$(V^*_{\infty,q} f)(x) = (B_{\infty,q} f)(r^*_q(x)) = \begin{cases} \sum_{k=0}^{\infty} p_{\infty,k}(q; r^*_q(x)) f(1-q^k), & \text{if} \quad x \in [0,1) \\ f(1), & \text{if} \quad x = 1, \end{cases}$$

where $r^*_q(x) = \frac{1}{2q}\left(-1 + q + \sqrt{x^2 + (1-q)^2}\right)$, $x \in [0,1]$.

3. The Case of q-Aldaz-Kounchev-Render Operators

The q-Aldaz-Kounchev-Render operators (see (1.12)) depend on the nodes

$$a_{n,k} \equiv a_{n,k}(q,j)$$
$$= \sqrt[j]{\frac{[k]_q [k-1]_q \ldots [k-j+1]_q}{[n]_q [n-1]_q \ldots [n-j+1]_q}} = \sqrt[j]{\frac{(1-q^k)(1-q^{k-1})\ldots(1-q^{k-j+1})}{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-j+1})}}, \quad k = 0,1,\ldots,n.$$

With the notation

$$a_{\infty,k} \equiv a_{\infty,k}(q,j) = \sqrt[j]{(1-q^k)(1-q^{k-1})\ldots(1-q^{k-j+1})}, \quad k = 0,1,2,\ldots,$$

the corresponding limit operator of the sequence of operators given by (1.12) is the following:

$$(U_{\infty,q} f)(x) = U_{\infty,q}(f;x) = \begin{cases} \sum_{k=0}^{\infty} p_{\infty,k}(q;x) f(a_{\infty,q}), & \text{if} \quad x \in [0,1) \\ f(1), & \text{if} \quad x = 1. \end{cases} \tag{3.25}$$

The next result contains the properties of the nodes $a_{n,k}$ and $a_{\infty,k}$.

Lemma 3.4. *Let $j \in \{2,3,\ldots\}$ be given and $n \geq j$. Then for the numbers $a_{n,k}$ $(k=0,1,\ldots,n)$ and $a_{\infty,k}$ $(k=0,1,2,\ldots)$, we have*

a) $a_{n,0}=a_{n,1}=\ldots=a_{n,j-1}=0$, $a_{n,n}=1$ *and* $a_{\infty,0}=a_{\infty,1}=\ldots=a_{\infty,j-1}=0$;

b) $0<a_{n,k}-a_{\infty,k}<q^{n-j+1}$ *for* $k=j,j+1,\ldots,n$;

c) $\frac{[j]_q}{j}\,q^{k-j+1}<1-a_{\infty,k}<q^{k-j+1}$ *for* $k=j,j+1,j+2,\ldots$

Proof. $a)$ It follows immediately from the definitions of $a_{n,k}$ and $a_{\infty,k}$. For $k=0,1,\ldots,j-1$, the value of $a_{n,k}$ can be calculated using its second form.
$b)$ Because $0<q^{k-i}<1$ and $0<q^{n-i}<1$ for $k\in\{j,j+1,\ldots,n\}$ and $i\in\{0,1,\ldots,j-1\}$, we get

$$a_{n,k}-a_{\infty,k} = \sqrt[j]{\frac{(1-q^k)(1-q^{k-1})\ldots(1-q^{k-j+1})}{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-j+1})}}\left(1-\sqrt[j]{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-j+1})}\right)>0.$$

On the other hand, by the inequality of geometric and harmonic means and the inequalities $1-q^n>1-q^{n-1}>\ldots>1-q^{n-j+1}$, $n\geq j$, we obtain

$$\begin{aligned}&\sqrt[j]{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-j+1})}\\&>\frac{j}{\frac{1}{1-q^n}+\frac{1}{1-q^{n-1}}+\ldots+\frac{1}{1-q^{n-j+1}}}\geq\frac{1}{\frac{1}{1-q^{n-j+1}}}=1-q^{n-j+1}.\end{aligned}\tag{3.26}$$

But $\frac{1-q^{k-i}}{1-q^{n-i}}\leq 1$ for $i\in\{0,1,\ldots,j-1\}$, since $q^{n-i}\leq q^{k-i}$, where $n\geq j$, $k\in\{j,j+1,\ldots,n\}$ and $i\in\{0,1,\ldots,j-1\}$. In conclusion $a_{n,k}-a_{\infty,k}<q^{n-j+1}$ for $k=j,j+1,\ldots,n$.
$c)$ By the inequality of arithmetic and geometric means, for $k=j,j+1,\ldots,n$, we have

$$\begin{aligned}&\sqrt[j]{(1-q^k)(1-q^{k-1})\ldots(1-q^{k-j+1})}\\&<\frac{1-q^k+1-q^{k-1}+\ldots+1-q^{k-j+1}}{j}=\frac{j-q^{k-j+1}(1+q+\ldots+q^{j-1})}{j}=1-\frac{[j]_q}{j}q^{k-j+1}\end{aligned}$$

or

$$\frac{[j]_q}{j}q^{k-j+1}<1-\sqrt[j]{(1-q^k)(1-q^{k-1})\ldots(1-q^{k-j+1})}=1-a_{\infty,k}.\tag{3.27}$$

Analogously to (3.26), we have for $k=j,j+1,\ldots$ that

$$\sqrt[j]{(1-q^k)(1-q^{k-1})\ldots(1-q^{k-j+1})}>1-q^{k-j+1}\quad\text{or}\quad 1-a_{\infty,k}<q^{k-j+1}.\tag{3.28}$$

Then, (3.27) and (3.28) imply the desired inequalities. □

We have the following result.

Theorem 3.3. *Let $j\in\{2,3,\ldots\}$ be given, $n\geq j$ and $0<q<1$. Then*

$$\begin{aligned}&|(U_{n,q}f)(x)-(U_{\infty,q}f)(x)|\\&\leq 2\left(1+\frac{2}{q(1-q)}\ln\left(\frac{1}{1-q}\right)\right)\omega\left(f;q^{n-j+1}\right)+|f(0)-f(1)|\frac{2}{q(1-q)}\ln\left(\frac{1}{1-q}\right)q^{n-j+1}\end{aligned}\tag{3.29}$$

for each $x\in[0,1]$ and $f\in C[0,1]$.

Proof. By (1.12) and (3.25), we have $(U_{n,q}f)(1) = f(1) = (U_{\infty,q}f)(1)$, which implies the validity of (3.29) for $x = 1$.

Now, let $x \in [0,1)$. In view of $\sum_{k=0}^{n} p_{n,k}(q;x) = 1$ (see [17, p. 268, (7.55)]), $\sum_{k=0}^{\infty} p_{\infty,k}(q;x) = 1$ (see [12, p. 103, (13)]) and Lemma 3.4, a), we have

$$\begin{aligned}
&|(U_{n,q}f)(x) - (U_{\infty,q}f)(x)| \\
&= \Big| \sum_{k=0}^{n} p_{n,k}(q;x) f(a_{n,k}) - \sum_{k=0}^{\infty} p_{\infty,k}(q;x) f(a_{\infty,k}) \Big| \\
&= \Big| \sum_{k=0}^{n} p_{n,k}(q;x) f(a_{n,k}) - \sum_{k=0}^{\infty} p_{\infty,k}(q;x) f(a_{\infty,k}) - f(1)\{\sum_{k=0}^{n} p_{n,k}(q;x) - \sum_{k=0}^{\infty} p_{\infty,k}(q;x)\} \Big| \\
&= \Big| \sum_{k=0}^{j-1} \{p_{n,k}(q;x) - p_{\infty,k}(q;x)\}\{f(0) - f(1)\} + \sum_{k=j}^{n} p_{n,k}(q;x) f(a_{n,k}) \\
&- \sum_{k=j}^{\infty} p_{\infty,k}(q;x) f(a_{\infty,k}) - f(1)\{\sum_{k=j}^{n} p_{n,k}(q;x) - \sum_{k=j}^{\infty} p_{\infty,k}(q;x)\} \Big| \\
&= \Big| \sum_{k=0}^{j-1} \{p_{n,k}(q;x) - p_{\infty,k}(q;x)\}\{f(0) - f(1)\} + \sum_{k=j}^{n} p_{n,k}(q;x)\{f(a_{n,k}) - f(a_{\infty,k})\} \\
&+ \sum_{k=j}^{n} \{p_{n,k}(q;x) - p_{\infty,k}(q;x)\}\{f(a_{\infty,k}) - f(1)\} - \sum_{k=n+1}^{\infty} p_{\infty,k}(q;x)\{f(a_{\infty,k}) - f(1)\} \Big| \\
&\le |f(0) - f(1)| \sum_{k=0}^{j-1} |p_{n,k}(q;x) - p_{\infty,k}(q;x)| + \sum_{k=j}^{n} p_{n,k}(q;x) |f(a_{n,k}) - f(a_{\infty,k})| \\
&+ \sum_{k=j}^{n} |p_{n,k}(q;x) - p_{\infty,k}(q;x)| |f(a_{\infty,k}) - f(1)| + \sum_{k=n+1}^{\infty} p_{\infty,k}(q;x) |f(a_{\infty,k}) - f(1)| \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.30}$$

First, we estimate I_1. In view of [20, p. 156, (2.9)], we have

$$\sum_{k=0}^{n} q^k |p_{n,k}(q;x) - p_{\infty,k}(q;x)| \le \frac{2q^n}{q(1-q)} \ln\left(\frac{1}{1-q}\right). \tag{3.31}$$

Therefore

$$\begin{aligned}
I_1 &= |f(0) - f(1)| q^{-j+1} \sum_{k=0}^{j-1} q^{j-1} |p_{n,k}(q;x) - p_{\infty,k}(q;x)| \\
&\le |f(0) - f(1)| q^{-j+1} \sum_{k=0}^{j-1} q^{k} |p_{n,k}(q;x) - p_{\infty,k}(q;x)| \\
&\le |f(0) - f(1)| q^{-j+1} \sum_{k=0}^{n} q^{k} |p_{n,k}(q;x) - p_{\infty,k}(q;x)|
\end{aligned}$$

$$\leq |f(0)-f(1)|\frac{2}{q(1-q)}\ln\left(\frac{1}{1-q}\right)q^{n-j+1}. \tag{3.32}$$

For the estimation of I_2, we apply Lemma 3.4, b):

$$I_2 \leq \sum_{k=j}^{n} p_{n,k}(q;x)\omega\left(f;a_{n,k}-a_{\infty,k}\right) \leq \sum_{k=j}^{n} p_{n,k}(q;x)\omega\left(f;q^{n-j+1}\right)$$
$$\leq \omega\left(f;q^{n-j+1}\right)\sum_{k=0}^{n} p_{n,k}(q;x) = \omega\left(f;q^{n-j+1}\right). \tag{3.33}$$

At the estimation of I_3 we use the property $\omega(f;\lambda\delta)\leq(1+\lambda)\omega(f;\delta)$, where $\lambda>0$, $\delta>0$. Then, by Lemma 3.4, c), we have for $k\in\{j,j+1,\ldots,n\}$ that

$$\begin{aligned}
|f(a_{\infty,k})-f(1)| &\leq \omega\left(f;1-a_{\infty,k}\right) \leq \left(1+q^{-n+j-1}(1-a_{\infty,k})\right)\omega\left(f;q^{n-j+1}\right)\\
&= q^{-n+j-1}\left(q^{n-j+1}+(1-a_{\infty,k})\right)\omega\left(f;q^{n-j+1}\right)\\
&\leq q^{-n+j-1}\left(q^{n-j+1}+q^{k-j+1}\right)\omega\left(f;q^{n-j+1}\right)\\
&\leq q^{n-j+1}\,2q^{k-j+1}\omega\left(f;q^{n-j+1}\right) = 2q^{k-n}\omega\left(f;q^{n-j+1}\right).
\end{aligned}$$

Then, by (3.31),

$$\begin{aligned}
I_3 &\leq 2q^{-n}\omega\left(f;q^{n-j+1}\right)\sum_{k=j}^{n} q^k|p_{n,k}(q;x)-p_{\infty,k}(q;x)|\\
&\leq 2q^{-n}\omega\left(f;q^{n-j+1}\right)\sum_{k=0}^{n} q^k|p_{n,k}(q;x)-p_{\infty,k}(q;x)|
\end{aligned}$$
$$\leq \frac{4}{q(1-q)}\ln\left(\frac{1}{1-q}\right)\omega\left(f;q^{n-j+1}\right). \tag{3.34}$$

If $k\geq n+1$, then, in view Lemma 3.4, c), we get

$$|f(a_{\infty,k})-f(1)| \leq \omega\left(f;1-a_{\infty,k}\right) \leq \omega\left(f;q^{k-j+1}\right) \leq \omega\left(f;q^{n-j+1}\right).$$

Hence, we obtain the estimation of I_4:

$$I_4 \leq \sum_{k=n+1}^{\infty} p_{\infty,k}(q;x)\omega\left(f;q^{n-j+1}\right) \leq \omega\left(f;q^{n-j+1}\right)\sum_{k=0}^{\infty} p_{\infty,k}(q;x) = \omega\left(f;q^{n-j+1}\right). \tag{3.35}$$

Combining (3.30), (3.32)-(3.35), we obtain (3.29). □

Theorem 3.4. *Let $j\in\{2,3,\ldots\}$ be given and $q\in(0,1)$. If $f\in C[0,1]$ and $x\in[0,1]$, then*

$$|(U_{\infty,q}f)(x)-f(x)| \leq \left(1+\sqrt{\frac{1}{2}+2(j-1)^2}\right)\omega\left(f;\sqrt{1-q}\right). \tag{3.36}$$

Proof. Analogously to the proof of Theorem 2.2, we have

$$|(U_{\infty,q}f)(x)-f(x)| \leq \omega(f;\delta)\left\{1+\delta^{-1}\left(\sum_{k=0}^{\infty} p_{\infty,k}(q;x)(a_{\infty,k}-x)^2\right)^{1/2}\right\}$$
$$= \omega(f;\delta)\left\{1+\delta^{-1}\left(U_{\infty,q}((e_1-xe_0)^2;x)\right)^{1/2}\right\}, \tag{3.37}$$

where $\delta>0$ and $x\in[0,1)$.

On the other hand, taking into account (1.12), the inequality $(a+b)^2 \le 2(a^2+b^2)$, $a, b \in \mathbb{R}$, $B_{n,q}((e_1 - xe_0)^2; x) = \frac{1}{[n]_q} x(1-x)$ [17, pp. 268-269, (7.55)-(7.57)] and Lemma 3.4, a), we get

$$\begin{aligned} U_{n,q}((e_1 - xe_0)^2; x) &= \sum_{k=0}^{n} p_{n,k}(q;x)(a_{n,k} - x)^2 \\ &\le 2\sum_{k=0}^{n} p_{n,k}(q;x)\left(a_{n,k} - \frac{[k]_q}{[n]_q}\right)^2 + 2\sum_{k=0}^{n} p_{n,k}(q;x)\left(\frac{[k]_q}{[n]_q} - x\right)^2 \\ &= 2\sum_{k=0}^{n} p_{n,k}(q;x)\left(a_{n,k} - \frac{[k]_q}{[n]_q}\right)^2 + \frac{2}{[n]_q} x(1-x) \\ &= 2\sum_{k=0}^{j-1} p_{n,k}(q;x)\left(\frac{[k]_q}{[n]_q}\right)^2 + 2\sum_{k=j}^{n} p_{n,k}(q;x)\left(a_{n,k} - \frac{[k]_q}{[n]_q}\right)^2 + \frac{2}{[n]_q} x(1-x) \\ &\le 2\sum_{k=0}^{j-1} p_{n,k}(q;x)\left(\frac{[j-1]_q}{[n]_q}\right)^2 + 2\sum_{k=j}^{n} p_{n,k}(q;x)\left(a_{n,k} - \frac{[k]_q}{[n]_q}\right)^2 \\ &\quad + \frac{2}{[n]_q} x(1-x). \end{aligned} \tag{3.38}$$

Because

$$\frac{[k-j+1]_q}{[n-j+1]_q} \le \ldots \le \frac{[k-1]_q}{[n-1]_q} \le \frac{[k]_q}{[n]_q}, \quad k \in \{j, j+1, \ldots n\},$$

we obtain that

$$\frac{[k-j+1]_q}{[n-j+1]_q} \le a_{n,k} = \sqrt[j]{\frac{[k]_q[k-1]_q \ldots [k-j+1]_q}{[n]_q[n-1]_q \ldots [n-j+1]_q}} \le \frac{[k]_q}{[n]_q},$$

where $k \in \{j, j+1, \ldots n\}$. Hence

$$\begin{aligned} 0 &\le \frac{[k]_q}{[n]_q} - a_{n,k} \le \frac{[k]_q}{[n]_q} - \frac{[k-j+1]_q}{[n-j+1]_q} \\ &= \frac{[k]_q[n-j+1]_q - [n]_q[k-j+1]_q}{[n]_q[n-j+1]_q} = q^{k-j+1}\frac{[n-k]_q}{[n-j+1]_q}\frac{[j-1]_q}{[n]_q} \le \frac{[j-1]_q}{[n]_q} \end{aligned}$$

for $k \in \{j, j+1, \ldots n\}$. Then, by (3.38) and $(B_{n,q}e_0)(x) = 1$ [17, p. 268, (7.55)], we find

$$\begin{aligned} &U_{n,q}((e_1 - xe_0)^2; x) \\ &\le 2\sum_{k=0}^{j-1} p_{n,k}(q;x)\left(\frac{[j-1]_q}{[n]_q}\right)^2 + 2\sum_{k=j}^{n} p_{n,k}(q;x)\left(\frac{[j-1]_q}{[n]_q}\right)^2 + \frac{2}{[n]_q} x(1-x) \\ &\le 2\left(\frac{[j-1]_q}{[n]_q}\right)^2 + \frac{1}{2[n]_q} \le \left(2[j-1]_q^2 + \frac{1}{2}\right)\frac{1}{[n]_q} \le \left(\frac{1}{2} + 2(j-1)^2\right)\frac{1}{[n]_q}. \end{aligned}$$

By passing to the limit $n \to \infty$, in view of Theorem 3.3 and $\lim_{n\to\infty} [n]_q = (1-q)^{-1}$, we get

$$U_{\infty,q}((e_1 - xe_0)^2; x) \le \left(\frac{1}{2} + 2(j-1)^2\right)(1-q).$$

Hence, by (3.37), we have

$$|(U_{\infty,q}f)(x)-f(x)| \le \omega(f;\delta)\left\{1+\delta^{-1}\sqrt{\frac{1}{2}+2(j-1)^2}\sqrt{1-q}\right\}.$$

Choosing $\delta=\sqrt{1-q}$, we obtain (3.36) for $x\in[0,1)$.

Finally, if $x=1$, then (3.25) implies that $(U_{\infty,q}f)(1)=f(1)$, which means that (3.36) is true for $x=1$. Thus the theorem is proved. □

References

[1] T. Acar, M. C. Montano, P. Garrancho and V. Leonessa: *On sequences of J. P. King-type operators*, J. Funct. Spaces, **2019** (2019), Article ID:2329060.

[2] A. M. Acu, H. Gonska and M. Heilmann: *Remarks on a Bernstein-type operator of Aldaz, Kounchev and Render*, J. Numer. Anal. Approx. Theory, **50** (1) (2021), 3–11.

[3] O. Agratini, T. Andrica: *Discrete approximation processes of King's type*, Nonlinear Analysis and Variational Problems, Springer Optimization and Its Applications 35, Springer, New York (2010), 3–12.

[4] J. M. Aldaz, O. Kounchev and H. Render: *Shape preserving properties of generalized Bernstein operators on extended Chebyshev spaces*, Numer. Math., **114** (1) (2009), 1–25.

[5] D. Cárdenas-Morales, P. Garrancho and I. Raşa: *Asymptotic formulae via a Korovkin-type result*, Abstr. Appl. Anal., **2012** (2012), Article ID:217464.

[6] Z. Finta: *Estimates for Bernstein type operators*, Math. Inequal. Appl., **15** (1) (2012), 127–135.

[7] Z. Finta: *Bernstein type operators having* 1 *and* x^j *as fixed points*, Cent. Eur. J. Math., **11** (12) (2013), 2257–2261.

[8] Z. Finta: *On approximation properties of q-King operators*, Topics in Mathematical Analysis and Applications, Springer Optimization and Its Applications 94, Springer, Switzerland (2014), 342–362.

[9] Z. Finta: *Note on a Korovkin-type theorem*, J. Math. Anal. Appl., **415** (2) (2014), 750–759.

[10] Z. Finta: *King operators which preserve* x^j, Constr. Math. Anal., **6** (2) (2023), 90–101.

[11] H. Gonska, P. Piţul: *Remarks on an article of J. P. King*, Comment. Math. Univ. Carolinae, **46** (4) (2005), 645–652.

[12] A. Il'inskii, S. Ostrovska: *Convergence of Generalized Bernstein Polynomials*, J. Approx. Theory, **116** (1) (2002), 100–112.

[13] J. P. King: *Positive linear operators which preserve* x^2, Acta Math. Hungar., **99** (3) (2003), 203–208.

[14] G. G. Lorentz: *Bernstein Polynomials*, Chelsea Publishing Company, New York (1986).

[15] S. Ostrovska: *q-Bernstein polynomials and their iterates*, J. Approx. Theory, **123** (2) (2003), 232–255.

[16] G. M. Phillips: *Bernstein polynomials based on the q-integers*, Ann. Numer. Math., **4** (1997), 511–518.

[17] G. M. Phillips: *Interpolation and approximation by polynomials*, Springer, New York (2003).

[18] G. Pólya, G. Szegő: *Problems and theorems in analysis, vol. I*, Springer, New York (1972).

[19] V. S. Videnskii: *On some classes of q-parametric positive linear operators*, Operator Theory: Advances and Applications, vol. 158, Birkhäuser, Basel (2005), 213–222.

[20] H. Wang, F. Meng: *The rate of convergence of q-Bernstein polynomials for* $0<q<1$, J. Approx. Theory, **136** (2) (2005), 151–158.

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