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Research Article

On the absence of an exceptional set in the main relation of Wiman-Valiron theory

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ABSTRACT. The object of investigation is a class of functions analytic in the right half-plane. The half-plane contains all points whose real part is greater than some fixed a and for all x from the interval $(a, +\infty)$ the corresponding supremum of modulus of the function F inside some vertical strip is finite. There is selected subclass consisting of those functions, for which the right-hand derivative of the supremum logarithm tends to infinity. For these functions for which there exists an auxiliary function with some local behavior. The following theorem is proved: If an analytic function belongs to the described class, then for each positive natural k the k-th order derivative equals k-th order power of the right-hand derivative of the supremum logarithm for every point.

Keywords: Analytic function, Wiman-Valiron theory, main relation, exceptional set.

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1. INTRODUCTION

Let f be an entire function of the form

(1.1) $$f(z) = f_0 + \sum_{k=1}^{+\infty} f_k z^k, \quad z \in \mathbb{C}.$$

For $r > 0$, we denote

$$M_f(r) = \max\{|f(z)|\colon |z| = r\},$$
$$\mu_f(r) = \max\{|f_k|r^k\colon k \geq 0\},$$
$$\nu_f(r) = \max\{k\colon |f_k|r^k = \mu_f(r)\}$$

the maximum modulus, maximal term and central index, respectively. The main theorem of the Wiman-Valiron theory [6, 7, 13, 23] states that for each non-constant entire function $f : \mathbb{C} \to \mathbb{C}$, for any point z, $|z| = r$, such that $|f(z)| = M_f(r)$, we have for given $n \in \mathbb{N}$

(1.2) $$f^{(n)}(z) \sim \left(\frac{\nu_f(r)}{z}\right)^n f(z),$$

as $|z| = r \to +\infty$, $r \notin E$, where E is some set of finite logarithmic measure (f.l.m.), that is

$$\int_{E\cap[1,+\infty)} d\ln r < +\infty.$$

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Let us denote

$$K_f(r)=r(\ln M_f(r))'_+,$$

where $(\ln M_f(r))'_+$ is the right-hand derivative. From the theorem obtained in [14] for the entire Dirichlet series, it follows that for each entire function f

$$K_f(r)\sim \nu_f(r)\quad (r\to+\infty,\ r\notin E),$$

where $E=E(f)$ is some set of f.l.m.

Various theorems about analogues of relation (1.2) for an absolutely convergent Dirichlet series, by additional conditions on its positive monotonically increasing to infinity sequence of exponents, can be found in [4, 11, 13, 14, 18, 19]. In [11, 13, 14] for the entire Dirichlet series, for absolutely convergent in a half-plane Dirichlet series in [4, 17, 18, 19]. On the other hand, R. Gorenflo [5] found a class of entire functions for which an exceptional set is absent in relation (1.2). He proved the following statement.

Theorem 1.1 ([5]). *If $f(z)=\sum\limits_{n=0}^{+\infty} f_nz^n,\ a_n\geq 0\ (n\geq 0)$ is an entire function with perfectly regular growth, that is*

$$(\exists\,\varrho\in(0,+\infty))(\exists\,\sigma\in(0,+\infty))\colon\ \lim_{r\to+\infty}\frac{\ln M_f(r)}{r^{\varrho}}=\sigma,$$

then for any fixed $A,B>0$, for each $n\in\mathbb{N},\ n<B$ one has

$$f^{(n)}(ze^{\tau})=(1+o(1))\Big(\frac{\nu_f(r)}{ze^{\tau}}\Big)^{n}e^{\nu_f(r)\tau}f(z)\ \text{as}\ r\to+\infty \tag{1.3}$$

for arbitrary $\tau,\ |\tau|<A/\nu_f(r)$, and every point z, $|z|=r$, such that the inequality

$$|f(z)|\geq\alpha M_f(r),$$

holds with a given arbitrary $\alpha\in(0,1]$.

Theorem 1.1 at $\tau=0$ implies that for every entire function with non-negative Taylor coefficients of perfectly regular growth relation (1.2) holds as $r\to+\infty$ at every point $z,\ |z|=r$ such that the inequality

$$|f(z)|\geq\alpha M_f(r)$$

is true with a given $\alpha\in(0,1]$.

R. R. London [8] obtained this result for a wider class of entire functions satisfying the following conditions: There exists a function $\phi\colon\mathbb{R}_+\to\mathbb{R}_+:=(0,+\infty)$ such that ϕ' is a positive and unbounded function, ϕ'' is a positive and continuous function,

$$(\exists\,\alpha,\beta\in\mathbb{R}_+)(\forall\,x\geq x_0)\colon\ \alpha\frac{\phi'(x)}{\phi(x)}<\frac{\phi''(x)}{\phi'(x)}<\beta\frac{\phi'(x)}{\phi(x)} \tag{1.4}$$

and

$$\ln M_f(r)\sim\phi(\ln r)\qquad(r\to+\infty). \tag{1.5}$$

In this paper, we will prove a slightly different result of this form. We essentially consider the class of entire functions which are bounded in the left half-planes

$$\Pi_b:=\{z\in\mathbb{C}\colon\ \mathrm{Re}\,z<b\}\ \text{for all}\ b\in\mathbb{R}.$$

Clearly, this class contains every function

$$F(z)=f(e^z),$$

where f is an entire function on the complex plane $\mathbb{C}$. Any absolutely convergent in the whole complex plane Dirichlet series with a sequence of positive exponents increasing to $+\infty$ represent the entire function, so it belongs to this class. The partial case of such series is the Riemann zeta function [9]. Note that a similar statement about the absence of an exceptional set was earlier proved in the case of validity of the asymptotic relation (Wiman's type theorem)

$$\begin{aligned} M(x,F) &= (1+o(1))B_F(x) \\ &= -(1+o(1))A_F(x) \end{aligned} \tag{1.6}$$

as $x \to +\infty$, where

$$B_F(x) = \sup\{\text{Re } F(z)\colon \text{Re } z = x\},$$

$$A_F(x) = \inf\{\text{Im } F(z)\colon \text{Re } z = x\}.$$

These results were presented in [3] and obtained in the class of entire functions which are bounded in the every left half-plane Π_b. For other results on the conditions under which the relation (1.6) holds as $x \to +\infty$ outside exceptional sets, see, for example, also [17, 20, 22]. In [20], this result is deduced for entire Dirichlet series with increasing to $+\infty$ of a sequence of positive exponents. A similar estimate was presented for functions represented by some positive integrals in [21]. In [17], the estimate was considered in the class of absolute convergent Dirichlet series in the left half-plane $\{z \in \mathbb{C}\colon x = \text{Re } z < 0\}$ as $x \to -0$ without some exceptional set.

2. Lemmas

Let $S(a)$, $-\infty \le a \le +\infty$, be a class of functions analytic in $\Pi(a) = \{z : a < \text{Re}\, z\}$ which are bounded in the vertical stripes, i.e.

$$(\forall x \in (a,+\infty))\colon\ M(x,F) := \sup\{|F(t+iy)| : a < t \le x, y \in \mathbb{R}\} < +\infty.$$

By Maximum Modulus Principle the supremum in the vertical stripe is attained at its boundary, i.e. at the vertical line. Therefore, the boundedness in the vertical stripe is replaced by the boundedness on vertical line. Moreover, by Maximum Modulus Principle the function

$$M(x,F) = \sup\{|F(x+iy)|\colon y \in \mathbb{R}\}$$

is non-decreasing on $(a,+\infty)$, and by Hadamard Three Lines Theorem the function $\ln M(x,F)$ is convex on $(a,+\infty)$ (see [15, pp.14-16], [22, p.145, p.266]). Therefore, for all $x \in (a,+\infty)$ there exists the non-decreasing right-hand derivative

$$L(x) = L(x,F) \overset{def}{=} (\ln M(x,F))'_+$$

at the interval $(a,+\infty)$. Let us denote by $S_\infty(a)$ the class of the functions $F \in S(a)$ such that

$$L(x,F) \to +\infty \quad (x \to +\infty).$$

Similarly as in [3], by S_0, we denote the class of functions $F \in S_\infty(0)$ for which there exists a function $\delta(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties

1. $\delta(x) \nearrow +\infty \ \ (0 \le x \uparrow +\infty)$;
2. $\frac{L(x,F)}{\delta(x)} \nearrow +\infty \ \ (0 \le x \uparrow +\infty)$;
3. for all $x \ge x_0$ with sufficiently large x_0 one has

$$\left| L\left(x \pm \frac{\delta(x)}{L(x,F)}, F\right) - L(x,F) \right| \le \frac{L(x,F)}{\delta(x)}. \tag{2.7}$$

One should observe that the class S_0 is non-empty. Indeed, for every $p \in \mathbb{N}$ the function

$$F_p(z) = \exp\{\exp\{z^p\}\}$$

belongs to the class S_0. In paper [3], it was essentially proved Lemma 2.1 and also Lemma 2.2 which are formulated below.

Lemma 2.1. *If a function $\delta(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

(1) $\delta(x) \nearrow +\infty$;
(2) $\frac{L(x,F)}{\delta(x)} \nearrow +\infty \ (0 \le x \uparrow +\infty)$

then condition (2.7) and

$$\left| L\left(x + \frac{\delta(x)}{L(x,F)}, F\right) - L(x,F) \right| \le \frac{L(x,F)}{\delta(x)} \quad (x \ge x_0) \tag{2.8}$$

are equivalent.

Lemma 2.2. *Let $F \in S_0$. Then, for all $x \ge x_0$ and every $\eta \in \mathbb{C}$, satisfying*

$$|\eta| \le \frac{\delta(x)}{L(x,F)},$$

the asymptotic relation

$$F(w+\eta) = (1+\omega(\eta))F(w)e^{\eta L(x,F)}$$

holds, where

$$|\omega(\eta)| < |\eta| c(x) \frac{L(x,F)}{\delta(x)},$$

$$c(x) := 1 + e(1+\varepsilon(x)),$$

and the point w is chosen such that $\mathrm{Re}\, w = x$ *and*

$$|F(w)| \ge \frac{M(x,F)}{1+\varepsilon(x)}$$

for a given function $\varepsilon(x)$ such that $\varepsilon(x) \to +0$ $(x \to +\infty)$.

We need also the modified Cauchy inequality. The complete proof of the following lemma can be found in the solution of Problem **236** (see, [10, Part III, Ch. 5, § 2, **236**, p.355]).

Lemma 2.3. *Let*

$$f(z) = \sum_{n=0}^{+\infty} f_n z^n$$

be an analytic function in the disk

$$\mathbb{D}_R = \{z\colon |z| < R\}, \quad R > 0.$$

If for all $z \in \mathbb{D}_R$ real part of the function f is upper bounded by some constant M, i.e.

$$\mathrm{Re}\ f(z) < M,$$

then for all $z \in \mathbb{D}_R$ one has

$$|f_n| R^n \le 2(M - \mathrm{Re}\ f_0).$$

3. Main Result

Let us prove the following main result of the present article.

Theorem 3.2. *Let $F \in S_0$. Then, for each $n \in \mathbb{N}$ the asymptotic relation*

$$F^{(n)}(w) = (1 + o(1))\, L^n(x, F)\, F(w) \qquad (x \to +\infty) \tag{3.9}$$

holds for every point w with $Re\, w = x$ such that the inequality

$$|F(w)| \geq \frac{M(x, F)}{1 + \varepsilon(x)}$$

is valid for a given positive function $\varepsilon(x)$ with $\varepsilon(x) \to +0$ $(x \to +\infty)$.

Proof. For proof we will use ideas from [4, 12, 16, 18, 19]. Denote

$$\psi(x) = \frac{\delta(x)}{L(x, F)}.$$

Let a given point w, and $x = Re\, w$ be such that condition

$$F(w) \geq \frac{M(x, F)}{1 + \varepsilon(x)} \ \ (x \geq x_0)$$

is satisfied. Then, for all $\eta \in \mathbb{C}$,

$$|\eta| < \frac{\psi(x)}{c(x)},$$

and for all $x \geq x_0$,

$$|\eta| c(x) \frac{L(x, F)}{\delta(x)} < 1,$$

from Lemma 2.2, we obtain

$$\begin{aligned} |F(w + \eta)| &\geq \big(1 - |\omega(\eta)|\big) |F(w) \mathrm{e}^{\eta L(x,F)}| \\ &\geq \left(1 - |\eta| c(x) \frac{L(x, F)}{\delta(x)}\right) |F(w) \mathrm{e}^{\eta L(x,F)}| > 0. \end{aligned}$$

Since the function

$$\frac{F^{'}(w + \tau)}{F(w + \tau)}$$

is an analytic function of the variable τ in the open disk

$$\left\{\tau : |\tau| < \frac{\psi(x)}{c(x)}\right\}$$

for given w, we yield that the following function

$$f(\eta) := \int_0^\eta \frac{F^{'}(w + \tau)}{F(w + \tau)} d\tau - \eta L(x, F) \text{ with } f(0) = 0$$

is an analytic function in the disc

$$\left\{\eta : |\eta| < \frac{\psi(x)}{c(x)}\right\}.$$

One should observe that

$$f^{'}(0) = \frac{F^{'}(w)}{F(w)} - L(x, F).$$

Further, let $q = q(x) \in \left(0, \frac{\psi(x)}{c(x)}\right)$ for fixed $x,\ x \geq x_0$. Then, for all η with $|\eta| \leq q$ we have

$$\operatorname{Re} f(\eta) = \ln \left| \frac{F(w+\eta)}{F(w)} e^{-\eta L(x,F)} \right| = \ln|1+\omega(\eta)|$$
$$\leq \ln(1+|\omega(\eta)|) \leq \ln\left(1+\frac{qc(x)}{\psi(x)}\right).$$

Consider the function f in the disc $\mathbb{D}_q = \{\eta\colon |\eta| \leq q\}$ with $q < \frac{\psi(x)}{c(x)}$. We will apply now Lemma 2.3 and obtain

$$\left|\frac{F'(w)}{F(w)} - L(x,F)\right| = |f'(0)| = |f_1|$$
$$\leq 2\max\{\operatorname{Re} f(\eta)\colon |\eta| = q\}$$
$$\leq 2\ln\left(1+\frac{qc(x)}{\psi(x)}\right) \leq \frac{2qc(x)}{\psi(x)}. \tag{3.10}$$

Therefore, for all $w,\ Re\, w = x$ such that

$$F(w) \geq \frac{M(x,F)}{1+\varepsilon(x)} \quad (x \geq x_0)$$

and $L(x,F) \to +\infty$ as $x \to +\infty$, one has

$$\left|\frac{F'(w)}{F(w)}\frac{1}{L(x,F)} - 1\right| \leq \frac{2c(x)}{L(x,F)\psi(x)} = \frac{c(x)}{\delta(x)} = o(1). \tag{3.11}$$

Further, applying Lemma 2.3 to the function f at the disc $\mathbb{D}_q$ with arbitrary $q \in \left(0, \frac{\psi(x)}{c(x)}\right)$, for given arbitrary $k \in \mathbb{N}$ we obtain

$$\frac{1}{k!}\left|f^{(k)}(0)\right| = |f_k| \leq \frac{2}{q^k}\max\{\operatorname{Re} f(\eta)\colon |\eta| = q\}$$
$$\leq 2q^{-k}\ln(1+qc(x)/\psi(x))$$
$$\leq 2q^{-k+1}\frac{c(x)}{\psi(x)}. \tag{3.12}$$

Continuing further, for all η with $|\eta| \leq q < \frac{\psi(x)}{c(x)}$ one has

$$F(w+\eta) = F(w)\exp\{f(\eta)+\eta L(x,F)\}$$
$$= F(w)\exp\left\{\frac{F'_A(w)}{F(w)}\eta + \sum_{k=2}^{+\infty}\frac{f^{(k)}(0)}{k!}\eta^k\right\}$$
$$:= F(w)F_0(\eta), \tag{3.13}$$

where $F_0(\eta)$ is an analytic function of the variable η and $F_0(0) = 1$. We put

$$F_0(\eta) = 1 + \sum_{k=1}^{+\infty} F_k\eta^k, \quad |\eta| \leq q. \tag{3.14}$$

Since for the function

$$f_1(\eta) := f(\eta) + \eta L(x,F)$$

we have

$$f_1^{(k)}(\eta) = f^{(k)}(\eta) \ \ (k \geq 2),$$

so

$$F_0(\eta) = 1 + \sum_{s=1}^{+\infty} \frac{1}{s!} \left(\frac{F'(z)}{F(z)}\eta + \sum_{k=2}^{+\infty} \frac{f^{(k)}(0)}{k!}\eta^k \right)^s$$

$$= 1 + \sum_{s=1}^{+\infty} \frac{1}{s!} \left(\sum_{k=1}^{+\infty} \frac{f_1^{(k)}(0)}{k!}\eta^k \right)^s. \tag{3.15}$$

Therefore, the Taylor coefficients in (3.14) are formed by sums of the following form

$$F_k = \sum_{\|\alpha\|_0 = k} B_\alpha \left(f_1'(0)\right)^{\alpha_1} \cdot \left(\frac{f_1''(0)}{2!} \right)^{\alpha_2} \cdot \ldots \cdot \left(\frac{f_1^{(k)}(0)}{k!} \right)^{\alpha_k}$$

with finite quantities of the summands, where $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_+^k$ is the multi-index, and $\|\alpha\|_0 := \sum_{j=1}^k j\alpha_j$ its weight, $B_\alpha \in \mathbb{R}_+$ is generated by finite sum of positive numbers by equating coefficients at η^k from (3.14) and (3.15). From the Taylor expansion

$$F(w+\eta) = \sum\nolimits_{k=0}^{+\infty} \frac{F^{(k)}(w)}{k!}\eta^k$$

and (3.13) it follows that

$$F^{(k)}(w) = F(w)k!F_k \ \ (k \geq 1).$$

Hence,

$$\frac{F^{(k)}(w)}{F(w)} \frac{1}{L^k(x,F)} - 1 = \left(\frac{F'(w)}{F(w)} \frac{1}{L(x,F)} \right)^k - 1 + \sum_{\substack{\|\alpha\|_0 = k \\ \alpha_1 < k}} B_\alpha \prod_{j=1}^{k} \left(\frac{f_1^{(j)}(0)}{j!} \right)^{\alpha_j}. \tag{3.16}$$

Since $c(x) = \mathcal{O}(1)$ and $\delta(x) \nearrow +\infty$ $(r \to +\infty)$, we have

$$\frac{c(x)}{\delta(x)} = o(1) \ \ (r \to +\infty).$$

So, from inequality (3.11), we have

$$\left| \left(\frac{F'(w)}{F(w)} \frac{1}{L(x,F)} \right)^k - 1 \right| = \left| \frac{F'(w)}{F(w)} \frac{1}{L(x,F)} - 1 \right| \sum_{j=0}^{k-1} \left| \frac{F'(w)}{F(w)} \frac{1}{L(x,F)} \right|^j$$

$$\leq \frac{c(x)}{\delta(x)} \sum_{j=0}^{k-1} \left(1 + \frac{c(x)}{\delta(x)} \right)^j$$

$$= \left(1 + \frac{c(x)}{\delta(x)} \right)^k - 1 = o(1) \tag{3.17}$$

as $x \to +\infty$. We put

$$q = \frac{1}{2} \frac{\psi(x)}{c(x)} = \frac{1}{2} \frac{\delta(x)}{c(x)L(x,F)}.$$

Using inequality (3.12), we get

$$\begin{aligned}\sum_{\substack{\|\alpha\|_0=k\\ \alpha_1<k}} B_\alpha \prod_{j=1}^{k}\left(\frac{f_1^{(j)}(0)}{j!}\right)^{\alpha_j} &\le \sum_{\substack{\|\alpha\|_0=k\\ \alpha_1<k}} |B_\alpha|(qL(x,F))^{-k}\left(\frac{2qc(x)}{\psi(x)}\right)^{\|\alpha\|}\\ &= \sum_{\substack{\|\alpha\|_0=k\\ \alpha_1<k}} |B_\alpha|\left(\frac{2c(x)}{\delta(x)}\right)^{k} = o(1)\end{aligned} \tag{3.18}$$

as $x \to +\infty$. By applying relations (3.17) and (3.18), from equality (3.16) we obtain

$$\left|\frac{F^{(k)}(w)}{F(w)}\frac{1}{L^k(x,F)} - 1\right| = o(1)$$

as $x \to +\infty$. Thus, asymptotic relation (3.9) is proved. □

4. Some Corollaries and Open Problems

Let f be an entire transcendental function of the form (1.1). Since at $x = \ln r$

$$L(x,F) = K_f(r) \nearrow +\infty \quad (r \to +\infty)$$

for the function $F(z) = f(e^z)$, the function F belongs to the class $S_\infty(0)$. From Theorem 3.2, it follows the following corollary.

Corollary 4.1. *Let f be an entire transcendental function of the form (1.1), and $F(z) = f(e^z)$. If $F \in S_0$, then for given $n \in \mathbb{N}$ and for any point z, $|z| = r$, such that $|f(z)| = M_f(r)$, the asymptotic relation*

$$f^{(n)}(z) \sim \left(\frac{K_f(r)}{z}\right)^n f(z),$$

holds as $|z| = r \to +\infty$.

The proof of the Corollary 4.1 can be obtained from the proof of Theorem 3.2 using simple calculations and transformations.

Remark 4.1. *The condition $F \in S_0$ for the function $F(z) = f(e^z)$ can be written as the following properties: $K_f(r) \to +\infty$ $(r \to +\infty)$ and there exists a function $\delta(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

(1) $\delta(r) \nearrow +\infty$ $(1 \le r \uparrow +\infty)$;
(2) $\frac{K_f(r)}{\delta(r)} \nearrow +\infty$ $(1 \le r \uparrow +\infty)$;
(3) for all $r \ge r_0$ with sufficiently large r_0 one has

$$\left|K_f\left(r \pm \frac{\delta(r)}{K_f(r)}\right) - K_f(r)\right| \le \frac{K_f(r)}{\delta(r)}.$$

Question 4.1. *A natural question arises: How are the conditions (1.4), (1.5) and the condition (2.7) (also (2.8)) related to each other?*

Question 4.2. *It is well known that in the general class of entire functions it is impossible to obtain a basic relation (1.2) without exceptional sets. This fact leads to the following question: How much can the conditions of Gorenflo's Theorem or Theorem 3.2 be weakened so that the exceptional set is absent?*

Question 4.3. *What is analog of Theorem 3.2 for analytic functions in a vertical strip?*

Question 4.4. *What are analogs of relations*

$$M(x,F) = (1+o(1))B_F(x) = -(1+o(1))A_F(x) \text{ as } x \to +\infty$$

and

$$F^{(n)}(w) = (1+o(1))\, L^n(x,F)\, F(w) \qquad (\mathrm{Re}\, w = x \to +\infty)$$

for the class of entire vector-valued functions [1] *and for the class of entire slice regular functions of quaternionic variable* [2]*?*

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