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Survey Article

A survey on recent notions of singularity for kernels of nonlinear integral operators

Dedicated to Professor Ioan Raşa, on the occasion of his 75th birthday

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ABSTRACT. In this paper, we review several notions of singularity for kernels of abstract nonlinear integral operators acting on functions defined over locally compact Hausdorff topological spaces or groups. A recent general definition of abstract nonlinear operators is also discussed, which includes some discrete operators, as the sampling series in a nonlinear form.

Keywords: Singularity, generalized Lipschitz conditions, nonlinear integral operators, locally compact topological spaces, modular spaces.

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1. INTRODUCTION

This article deals with issues related to approximation with nonlinear operators in function spaces. Although these topics are not strictly related to those successfully developed by Prof. Ioan Rasa over many years, they have found their foundation and inspiration in the classical theory of approximation with linear operators, to which he made numerous and fundamental contributions. It is impossible to select the main articles from his vast scientific production here, as they are all not only of a high standard but also very varied.

We begin by considering the classical linear framework, so as to highlight the foundations that inspired the theory of approximation with nonlinear operators, first introduced by Prof. Julian Musielak.

In classical Fourier analysis, one of the main tasks is the problem of reconstruction of a periodic function f by means of its Fourier coefficients (see e.g. [29]). The classical methods are based on the approximation properties of sequences of convolution integral operators of type

$$(T_n f)(x) := \int_{-\pi}^{\pi} K_n(t) f(x-t)dt \qquad (x \in \mathbb{R}), \tag{1.1}$$

where K_n is a (measurable) 2π-periodic function and the sequence $(K_n)_{n\in\mathbb{N}}$, satisfies some assumptions that characterize it as an "approximate identity". Denoted by $L^1_{2\pi}$ the space comprising all 2π-periodic functions which are (locally) integrable on $\mathbb{R}$, endowed with the usual norm $\|g\|_{L^1_{2\pi}} := \int_{-\pi}^{\pi} |g(x)|dx$, these assumptions can be summarized in the following:

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(p.1) there exists a constant $M > 0$ such that $\|K_n\|_{L^1_{2\pi}} \leq M$ for every $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} K_n(t)dt = 1,$$

(p.2) for every $\delta \in]0, \pi[$, it holds

$$\lim_{n\to\infty} \int_{\delta \leq |t| \leq \pi} |K_n(t)|dt = 0.$$

The above assumptions define what in the language of approximation theory is called "singularity" of the sequence $(K_n)_n$.

An analogous problem in Fourier analysis is the reconstruction of a (measurable) function $f : \mathbb{R} \to \mathbb{R}$ by means of its Fourier transform. In this case, the reconstruction is based on the approximation properties of convolution integral operators of type

$$(1.2) \qquad (T_n f)(x) := \int_{-\infty}^{\infty} K_n(t) f(x-t)dt \qquad (x \in \mathbb{R}).$$

Here, K_n is a measurable function over $\mathbb{R}$. Denoting by $L^1(\mathbb{R})$ the usual Lebesgue space of all the integrable functions on $\mathbb{R}$, endowed with the norm $\|g\|_1 = \int_{\mathbb{R}} |g(t)|dt$, the notion of singularity for the sequence $(K_n)_{n\in\mathbb{N}}$ is characterized by the following assumptions:

(r.1) there exists a constant $M > 0$ such that $\|K_n\|_1 \leq M$, for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} K_n(t)dt = 1,$$

(r.2) for every $\delta > 0$,

$$\lim_{n\to\infty} \int_{|t|\geq\delta} |K_n(t)|dt = 0.$$

In the study of the approximation properties of the operators (1.1) and (1.2) a relevant role is played by the translation operator $(\tau_h f)(x) := f(x-h)$, $x, h \in \mathbb{R}$, that is invariant with respect to the Lebesgue measure.

More recently, Paul Butzer and Stephan Jansche (see [24]-[28]), introduced a modern approximation theory in the framework of Mellin analysis, which is based on sequences of convolution integral operators of type

$$(1.3) \qquad (T_n f)(x) := \int_0^{\infty} K_n(xt^{-1}) f(t) \frac{dt}{t} \qquad (x \in \mathbb{R}^+),$$

for functions $f \in L^1_\mu(\mathbb{R}^+)$ where μ is the measure defined on the Borel $\sigma-$algebra of $\mathbb{R}^+$, denoted by $\mathcal{B}(\mathbb{R}^+)$, defined by

$$\mu(A) := \int_A \frac{dt}{t} \qquad (A \in \mathcal{B}(\mathbb{R}^+)).$$

For a complete treatment of the approximation by the operators (1.3) in the framework of Mellin analysis see also the new monograph [6]. The space $L^1_\mu(\mathbb{R}^+)$ is endowed with the norm

$$\|g\|_{L^1_\mu} := \int_0^{\infty} |g(t)| \frac{dt}{t} \qquad (g \in L^1_\mu(\mathbb{R}^+)).$$

Here, $K_n : \mathbb{R}^+ \to \mathbb{R}$ is a measurable function and the singularity assumptions of the sequence $(K_n)_n$ are now expressed by the following:

(m.1) there is a constant $M > 0$ such that $\|K_n\|_{L^1_\mu} \leq M$ for every $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \int_0^\infty K_n(t)\frac{dt}{t} = 1,$$

(m.2) for any $\delta > 1$, setting $U_\delta :=]1/\delta, \delta[$, it holds

$$\lim_{n\to\infty} \int_{\mathbb{R}^+ \setminus U_\delta} |K_n(t)|\frac{dt}{t} = 0.$$

Here, the translation operator is indeed a "dilation" operator, defined by $(\widetilde{\tau}_h f)(x) := f(xh^{-1})$, for $x, h \in \mathbb{R}^+$, that is invariant with respect to the measure μ. All operators (1.1), (1.2) and (1.3), admit a unique approach, using the language of locally compact topological groups and its associated Haar measure μ defined over the Borel sets (see [35, Chapter 10] and [36]). For example, in the periodic case, the underlying measure space can be identified with the group

$$S^1 := \{z \in \mathbb{C} : z = e^{it},\ t \in [0, 2\pi]\}.$$

For operators (1.2) the underlying space is the additive group $(\mathbb{R}, +)$ while for operators (1.3) it is the multiplicative group $(\mathbb{R}^+, \cdot)$.

The general approach consists in the study of convolution integral operators of type

$$(T_n f)(x) := \int_G K_n(t) f(x - t) d\mu(t) \qquad (x \in G), \tag{1.4}$$

where G is a locally compact topological group, endowed with its operation "+" (we adopt here the additive notation) and with its Haar measure μ, that is invariant with respect to the group operation, and $f : G \to \mathbb{R}$ is a μ-measurable function.

In order to define the notion of singularity for the sequence $(K_n)_n$, we assume here that the locally compact group $(G, +)$ is abelian. We denote by $\mathcal{U}$ a base of compact neighborhood of the neutral element θ of G. Let us denote by $L^1(G, \mu)$ the usual Lebesgue space comprising all the integrable functions g with respect the measure μ, endowed with the norm $\|g\|_{L^1(G,\mu)} := \int_G |g(t)| d\mu(t)$. The notion of singularity reads now as follows:

(g.1) there exists a constant $M > 0$ such that $\|K_n\|_{L^1(G,\mu)} \leq M$, for every $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \int_G K_n(t) d\mu(t) = 1,$$

(g.2) for every $U \in \mathcal{U}$ it holds

$$\lim_{n\to\infty} \int_{G \setminus U} |K_n(t)| d\mu(t) = 0.$$

This formulation is the starting point for a general treatment of the notion of singularity for kernels of nonlinear integral operators of convolution type, which was developed starting from the pioneering work by Julian Musielak (see [40], [42]) and then developed in various directions in the framework of approximation theory, see e.g. [3], [7]-[14], [16, 17, 18, 21, 39, 44, 46], see also the monograph [19].

The main problems to be solved in order to obtain a suitable approximation theory by means of nonlinear operators are essentially the following two: a) How to replace the linearity? b) How we can extend the notion of singularity for the kernels in a consistent way? The first problem can be adequately solved employing certain generalized Lipschitz conditions on the kernel functions K_n. This is quite usual in the theory of nonlinear integral operators, for example in solving nonlinear integral equations and in fixed point theory. The second problem represents the main objective of this survey article: we report the original definitions of singularity introduced by Musielak. Then we describe some useful generalizations that enable us to obtain more easily examples of singular sequences. In particular, we consider singularity for kernels of nonlinear integral operators of Urysohn type, for which it is not necessary to employ any algebraic structure, so that our framework will be an abstract Hausdorff and locally compact topological space. Finally, we discuss the final form of singularity for a general class of nonlinear operators, which includes also nonlinear discrete operators of sampling type, (see [1, 13, 14, 22, 23, 44]). Since the theory developed in the last years is devoted to the study of convergence properties in abstract spaces, we premise a section in which we introduce the notion of modular space, that includes large classes of function spaces (L^p-spaces, Orlicz spaces, Musielak-Orlicz spaces, etc).

2. Basic Notions

Let $(G, +)$ be a locally compact and σ-compact Hausdorff topological group, provided with its Haar measure μ defined on the σ-algebra Σ of all Haar measurable sets (see e.g [35, 36]). For any measurable set $A \subset G$, we denote by χ_A the characteristic function of the set A. For a sake of simplicity, we will assume that $(G, +)$ is abelian. Let $\mathcal{U}$ the family of (measurable) neighborhoods of the neutral element θ.

Let $L^0(G, \mu) \equiv L^0(G)$ be the space of all the μ-measurable functions defined on G, finite $\mu-$a.e. and with equality a.e.

Let $\varrho : L^0(G) \to \widetilde{\mathbb{R}}^+ = [0, +\infty]$ be a measurable modular functional, that is ϱ satisfies the following assumptions:

1. $\varrho(f) = 0$ if and only if $f = 0$, a.e.
2. $\varrho(-f) = \varrho(f)$,
3. $\varrho(\alpha f + \beta g) \leq \varrho(f) + \varrho(g)$, for every $f, g \in L^0(G)$, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.
4. $\varrho(F(t, \cdot))$ is a $\Sigma-$measurable function of $t \in G$ for any $\mu \otimes \mu-$measurable function $F : G \times G \to \mathbb{R}^+$.

In what follows a μ-measurable (or $\mu \otimes \mu$-measurable) set or function will be called simply "measurable", when the context is clear. By means of the functional ϱ we introduce the vector subspace of $L^0(G)$, denoted by $L^\varrho(G, \mu) \equiv L^\varrho(G)$, defined by

$$L^\varrho(G) = \{f \in L^0(G) : \lim_{\lambda \to 0^+} \varrho(\lambda f) = 0\}.$$

The subspace $L^\varrho(G)$ is called the modular space generated by ϱ. A general theory of modular spaces can be found in [41] (see also [19], [38]).

The following notions on measurable modulars will be used (see [19, Chapters 1, 2])

a. ϱ is quasi-convex if there is a constant $M \geq 1$ such that

$$\varrho\left[\int_G p(t)\, h(t, \cdot)\, d\mu(t)\right] \leq M \int_G p(t)\, \varrho(Mh(t, \cdot))\, d\mu(t),$$

for every $p \in L^1(G, \mu)$, $p \geq 0$, with $\|p\|_{L^1(G,\mu)} = 1$ and for any (globally) measurable $h : G \times G \to \mathbb{R}^+_0$.

b. ϱ is monotone if $f, g \in L^0(G)$, $|f| \leq |g|$ implies $\varrho(f) \leq \varrho(g)$.
c. ϱ is finite if $\chi_A \in L^\varrho(G)$ whenever A is a measurable subset of G such that $\mu(A) < \infty$.
d. ϱ is absolutely continuous if there exists an $\alpha > 0$ such that for every $f \in L^0(G)$, with $\varrho(f) < +\infty$, the following two conditions are satisfied:
   i. for every $\epsilon > 0$, there is a measurable compact subset $V \subset G$ such that $\mu(V) < \infty$ and $\varrho(\alpha f \chi_{G \setminus V}) < \epsilon$

ii. for every $\epsilon > 0$ there is a $\delta > 0$ such that $\varrho(\alpha f \chi_B) < \epsilon$, for any measurable subset $B \subset G$ with $\mu(B) < \delta$.

Among the examples of modular spaces, we quote here the Lebesgue spaces $L^p(G,\mu) \equiv L^p(G)$, the Orlicz spaces $L^\varphi(G,\mu) \equiv L^\varphi(G)$ generated by a φ-function, the generalized Orlicz spaces, and many others (for details see [19], [41]). We will say that a net of functions $(f_w)_{w>0} \subset L^\varrho(G)$ is modularly convergent to a function $f \in L^\varrho(G)$, if there is $\lambda > 0$, such that

$$\lim_{w\to\infty} \varrho[\lambda(f_w - f)] = 0.$$

We denote this by $f_w \to f\ (\varrho)$. This notion extends the norm-convergence in L^p-spaces. Moreover it is weaker than the norm-convergence induced by the Luxemburg norm generated by ϱ and they are equivalent if and only if the modular satisfies a Δ_2-condition, (see [19, p. 9], [41]).

3. Singularity for Kernels of Nonlinear Integral Operators of Convolution Type

J. Musielak (cfr. [40]) introduced for the first time the study of approximation theory by means of nonlinear convolution integral operators, defined by

$$(T_w f)(x) := \int_a^b K_w\big(x - t, f(t)\big)dt \qquad (x \in \mathbb{R}).$$

Here, the family $(K_w)_{w>0}$ is a family of kernel functions $K_w : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, with $w \in \mathbb{R}^+$, that satisfies a generalized Lipschitz condition with respect to the second variable of type

$$|K_w(t,u) - K_w(t,v)| \le L_w(t)|u - v| \qquad (t \in [a,b],\ u,v \in \mathbb{R}),$$

and the functions $K_w(\cdot,u)$ are extended periodically outside the interval $[a,b]$. These operators were subsequently generalized to obtain the form

$$(T_w f)(x) := \int_G K_w\big(x - t, f(t)\big)d\mu(t) \qquad (x \in G), \tag{3.5}$$

where G is locally compact topological abelian group, endowed with its Haar measure on the Borel sets.

Let $\mathcal{P}$ be the class comprising all the (globally) measurable functions $\psi : G \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that the following assumptions hold:

1. $\psi(t,\cdot)$ is a continuous, nondecreasing function, for every $t \in G$.
2. $\psi(t,0) = 0$, $\psi(t,u) > 0$ for every $u > 0$ and $t \in G$.

The notion of singularity for the family $(K_w)_{w>0}$, $K_w : G\times\mathbb{R} \to \mathbb{R}$, is expressed in the following way (see [19]). First, we will assume that K_w is globally measurable with respect to the σ-algebra in the product space $G \times \mathbb{R}$, $K_w(\cdot,u) \in L^1(G,\mu)$ and $K_w(t,0) = 0$, for every $w > 0$. Setting $\mathbb{K} = (K_w)_{w>0}$, we denote by $\mathcal{K}$ the class of all families $\mathbb{K}$ whose elements K_w satisfy the above assumptions. We say that $\mathbb{K} \in \mathcal{K}$ satisfies an $(L,1)$-Lipschitz condition, or that $\mathbb{K}$ is $(L,1)$-Lipschitz, if for every $w > 0$ there is a nonnegative measurable function $L_w : G \to \mathbb{R}_0^+$ such that

$$|K_w(t,u) - K_w(t,v)| \le L_w(t)|u - v| \qquad (t \in G,\ u,v \in \mathbb{R}).$$

For a fixed function $\psi \in \mathcal{P}$, we say that $\mathbb{K}$ is (L,ψ)-Lipschitz if for every $w > 0$ there exists a nonnegative measurable function $L_w : G \to \mathbb{R}_0^+$ such that

$$|K_w(t,u) - K_w(t,v)| \le L_w(t)\psi(t,|u - v|) \qquad (t \in G,\ u,v \in \mathbb{R}).$$

The corresponding notions of singularity of $\mathbb{K}$ associated with the above Lipschitz conditions, are as follows:

Definition 3.1. *We say that $\mathbb{K} \in \mathcal{K}$ is singular if*

(s.1) $L_w \in L^1(G, \mu)$ *and there exists a constant $D > 0$ such that $\|L_w\|_{L^1(G,\mu)} \leq D$ for every $w > 0$,*

(s.2) *For any $n \in \mathbb{N}$, it holds*

$$r_n(w) := \sup_{1/n \leq |u| \leq n} \left| \frac{1}{u} \int_G K_w(t, u) d\mu(t) - 1 \right| \to 0 \qquad (w \to \infty),$$

(s.3) *Setting $p_w(t) := \dfrac{L_w(t)}{\|L_w\|_{L^1(G,\mu)}}$, for $t \in G$ and $w > 0$, it holds*

$$\lim_{w \to \infty} \int_{G \setminus U} p_w(t) d\mu(t) = 0,$$

for every $U \in \mathcal{U}$.

Definition 3.2. *We say that $\mathbb{K}$ is strongly singular if the assumption $(s.2)$ is replaced by the stronger one*

$$\sup_{n \in \mathbb{N}} r_n(w) \to 0 \qquad (w \to \infty).$$

For the above definitions and their consequences see [19, Chapter 3].

Remark 3.1. *1. If we define $K_w(t, u) := K_w^*(t)u$, for $t \in G$ and $u \in \mathbb{R}$, where $(K_w^*)_{w>0}$ is a net of measurable functions, then taking $L_w(t) := |K_w^*(t)|$ we see that the notion of singularity for kernels of linear operators is a particular case of the strong singularity expressed in Definition 3.2.*

2. The notion of singularity given in Definition 3.1 is strictly connected with the Lipschitz conditions. This is quite natural, due the nature of the above definitions. In particular the assumption $(s.2)$ in Definition 3.1 may be interpreted as a statement that $K_w(t, u)$ behaves nearly u for large values of w. This would imply that a generalized Lipschitz condition with a function $\psi \in \mathcal{P}$, is nearly an usual $(L, 1)$-Lipschitz condition. As showed in [17] by an example, this is not true.

Using the above notions in [19, Chapter 3] an approximation theory with the nonlinear convolution operators (3.5) was developed for functions in a modular space $L^\rho(G)$ generated by a modular ρ satisfying certain specific conditions. For details see [19].

The weak point of the definition of singularity for kernels satisfying (L, ψ)-Lipschitz conditions, with a general $\psi \in \mathcal{P}$, lies in the difficulty of finding examples that connect the Lipschitz condition with assumption $(s.2)$ of Definition 3.1 without leading us back to a Lipschitz condition of type $(L, 1)$. In the next section, we will describe a generalization of the notion of singularity which give a contribution to solve the above difficulties.

4. A Generalization of Singularity for Nonlinear Convolution Integral Operators

In this section, we report a generalized notion of singularity for kernels $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}$ of nonlinear integral operators of convolution type (3.5). This generalization, introduced in [10], is based on the notion of "regular nets" of functions in modular spaces. As a first step, we modify slightly the generalized Lipschitz condition.

Definition 4.3. *Let $\Psi = (\psi_w)_{w>0}$ be a family of functions from the class $\mathcal{P}$. We say that $\mathbb{K}$ is (L, Ψ)-Lipschitz if for every $w > 0$, there is a nonnegative measurable function $L_w : G \to \mathbb{R}_0^+$ such that*

$$|K_w(t, u) - K_w(t, v)| \leq L_w(t)\psi_w(t, |u - v|) \qquad (t \in G,\ u, v \in \mathbb{R}).$$

4.1. **Regular Nets in Modular Spaces.** Let ϱ be a modular on $L^0(G)$ and let $f \in L^\varrho(G)$ be a fixed function.

Definition 4.4. *A net $(\xi_w)_{w>0}$ of measurable functions $\xi_w : \mathbb{R} \to \mathbb{R}$ is said to be $\varrho-$regular for f if we have $\xi_w \circ f \to f\ (\varrho)$, $w \to \infty$.*

In order to give some sufficient conditions in order that a net $(\xi_w)_{w>0}$ is $\varrho-$regular, we begin with the following definition.

Definition 4.5. *We say that a family of functions $(g_w)_{w>0} \subset L^\varrho(G)$ is $\varrho-$ uniformly equicontinuous if there is a constant $\beta > 0$ such that:*

1. *For every $\epsilon > 0$ there is a compact neighborhood of θ, $U \in \mathcal{U}$, such that*
$$\varrho(\beta g_w \chi_{G\setminus U}) < \epsilon,$$
for every $w > 0$.
2. *We have $\varrho[\beta g_w \chi_{B_k}] \to 0$ as $k \to \infty$, for every sequence $\{B_k\}$ with $B_k \in \Sigma$ for every $k \in \mathbb{N}$, $B_{k+1} \subset B_k$, and $\mu(B_k) \to 0$.*

We have the following (see [10]).

Theorem 4.1. *Let ϱ be a finite, monotone, absolutely continuous modular on $L^0(G)$, $f \in L^\varrho(G)$ and let $\xi_w : \mathbb{R} \to \mathbb{R}$ be a net of functions with the following properties*

1. *$\lim_{w\to\infty} \xi_w(u) = u$ uniformly on the compact sets of $\mathbb{R}$.*
2. *The family $(g_w)_{w>0}$, with $g_w(s) = \xi_w \circ f(s)$ is a $\varrho-$uniformly equicontinuous net of functions in $L^\varrho(G)$.*

Then $\xi_w \circ f \to f\ (\varrho)$.

The following corollary provides an interesting special case.

Corollary 4.1. *Let ϱ be a finite, monotone, absolutely continuous modular on $L^0(G)$, $f \in L^\varrho(G)$ and let $\xi_w : \mathbb{R} \to \mathbb{R}$ be a net of measurable functions with the following properties*

1. *$\lim_{w\to\infty} \xi_w(u) = u$ uniformly on the compact sets of $\mathbb{R}$.*
2. *$|\xi_w(u)| \leq \gamma(u)$ for every $u \in \mathbb{R}$, where $\gamma : \mathbb{R} \to \mathbb{R}_0^+$ is a measurable function such that $\gamma \circ f \in L^\varrho(G)$.*

Then $(\xi_w)_{w>0}$ is $\varrho-$regular for f.

The proof is an immediate consequence of Theorem 4.1; indeed, if $\gamma \circ f \in L^\varrho(G)$, then it is easy to show, from absolute continuity of ϱ, that $(\xi_w \circ f)_{w>0}$ is $\varrho-$uniformly equicontinuous. As an example, let $f \in L^\varrho(G)$ and let ξ_w be defined by $\xi_w(u) = u^{1-\frac{1}{w}}$, for $u \geq 0$ and extended in odd way to the real axis for sufficiently large w. Put $\gamma(u) = \sup_{w \geq \widetilde{w}} |\xi_w(u)|$. If $\gamma \circ f \in L^\varrho(G)$, then $\{\xi_w \circ f\}_{w>0}$ is $\varrho-$uniformly equicontinuous.

4.2. **A Generalization of Singularity via Regular Nets of Functions.** In this section, we report the notion of singularity defined employing regular nets of functions, studied in [10].

Let ϱ be a modular on $L^0(G)$. Let $\Xi = (\xi_w)_{w>0}$, $\xi_w : \mathbb{R} \to \mathbb{R}$, be a net of measurable functions such that $\xi_w(u) \neq 0$ if $u \neq 0$. We denote by $L^\varrho_\Xi(G)$ the class of all functions $f \in L^\varrho(G)$ such that $(\xi_w)_{w>0}$ is $\varrho-$regular with respect to f.

Let $\gamma : \mathbb{R} \to \mathbb{R}_0^+$ be a fixed measurable function such that $|\xi_w(u)| \leq \gamma(u)$ for every $w > 0$ and $u \in \mathbb{R}$. Setting
$$L^\varrho_\gamma(G) := \{f \in L^\varrho(G) : \gamma \circ f \in L^\varrho(G)\},$$
it is easy to see that $L^\varrho_\Xi(G) \supset L^\varrho_\gamma(G)$ whenever $(\xi_w)_{w>0}$ satisfies (i) of Corollary 4.1. For a class $\Psi = (\psi_w)_{w>0} \subset \mathcal{P}$, let $\mathbb{K} = \{K_w\}_{w>0} \in \mathcal{K}$ be a (L, Ψ)-Lipschitz kernel.

Definition 4.6. *We say that $\mathbb{K}$ is Ξ-singular if the following assumptions hold:*

(Ξ.1) *for every $w > 0$, $L_w \in L^1(G, \mu)$, and there is $D > 0$ such that $\|L_w\|_{L^1(G,\mu)} =: D_w \leq D$, for every $w > 0$.*

(Ξ.2) *Setting for every $n = 1, 2, \dots,$*

$$r_w(n) = \sup_{(1/n)\leq|u|\leq n} \left| \frac{1}{\xi_w(u)} \int_G K_w(t, u)\, d\mu(t) - 1 \right|,$$

for every $w > 0$, it results $\lim_{w\to\infty} r_w(n) = 0$, for every $n \in \mathbb{N}$.

(Ξ.3) *Setting $p_w(t) = L_w(t)/D_w$, for every $w > 0$, it results*

$$\lim_{w\to\infty} \int_{G\setminus U} p_w(t) d\mu(t) = 0,$$

for every $U \in \mathcal{U}$.

Next, we say that $\mathbb{K}$ is $\Xi-$*strongly singular*, if (Ξ.2) is replaced with $r_w := \sup_{n\in\mathbb{N}} r_w(n) \to 0$ as $w \to \infty$. It is clear that if $\mathbb{K}$ is $\Xi-$strongly singular then it is also $\Xi-$singular.

Using this generalized notion of singularity for the kernel of the nonlinear operator (3.5) it is proved a general modular approximation thorem in modular space, under certain side conditions. For details see [10]. In order to compare this new definition of singularity, with the previous one with (L, Ψ)-Lipschitz kernels (see Definition 3.1 and Definition 4.3), we have the following proposition (see [10]).

Proposition 4.1. *Let $\Xi = (\xi_w)_{w>0}$ be a family of measurable functions $\xi_w : \mathbb{R} \to \mathbb{R}$ such that $\xi_w(u) \neq 0$, for any $u \neq 0$ and $\xi_w(u) \to u$ uniformly on every compact set in $\mathbb{R}$. Let $\mathbb{K}$ be an (L, Ψ)-Lipschitz kernel, $\Psi \subset \mathcal{P}$, such that $(L_w(\cdot)\psi_w(\cdot, n))_{w>0}$ is uniformly bounded in $L^1(G, \mu)$ for every $n \in \mathbb{N}$. Then $\Xi-$singularity and singularity of $\mathbb{K}$ are equivalent.*

As an example, let $\Xi = (\xi_w)_{w\geq0}$ be a family of Holderian functions, with degree α_w, with $\alpha_w \to 1$, as $w \to \infty$. Put

$$K_w(t, u) = L_w(t)\xi_w(u).$$

In this case, all the assumptions of Proposition 4.1 are satisfied and so for such kernel the equivalence between the two definitions of singularity holds. In particular if $L_w \geq 0$ and $\|L_w\|_{L^1(G,\mu)} = 1$, and $\xi_w(u) = u^{1-1/w}$ for $u \geq 0$, and $\xi_w(u) = -|u|^{1-1/w}$ for $u \leq 0$, with sufficiently large $w > 0$, we obtain an example of singular and $\Xi-$singular kernel. Note that the equivalence given in Proposition 4.1 is not true for strong singularity.

5. Singularity for Kernels of Urysohn Integral Operators

In this section, we consider nonlinear integral operators which have no convolution structures, of type

$$(U_w f)(s) := \int_G K_w(s, t, f(t)) d\mu(t) \qquad (w > 0,\ s \in G).$$

Since we do not use a translation operator, we do not need an algebraic structure on G. Therefore our framework is the class of locally compact Hausdorff topological spaces. However, we will need also a uniform structure generating the topology of G. This enables us to employ some features of the topology of a topological group.

Let G be a locally compact Hausdorff topological space, provided with a regular measure μ on the $\sigma-$algebra $\mathcal{B}$ of the Borel sets of G. As before, by $L^0(G)$ we will denote the set of all measurable real functions on G (and as usual we identify functions which agree on a measurable subset H with $\mu(G \setminus H) = 0$). As before, by $L^1(G, \mu) \equiv L^1(G)$ we denote the Lebesgue space comprising all the integrable functions with respect to the measure μ.

We will assume that the topology of G is uniformizable, i.e. there is a (diagonal) uniform structure $\mathcal{V} = \{V\} \subset G \times G$, which generates the topology of G, (see, e.g. [47]). We recall that for $s \in G$, a basic neighborhood of s is given by

$$V_s = \{t \in G : (s,t) \in V\}$$

for any $V \in \mathcal{V}$. By local compactness, we will assume that for every $s \in G$, the base $\{V_s : V \in \mathcal{V}\}$ contains compact sets. Note that the uniform structure is employed only in order to define uniform continuity and to study uniform convergence.

Let $\mathcal{L}$ be the class of all $\mu \otimes \mu-$measurable functions $L : G \times G \to \mathbb{R}_0^+$ such that the sections $L(\cdot, t)$ and $L(s, \cdot)$ belong to $L^1(G, \mu)$, for every $t \in G$ and $s \in G$ respectively.

Let $\mathcal{P}^*$ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that ψ is a continuous (nondecreasing) function and $\psi(0) = 0,\ \psi(u) > 0,$ for $u > 0$. We will denote by $\mathcal{K}^*$ the class comprising the families $\mathbb{K} = (K_w)_{w>0}$, where $K_w : G \times G \times \mathbb{R} \to \mathbb{R}$, such that K_w is globally measurable and $K_w(s,t,0) = 0$ for every $(s,t) \in G \times G$.

We introduce the following slight modification of the generalized Lipschitz condition considered in [13, 14], (see also [15], in a more general case).

Definition 5.7. *Let $\Psi = (\psi_w)_{w>0} \subset \mathcal{P}^*$. We will say that $\mathbb{K} \in \mathcal{K}$ is (L, Ψ)-Lipschitz, if for every $w > 0$ there exists $L_w \in \mathcal{L}$ such that*

$$|K_w(s,t,u) - K_w(s,t,v)| \leq L_w(s,t)\psi_w(|u - v|) \qquad (s,t \in G).$$

If $\psi_w(u) = u$, for $u \geq 0$, and for every $w > 0$, we will say that $\mathbb{K}^*$ is $(L, 1)$-Lipschitz. We are now ready to introduce the following notion of singularity.

Definition 5.8. *We say that a (L, Ψ)-Lipschitz family $\mathbb{K}$ is singular if the following assumptions hold:*

(u.1) *There is $D > 0$ such that, for every $t, s \in G,\ w > 0$, we have*

$$\alpha_w(s) := \|L_w(s,\cdot)\|_{L^1(G,\mu)} \leq D,\ \ \widetilde{L}_w(t) := \|\frac{L_w(\cdot,t)}{\alpha_w(\cdot)}\|_{L^1(G,\mu)} \leq D,$$

(u.2) *For every $s \in G$ and $u \in \mathbb{R}$, we have*

$$\lim_{w\to\infty} \int_G K_w(s,t,u)d\mu(t) = u.$$

(u.3) *For every $s \in G$, and for every neighborhood V_s of s we have*

$$\lim_{w\to\infty} \int_{G\setminus V_s} L_w(s,t)d\mu(t) = 0.$$

We say that $\mathbb{K}$ is uniformly singular if conditions (u.2) and (u.3) are replaced by the following ones:

(u.2)′ *We have*

$$\lim_{w\to\infty} \int_G K_w(s,t,u)d\mu(t) = u,$$

uniformly with respect to $s \in G$ and $u \in C$, where C is any compact subset of $\mathbb{R} \setminus \{0\}$.

(u.3)′ *For every $V \in \mathcal{V}$ we have*

$$\lim_{w\to\infty} \int_{G\setminus V_s} L_w(s,t)d\mu(t) = 0$$

uniformly with respect to $s \in G$.

Remark 5.2. *1. Let us remark that in particular, in assumption (u.1) we can assume that there are constants $d, D > 0$ such that $d \leq \alpha_w(s) \leq D$ for every $w > 0, s \in G$ and $\|L_w(\cdot,t)\|_{L^1(G,\mu)} \leq D$, for every $w > 0$, $t \in G$. In this case it is easily seen that the second inequality in assumption 1 holds with the constant D/d. Moreover the above relations can be satisfied only for sufficiently large $w > 0$.*

2. *If G is a locally compact topological abelian group, setting $K_w(s,t,u) := K_w(s-t,u)$, for $s, t, \in G$ and $u \in \mathbb{R}$, in order to compare the definition of singularity given above with the one used for convolution operators in topological groups, considered in the previous sections, it should be noted that the present definition is more general and has the advantage to obtain examples of singular kernels in an easier way.*
3. *Employing the above definition of singularity, under the notion of regular families of measures, introduced in [13, 14] and some other side conditions, pointwise and uniform convergence theorems are obtained, and also a modular convergence theorem is established.*

6. Singularity for Kernels of Abstract Sampling-Type Operator

In recent years, it was introduced a general approach for the study of the convergence properties of sequences or nets of linear or nonlinear operators which include various kind of operators: from integral operators, to discrete operators of sampling type (see [1, 2, 4, 13, 14, 22, 23]). This approach allows us to develop a unitary treatment of the approximation properties of abstract operators in various functional spaces. Once again, the concept of singularity plays a fundamental role. In this section, we report the notion of singularity suitable for the study of these abstract operators.

Let G be a locally compact Hausdorff topological space, provided with its family of Borel sets $\mathcal{B}$ of G. Here we denote by μ_G be a regular and $\sigma-$finite measure defined on $\mathcal{B}$. As in the previous section, we will assume that the topology of G is uniformizable, i.e. there is a uniform structure $\mathcal{U} \subset G \times G$ which generates the topology of G. For every $U \in \mathcal{U}$, we put $U_s = \{t \in G : (s,t) \in U\}$. By local compactness, we will assume that for every $s \in G$, the base $\{U_s : U \in \mathcal{U}\}$ contains compact sets.

Let now $\{H_w\}_{w>0}$ be a net of nonempty closed subsets of G such that $G = \overline{\cup_{w>0} H_w}$.

For every $w > 0$ we will denote by μ_w a regular and σ-finite measure on H_w, defined on the Borel σ-algebra generated by the family $\{A \cap H_w : \text{A open subset of } G\}$.

Let $\mathcal{L}$ be the class of all the families of globally measurable functions $\mathbb{L} = (L_w)_{w>0}$, $L_w : G \times H_w \to \mathbb{R}_0^+$ such that for every $w > 0$ the sections $L_w(\cdot,t)$ and $L_w(s,\cdot)$ belong to $L^1(G,\mu)$ for every $t \in H_w$ and to $L^1(H_w,\mu_w)$ for every $s \in G$ respectively. If $\mathcal{P}^*$ is the class introduced in Section 5, let $\Psi := (\psi_w)_{w>0} \subset \mathcal{P}^*$ be a family of functions such that the following two assumptions hold:

1. $(\psi_w)_{w>0}$ is equicontinuous at $u = 0$,
2. for every $u \geq 0$ the net $(\psi_w(u))_{w>0}$ is bounded.

We denote by $\widetilde{\mathcal{K}}$ the class of all families of functions $\mathbb{K} = (K_w)_{w>0}$, where the functions $K_w : G \times H_w \times \mathbb{R} \to \mathbb{R}$, are such that for any $w > 0$, $K_w(\cdot,\cdot,u)$ is globally measurable on $G \times H_w$ for every $u \in \mathbb{R}$ and $K_w(s,t,0) = 0$, for every $(s,t) \in G \times H_w$.

Definition 6.9. *We say that $\mathbb{K} = (K_w)_{w>0}$ is (L,Ψ)-Lipschitz if there are a family $\mathbb{L} = (L_w)_{w>0} \in \mathcal{L}$ and a constant $D > 0$ such that*

$$0 < \beta_w(s) := \int_{H_w} L_w(s,t) d\mu_w(t) \leq D$$

for all $s \in G, w > 0$ *and*

$$|K_w(s,t,u) - K_w(s,t,v)| \leq L_w(s,t)\psi_w(|u-v|)$$

for every $s \in G,\ t \in H_w$ *and* $u, v \in \mathbb{R}$.

For a given (L,Ψ)-Lipschitz family $\mathbb{K} = (K_w)_{w>0} \in \widetilde{\mathcal{K}}$ we introduce the following family of nonlinear integral operators $\mathbf{T} = (T_w)_{w>0}$ given by

$$(6.6) \qquad (T_w f)(s) = \int_{H_w} K_w(s,t,f(t))d\mu_w(t) \;\; s \in G,\ w > 0$$

where $f \in Dom\mathbf{T} = \cap_{w>0} DomT_w;$ here $DomT_w$ is the subset of $L^0(G)$ on which $T_w f$ is well defined as a μ_G-measurable function of $s \in G$. The notion of singularity is now modified in the following way.

Definition 6.10. *We say that the* (L,Ψ)*-family* $\mathbb{K} = (K_w)_{w>0} \in \widetilde{\mathcal{K}}$ *is singular if the following assumptions hold*

(s.1) *There is a net* $(\zeta_w)_{w>0}$ *of positive real numbers such that for every* $w > 0$ *and* $t \in H_w$ *we have*

$$\int_G L_w(s,t)d\mu_G(s) \leq \zeta_w \leq D.$$

(s.2) *For every* $s \in G$ *and for every* $u \in \mathbb{R}$ *we have*

$$\lim_{w\to\infty} \int_{H_w} K_w(s,t,u)d\mu_w(t) = u.$$

(s.3) *For every* $s \in G$ *and for every* $U \in \mathcal{U}$ *we have*

$$\lim_{w\to\infty} \int_{H_w \setminus U_s} L_w(s,t)d\mu_w(t) = 0.$$

We say that $\mathbb{K}$ *is uniformly singular if conditions (s.2) and (s.3) are replaced by the following ones*

(s.2′) we have

$$\lim_{w\to\infty} \int_{H_w} K_w(s,t,u)d\mu_w(t) = u,$$

uniformly with respect to $s \in G$ *and* $u \in C,$ *where* C *is any compact subset of* $\mathbb{R} \setminus \{0\}$.

(s.3′) for every $U \in \mathcal{U}$ *we have*

$$\lim_{w\to\infty} \int_{H_w \setminus U_s} L_w(s,t)d\mu_w(t) = 0$$

uniformly with respect to $s \in G$.

For the above concepts see [13, 14]. By specializing G and $H_w,$ we obtain very interesting families of operators of discrete type. In particular, we can apply these results to nonlinear discrete operators (sampling series) of the form ([14])

$$(S_w f)(s) = \sum_{k\in\mathbb{Z}} L(ws-k)J_w\big(f\big(\frac{k}{w}\big)\big) \quad s \in \mathbb{R},\ w > 0.$$

Related results about nonlinear sampling type operators can be found e.g. in [32, 45]. Here, $G = \mathbb{R}$ endowed with the Lebesgue measure, $H_w = (1/w)\mathbb{Z}$ with counting measure, and the family of kernel is defined by

$$K_w(s,t,u) = L_w(s,t)J_w(u) \qquad (w > 0,\ s \in \mathbb{R},\ t \in H_w),$$

with $L_w(s,t) = L(w(s-t))$, where $L : \mathbb{R} \to \mathbb{R}$ is a continuous function with compact support such that

$$\sum_{k\in\mathbb{Z}} L(u-k) = 1 \qquad \text{(for every } u \in \mathbb{R})$$

and the family $(J_w)_{w>0}$ satisfies a $(L,\Psi)-$condition. Using this structure, reconstruction of the function f is obtained, in various sense (pointwise, uniform, modular). Analogously, taking $G = \mathbb{R}^+$ endowed with the logarithmic measure $d\mu(t) = dt/t$ and $H_w = \{t = e^{k/w} : k \in \mathbb{Z}\}$, endowed with the counting measure, we obtain nonlinear version of exponential sampling series (see [30, 31, 37]).

Other examples are given by the classical nonlinear operators defined over the euclidean space R^N, where we take $G = H_w = \mathbb{R}^N$, endowed with the Lebesgue measure, or for Mellin-type integral operators, for which we take $G = H_w = \mathbb{R}^+$, endowed with the logarithmic measure.

7. Other Developments

There exist other definitions of singularity which take into account certain properties of the kernels. In [7], there is considered a class of nonlinear integral operators, acting on functions defined over the positive real line, whose kernels satisfy some generalized Lipschitz conditions, in which the functions L_w satisfy certain general homogeneity assumptions with respect to a continuous function. To be precise, let $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that

$$(\eta(t))^{-1}\eta(ts) \leq C\eta(s) + D \qquad (t, s \in \mathbb{R}^+)$$

for suitable constants C and D. Then we say that the function $L_w : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+_0$ is $\eta-$homogeneous, if for any $\lambda > 0$, we have

$$\eta(t)L_w(\lambda s, \lambda t) = \eta(\lambda t)L_w(s,t),$$

for every $w > 0$, $s, t \in \mathbb{R}^+$. For this kind of kernels, the singularity is defined through the homogeneity function η. For example, assumption like (u.2) of Section 5, is now expressed as follows:

$$\lim_{w\to\infty} \int_0^\infty K_w(s,t,u)\frac{dt}{t} = \eta(s)u$$

for every $s \in \mathbb{R}^+$ and $u \in \mathbb{R}$. Employing this notion of singularity, approximation results in pointwise and uniform sense were obtained for the function $g := \eta f$. Another notion of singularity is linked with the study of quantitative estimates of convergence, for which the assumptions like (u.2) and (u.3) are given with a prescribed order (see [7]).

Related definitions are given in [5] for the nonlinear abstract sampling type operators considered in Section 6, when the convergence is studied with respect to a filter $\mathcal{F}$. In this case the singularity takes into account of this kind of convergence, and it is named $\mathcal{F}-$singularity. We shortly report this abstract approach. For a sake of simplicity, we consider now sequences instead of nets, that is we take $\mathbb{N}$ as the set of indices (instead of $\mathbb{R}^+$). For details and further references see [5] (see also [33]). We begin with some preliminary definitions.

Definition 7.11. *A nonempty family $\mathcal{F}$ of subsets of $\mathbb{N}$ is called a filter of $\mathbb{N}$ iff $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and $B \supset A$ we get $B \in \mathcal{F}$.*

As classical examples of filters we quote the filter $\mathcal{F}_{\text{cofin}}$ of all subsets of $\mathbb{N}$ whose complement is finite and the filter $\mathcal{F}_{\text{d}}$ associated with the statistical convergence, that is the class of all subsets of $\mathbb{N}$ whose asymptotic density is 1. We recall here this notion

Definition 7.12. *The asymptotic density of a set $A \subset \mathbb{N}$ is defined as*

$$d(A) = \lim_n \frac{\sharp(A \cap \{1, \ldots, n\})}{n}$$

(provided that this limit exists). Here by $\sharp$ we denote the cardinality of the set in brackets (see e.g [34, 43]).

Definition 7.13. *1. A sequence $(x_n)_n$ in $\mathbb{R}$ is said to be $\mathcal{F}$-bounded iff there exists an $M > 0$ such that $\{n \in \mathbb{N} : |x_n| \leq M\} \in \mathcal{F}$.*
2. A sequence $(x_n)_n$ in G is $\mathcal{F}$-convergent to $x \in G$ (and we write $x = (\mathcal{F}) \lim_n x_n$) iff $\{n \in \mathbb{N} : (x_n, x) \in U\} \in \mathcal{F}$ whenever $U \in \mathcal{U}$.
3. A sequence $f_n : G \to \mathbb{R}$, $n \in \mathbb{N}$, is called $\mathcal{F}$-exhaustive at $s \in G$ iff for every $\varepsilon > 0$ there exist a neighborhood U_s of s and $A \in \mathcal{F}$ with $|f_n(z) - f_n(s)| \leq \varepsilon$, whenever $n \in A$ and $z \in U_s$.
4. A sequence $f_n : G \to \mathbb{R}$, $n \in \mathbb{N}$, is said to be $\mathcal{F}$-convergent uniformly to f on G iff

$$(\mathcal{F}) \lim_n [\sup_{t \in G} |f_n(t) - f(t)|] = 0.$$

Using the notations of the previous section, let $\mathbb{K} := (K_n)_{n \in \mathbb{N}} \in \widetilde{\mathcal{K}}$ be a (L, Ψ)-Lipschitz family.

Definition 7.14. *We say that $\mathbb{K}$ is $\mathcal{F}$-singular iff*

(f.1) there is a $D_1 > 0$ with

$$\Lambda := \Big\{n \in \mathbb{N} : \int_{H_n} L_n(s,t)\, d\mu_n(t) \leq D_1 \text{ for all } s \in G\Big\} \in \mathcal{F},$$

(f.2) for every $s \in G$ and for each neighborhood $U_s \subset G$ we get

$$(\mathcal{F}) \lim_n \int_{H_n \setminus U_s} L_n(s,t)\, d\mu_n(t) = 0,$$

(f.3) for every $s \in G$ and $u \in \mathbb{R}$ we have

$$(\mathcal{F}) \lim_n \int_{H_n} K_n(s,t,u)\, d\mu_n(t) = u.$$

We say that $\mathbb{K}$ is strongly $\mathcal{F}$-singular iff it fulfills (f.1) and

(f.2)′ for any $s \in G$ and for every neighborhood $U_s \subset G$ there is a neighborhood $Z_s \subset G$, with

$$(\mathcal{F}) \lim_n \Big[\sup_{z \in Z_s} \int_{H_n \setminus U_s} L_n(z,t)\, d\mu_n(t)\Big] = 0,$$

(f.3)′ for any $s \in G$ and $u \in \mathbb{R}$ there are two neighborhoods U_s of s and W of u respectively, with

$$(\mathcal{F}) \lim_n \Big[\sup_{z \in U_s, v \in W} \Big(\int_{H_n} K_n(z,t,v)\, d\mu_n(t) - v\Big)\Big] = 0.$$

With the above notion of convergence, in [5] several convergence theorem are established including also the case of statistical convergence for sequence of operators T_n (6.6).

As a final issue, we recall also the particular case of singularity for kernels of nonlinear operators acting of real functions with bounded variation (in its various sense, Jordan variation, φ-variation, and other generalizations). In [3, 4] (see also [20]), some notions of singularity are given for kernels of type

$$K_w(t,u) := L_w(t) H_w(t) \quad (w > 0,\ t \in \mathbb{R}, u \in \mathbb{R}),$$

where $L_w \in L^1(\mathbb{R})$ or $L_w \in L^1_{2\pi}$, being $L^1_{2\pi}$ the space comprising all the Lebesgue 2π-periodic summable functions with respect to the Lebesgue measure, and H_w satisfies a strong Lipschitz condition of type $|H_w(u) - H_w(v)| \leq C|u - v|$, for every $u, v \in \mathbb{R}$, and an absolute constant $C > 0$. In the periodic case, the notion of singularity in BV spaces is as follows.

Definition 7.15. *We say that $\mathbb{K} := \{K_w\}_{w>0}$ is BV-singular if $\|L_w\|_1 \leq A$ for an absolute constant $A > 0$, $A_w := \int_{-\pi}^{\pi} L_w(t)dt \to 1$ as $w \to \infty$ and for any $\delta > 0$ one has*

$$\lim_{w\to\infty} \int_{\delta\leq|t|\leq\pi} |L_w(t)|dt = 0,$$

and setting $G_w(u) := H_w(u) - u$, for $u \in \mathbb{R}$, one has $V_J(G_w) \to 0$ as $w \to \infty$, where V_J denotes a notion of variation over a bounded interval $J \subset \mathbb{R}$.

With the above notion of BV-singularity, convergence theorems with respect the total variation (in various sense) are obtained ([3, 4, 20]).

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