MODERN MATHEMATICAL METHODS
4 (2026), No. 2, pp. 134-148
https://modernmathmeth.com/
ISSN 3023 - 5294

Research Article

Indefinite proximities inherent in dynamical systems: An axiomatic approach

JAMES F. PETERS, TANE VERGILI*, FATIH UÇAN, AND DIVAGAR VAKEESAN

ABSTRACT. This paper introduces indefinite proximities inherent in every collection of physical objects found in a dynamical system. The number of characteristics in the description of any dynamical system is unknown (Axiom 2.1 and Axiom 2.2). Hence, the description of a dynamical system is indefinite. Axiomatically, these indefinite proximities lead to a new form of Hausdorff topology, which is indefinite descriptively. The main results in this paper are

1. Every descriptive proximity space on a dynamical system is indefinite (Theorem 3.1).
2. Every dynamical system has an indefinite descriptive Hausdorff topology (Theorem 3.3).
3. The energy of a dynamical system varies with every clock tick (Theorem 5.4).

An application of these results is given in terms of the detection of those portions of a dynamical system that are stable and that have low energy dissipation.

Keywords: Characteristic, descriptive proximity space, dynamical system, Hilbert envelope, indefinite.

2020 Mathematics Subject Classification: 54E05, 54C50.

1. INTRODUCTION

This paper introduces an axiomatic approach in the study of measurable descriptive proximities that are inherent in the self-similarities in the parts, behaviours and waveforms of complex dynamical system. The consequences of this approach carry forward recent work on the descriptive approach in the study of dynamical systems [6, 7, 20, 23], especially chaotic dynamical systems [5]. This approach leads to the introduction of a number of new descriptive proximity relations that represent advances in proximity space theory [3, 14, 18] useful in the detection and measurement of those portions of dynamical systems that are stable and that have low energy dissipation.

All proximities between the parts and waveforms of dynamical systems are indefinite. This observation leads to the introduction of an indefinite descriptive proximity $\delta_{lim\Phi}$, which is a refinement of the relaxed descriptive proximity δ_{Φ_o} [6]. We observe that every descriptive proximity space on a dynamical system is indefinite (Theorem 3.1). Important results stemming from the introduction of an indefinite descriptive distance $d^{lim\Phi}$ (Definition 2.7) are given, namely, Indefinite Descriptive Hausdorff Topology (Lemma 3.2) and every dynamical system has an indefinite Hausdorff topology (Theorem 3.3). This paper also includes an application of $\delta_{lim\Phi}$ in detecting as well as measuring the stability, low energy dissipation portions of dynamical system waveforms.

Received: 13.11.2025; Accepted: 28.07.2026; Published Online: 08.08.2026
*Corresponding author: Tane Vergili; tane.vergili@ktu.edu.tr
DOI: 10.64700/mmm.87

Symbol	Meaning
d^{Φ}	Descriptive proximity distance: Def. 2.2
d_H^{Φ}	Descriptive Hausdorff distance: Def. 2.5
$(\mathcal{K}_{\Phi}, d_H^{\Phi}, \tau_H^{\Phi})$	Descriptive Hausdorff Topological Space: Def. 2.6
$d^{lim\Phi}$	Indefinite descriptive distance: Def. 2.7
(X, f, Φ)	Descriptive dynamical system: Def. 4.12
$Per(f, \Phi)$	Descriptive periodic points: Def. 4.14
δ_{Φ_o}	Relaxed descriptive proximity: Def. 5.17
$E_{m(t)}$	Waveform $m(t)$ energy: Def. 5.19
$E_{diss}(loc, t)$	Energy dissipation at location loc at time t: Def. 5.21

TABLE 1. Principal symbols used in this paper.

2. PRELIMINARIES

This section introduces characteristic-based proximities as well as characteristically proximal self-similarities in dynamical system behaviors. The earlier introduction of probe functions [14, p. 67] as a basis for the description of a nonempty set has been refined, based on earlier work on holomorphic functions [25].

Definition 2.1 (Characteristic). *Let 2^X be a collection of subsets of a nonempty set X and $A \in 2^X$. Also, let $\mathbb{C}$ denote the complex plane. A characteristic of A is a mapping $\phi : 2^X \to \mathbb{C}$ defined by $\phi(A) = k \in \mathbb{C}$.*

A complete description of A with n characteristics is a set $\Phi(A)$ where

$$\Phi(A) = \{(\phi_1(a), \phi_2(a), \ldots, \phi_n(a)) \ : \ a \in A\} \subseteq \mathbb{C}^n.$$

Notice that $\Phi(A) \cap \Phi(B) \neq \emptyset$ implies there exist $a \in A$ and $b \in B$ such that $\Phi(\{a\}) = \Phi(\{b\})$. Throughout the rest of the paper, we will simply use $\Phi(a)$ instead of $\Phi(\{a\})$.

Definition 2.2 (Characteristically proximal sets). *Let $\Phi = \{\phi_1, \ldots, \phi_n\}$ (set of characteristic maps) and let X be a nonempty set with $A, B \in 2^X$. Consider the characteristic proximity mapping $d^{\Phi} : 2^X \times 2^X \to \mathbb{R}$ defined by*

$$d^{\Phi}(A, B) = \inf_{\substack{a \in A \\ b \in B}} \{|\Phi(a) - \Phi(b)|\} = r \in \mathbb{R}^{\geq 0}.$$

Then, we say that nonempty sets A, B are characteristically proximal, which is denoted by $A \ \delta_{\Phi} \ B$, provided $d^{\Phi}(A, B) = 0$.

Example 2.1. *Let $\boldsymbol{Fib} = \{\boldsymbol{1, 1, 2, 3, 5, 8, 13, 21, \ldots}\}$ (sequence of Fibonacci numbers). The pair of intersecting surfaces $\boldsymbol{S}_{\mathbb{R}} \cup \boldsymbol{S}_{\mathbb{C}}$ in Fig. 1 represent the real and the imaginary parts of a Hilbert cyclomatic exponential [12], defined over sequences of reals x, y and a subsequence η of the Fibonacci numbers, e.g.,*

$$\boldsymbol{S}_{\mathbb{R}} = \boldsymbol{Re}\left[\boldsymbol{x^{0.003} e^{i\pi y/2\eta+1}}\right].$$

In Fig. 1, let $\boldsymbol{imS}$ denote the surface derived from the imaginary part of the Hilbert cyclotomic exponential and let $\boldsymbol{reS}$ the surface derived from the real part of the Hilbert cyclotomic exponential. Further, let $\boldsymbol{reS^{+ve}}$ denote a characteristic of the upper surface peaks and let $\phi_{reS_{-ve}}$ denote a characteristic of the lower surface peaks. Also, let $X \subset reS^{+ve}$.

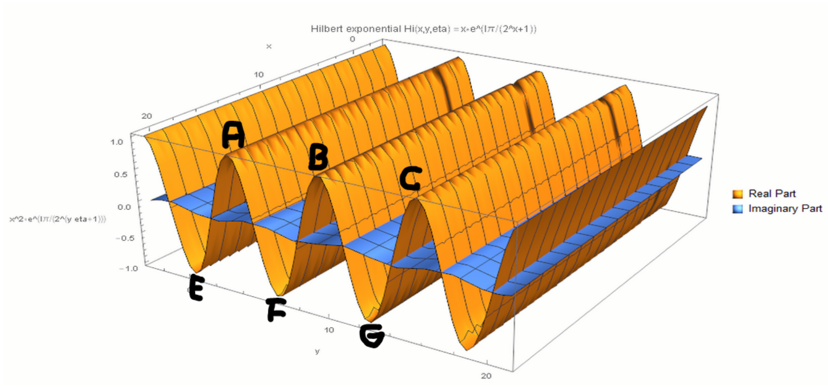

FIGURE 1. Hilbert sub-surfaces generated by a cyclomatic exponential [12].

Example 2.2 (Characteristics of Hilbert cyclomatic surface proximities)**.** *Consider, for the example, the following characteristics of the Hilbert cyclomatic surfaces in Fig. 1:*

$$\phi_{reS^{+ve}}(X) = 1, if\ X \subset reS^{+ve}, else\ \phi_{reS^{+ve}}(X) = 0 \text{ (upper surface characteristic).}$$

$$\phi_{reS^{-ve}}(X) = 1, if\ X \subset reS^{-ve}, else\ \phi_{reS^{-ve}}(X) = 0 \text{ (lower surface characteristic).}$$

Consider the following characteristic proximities

charProx-1: $\phi_{reS^{+ve}}(A) = \phi_{reS^{+ve}}(B) = \phi(C)_{reS^{+ve}} = 1$, *since* $X = \{A, B, C\} \subset reS^{+ve}$.

charProx-2: *From Def. 2.2 and charProx-1, we have* $A\ \delta_\Phi\ B$ *and* $B\ \delta_\Phi\ C$.

charProx-3: *Observe*

$$d^\Phi(A, B) = \inf_{\substack{a \in A \\ b \in B}} \{|\boldsymbol{\phi_{reS^{+ve}}(A) - \phi_{reS^{+ve}}(B)}|\} = 0.$$

Hence, our observation $A\ \delta_\Phi\ B$ *in charProx-2 is reinforced.*

charProx-4: $\phi_{reS^{+ve}}(E) = \phi_{reS^{+ve}}(F) = \phi_{reS^{+ve}}(G) = 0$, *since* $X = \{E, F, G\} \not\subset reS^{+ve}$.

charProx-5: $\phi_{reS^{-ve}}(E) = \phi_{reS^{-ve}}(F) = \phi(G)_{reS^{-ve}} = 1$, *since* $\{E, F, G\} \subset reS^{-ve}$. *Hence,* $E\ \delta_\Phi\ F$ *and* $F\ \delta_\Phi\ G$.

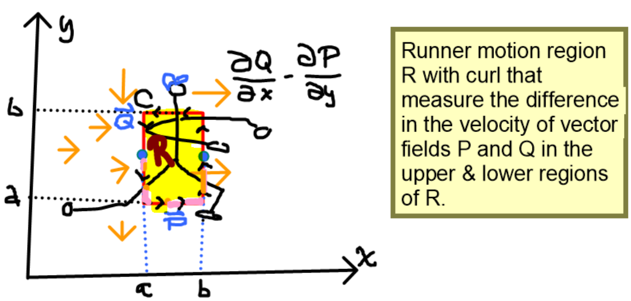

FIGURE 2. Runner vector field curl.

The following axioms enunciate the observations we have made about the characteristics of physical objects in dynamically changing systems.

Axiom 2.1 (Indefinite number of characteristics of a physical object). *Let $\Phi(A)$ be a description of a nonempty set of physical objects A. The total number of characteristics of A is indefinite, i.e., not known.*

Axiom 2.2 (Unknown maximum size of the differences between sets of characteristics of physical objects). *Let $|\Phi(A)|, |\Phi(B)|$ be the number of characteristics for nonempty sets A, B. The maximum size of the difference $|\Phi(A)| - |\Phi(B)|$ is not known.*

Remark 2.1 (N.B.). *We use N.B. abbreviation for Latin nota bene (note well) to call attention to something important. This expression first appeared in 1711, see* [1].

N.B. 2.1 (Curl of a dynamically changing system (see [24, §11-4])). *Observe, for example, in Fig. 2, that the total number of characteristics is unknown for a dynamically changing system represented by the curl of the runner vector field $\vec{V}f$, where*

$$\boldsymbol{curl\vec{V}f = \nabla \times \vec{V}f = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}},$$

which is the curl over a system surface region composed of sub-regions P, Q.

In describing the proximities between self-similarities in a physical chaotic system represented by the collection of subsets 2^X for a set X of system objects, the description of a subsets $P, Q \in 2^X$ would typically be incomplete. This is consistent with the view that the number of known characteristics of every physical object is indefinite (see Axiom 2.1) and the maximum number of characteristics of every pair of physical objects is also not known (see Axiom 2.2).

N.B. 2.2 (Origin of characteristic proximity). *Characteristic functions were introduced by W. R. Hamilton* [9] *in 1837 as a means of deducing the properties of a system. Characteristically proximal sets were introduced informally* [16] *in 2012. This form of proximity is largely due to a collaboration with S. A. Naimpally that culminated in the topology of near sets* [14] *in 2013 and the introduction of object characteristics in the context of computational proximity* [17] *in 2016.*

Definition 2.3 (Characteristic proximity space, [14]). *For a nonempty set X, $\left(2^X, \delta_\Phi\right)$ is a characteristic proximity space.*

Definition 2.4 (Hausdorff distance, [10, 11]). *The Hausdorff distance, $d_H(Q,S)$, between a pair of compact subsets Q and S in $\mathbb{R}^m$ is defined by*

$$d_H(Q,S) = \max\left\{\sup_{q\in Q} D(q,S), \sup_{s\in S} D(Q,s)\right\} \geq 0,$$

where $D(p,-)$ or $D(-,p)$ denote the distance between a single point p and a given set.

One can also measure the descriptive Hausdorff distance of A and B in X by assuming that their complete descriptions $\Phi(A)$ and $\Phi(B)$ are compact subsets of $\mathbb{R}^n$.

Definition 2.5 (Descriptive Hausdorff distance). *Let (X, δ_Φ) be a descriptive proximity space with a collection of n characteristics, and $A, B \in 2^X$ with compact complete descriptions $\Phi(A), \Phi(B)$ in $\mathbb{R}^m$. The descriptive Hausdorff distance $d_H^\Phi(A,B)$ between A and B is defined by*

$$d_H^\Phi(A,B) = d_H(\Phi(A), \Phi(B)).$$

Remark 2.2. *We assume that*

$$\boldsymbol{d^\Phi(A,B) = d_H(\Phi(A), \Phi(B)) = r} \in \mathbb{R}^{\geq 0}$$

for a set of known characteristics for $A, B \in 2^X$ for a set X. In a dynamical system X that is chaotic, X is inherently self-symmetric.

Given a descriptive proximity space (X, δ_Φ), let $\mathcal{K}_\Phi$ be a collection of all subsets of X with compact complete descriptions

$$\mathcal{K}_\Phi = \{A \in 2^X \ : \ \Phi(A) \text{ is compact}\}.$$

Then, d_H^Φ defines a metric on $\mathcal{K}_\Phi$.

Definition 2.6 (Descriptive Hausdorff topological space)**.** *Let the set I be an arbitrary index set. A descriptive Hausdorff topology, τ_H^Φ, induced by the descriptive Hausdorff metric d_H^Φ on $\mathcal{K}_\Phi$ has the following properties*

1º $\mathcal{K}_\Phi, \emptyset \in \tau_H^\Phi$.
2º $\{\mathcal{A}_i\}_{i\in I} \subseteq \tau_H^\Phi$ *implies* $\bigcup_{i\in I} \mathcal{A}_i \in \tau_H^\Phi$.
3º $\mathcal{A}, \mathcal{B} \in \tau_H^\Phi$ *implies* $\mathcal{A} \cap \mathcal{B} \in \tau_H^\Phi$.

The triple $(\mathcal{K}_\Phi, d_H^\Phi, \tau_H^\Phi)$ is called a descriptive Hausdorff topological space.

Corollary 2.1. *There is a descriptive Hausdorff topology τ_H^Φ on every collection of compact complete descriptions of a descriptive proximity space (X, δ_Φ).*

N.B. 2.3 (New form of descriptive proximity)**.** *From Axiom 2.1, the number of characteristics of a physical objects is indefinite. For this reason, we introduce a new form of descriptive proximity of a pair of sets of physical objects A, B in terms of the differences of the descriptions of A and B converging to 0 in the limit (see Definition 2.7).*

Definition 2.7 (Indefinite characteristic distance)**.** *Let X be a nonempty set. $A, B \in 2^X$ are indefinitely characteristically near sets of physical objects, provided*

$$d^{\lim \Phi}(A, B) = \lim_{|\phi(a)-\phi(b)|\to 0} d^\Phi(A, B) = 0.$$

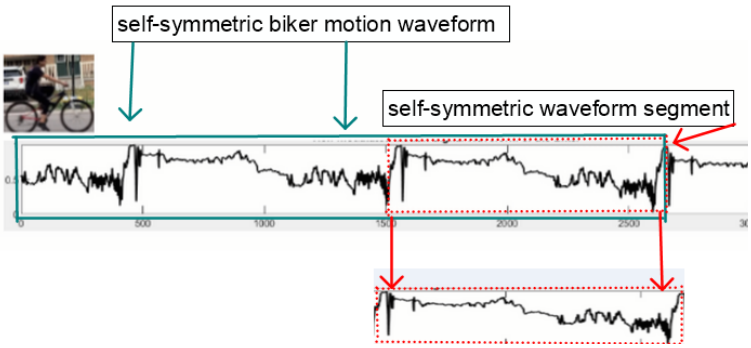

FIGURE 3. Self-similar biker motion waveform.

3. CHAOTIC DYNAMICAL SYSTEM

Dynamical chaos is different from randomness or disorder. Chaos is deterministic in descriptive set theory [23]. For example, vibrations of a mechanical system are quasi-periodic, changing from period doubling to chaos with period doubling [26]. Chaos in dynamics occurs if the cloud of representative points in the course of evolution in the phase space undergoes repetitive deformations of stretching, folding, and transversal compression [13]. Typically, nonlinear motion cascades to chaos [27].

Definition 3.8 (System)**.** *A system is a configuration of components functioning together as a whole and their relationships.*

Definition 3.9 (Dynamical system)**.** *A dynamical system is a physical system (collection of physical objects) that changes over time.*

Definition 3.10 (Chaotic dynamical system)**.** *A chaotic dynamical system (CDS) is a system that has the following properties*

1º The waveform of a CDS is self-similar, i.e., the CDS contains parts with the same structure as the complete CDS.

2º The self-similar fractals in the CDS satisfy the open set condition [5]*, i.e., there exists an integer n so that for every piece $A_i \in CDS$ with diameter ε, there are at most incomparable pieces $A_{j\leq n} \in CDS$ with diameter $\geq \varepsilon$ with distance ε from A_i* [22]*. Let $X \in 2^Y$ in a CDS with subsets 2^X in a descriptive Hausdorff proximity space $(\mathcal{K}^Y_\Phi, d^\Phi_H, \tau^\Phi_H)$. A description $\Phi(X)$ of a CDS X in a space Y contains the description of all the subsets of $A \in 2^X$ such that $d_H(A, B) < \varepsilon$ for all $B \in 2^Y$.*

3º The number of known characteristics of the CDS is indefinite.

Example 3.3. *A vibrating system is an example of CDS. For example, in Fig. 3 a biker is a CDS with the following properties*

1º The biker waveform in Fig. 3 is self-similar, i.e., the complete motion waveform pattern is repeated in its segments.

2º The biker waveform satisfies the open set condition. To see this, let $X \in 2^Y$ in a biker motion system with subsets 2^X in a collection Y of moving systems (e.g., walkers and vehicles). This biker CDS resides in a descriptive Hausdorff proximity space $\mathcal{K}^X_\Phi$ that is a subspace of $\mathcal{K}^Y_\Phi$. Let $B \in 2^Y$. A description $\Phi(A)$ of a subset $A \in 2^X$ in space Y contains the description of all the subsets of $A \in 2^X$ such that $d_H(A, B) < \varepsilon$ for all $B \in 2^Y$.

3º The number of known characteristics of the biker system is indefinite.

Lemma 3.1. *Every descriptive proximity space on a collection of physical objects is indefinite.*

Proof. Let (X, δ_Φ) be a descriptive proximity space on a collection of physical objects in 2^X. From Axiom 2.1, the number of known characteristics of subsets $A, B \in 2^X$ is indefinite. Then, from Definition 2.7, $d^\Phi(A, B) = d^{\lim \Phi}(A, B)$. Hence, $(2^X, d^\Phi)$ is indefinite. □

Theorem 3.1. *Every descriptive proximity space on a dynamical system is indefinite.*

Proof. Immediate from Axiom 2.1 and Lemma 3.1. □

Theorem 3.2. *Every descriptive proximity space on a chaotic dynamical system is indefinite descriptively.*

Proof. Immediate from Definition 3.10 and Theorem 3.1. □

Lemma 3.2 (Indefinite descriptive Hausdorff topology)**.** *Every descriptive Hausdorff topology on a sets of objects in a physical system is indefinite.*

Proof. Let 2^X be collections of subsets of a dynamical system X, $A, B \in 2^X$. Let $(X, \delta_{\lim \Phi})$ be an indefinite descriptive proximity space. From Axiom 2.2, all descriptive Hausdorff distances $d_H(\Phi(A), \Phi(B))$ approach zero, i.e., descriptive Hausdorff proximities between sets of physical objects in a dynamical system are indefinite. By replacing 2^X in Definition 2.6, and obtain an indefinite descriptive Hausdorff topology. □

Theorem 3.3. *Every dynamical system has an indefinite descriptive Hausdorff topology.*

Proof. Immediate from Definition 3.9 and Lemma 3.2. □

4. Descriptive Case

There are many definitions of chaos in the literature. Among these definitions, we often encounter definition of chaos in sense of R. L. Devaney made in 1986 [4], followed by studies of the relationship between individual chaos and collective chaos (see, e.g., [5, 8, 15, 22]). In this section, our aim is to analyze a chaotic dynamical systems in the descriptive sense by adding new features to it and observe them.

4.1. **Descriptive Dynamical System.** The fabric of a descriptive dynamical system is a family of interactions on a nonempty set of points (quanta) X. There is an orbit of each point $x \in X$ defined by a family of interactions on x with characteristics such as vector tail or vector tip. In this subsection, we consider X as a topological space together with a real-valued characteristic function $\Phi : X \to \mathbb{C}^m$ on it.

Definition 4.11 (Family of iterations). *The dynamical system (X, f) on X defined by f is the family of iterations $\{f^n\}_{n\in\mathbb{Z}^+}$ with each f^n mapping from X to itself. For $x \in X$ and $n \in \mathbb{Z}^+$ the orbit of x is the set of points*

$$x, f(x), f^2(x), \ldots, f^n(x), \ldots$$

and is denoted by

$$Orb_f(x) = \{f^n(x) : n \in \mathbb{Z}^+\}.$$

Definition 4.12 (Descriptive dynamical system). *A descriptive dynamical system is a pair $(\boldsymbol{S}, \boldsymbol{\Phi})$, which consists of a nonempty set S accompanied by a nonempty set characteristic functions such that*

$$\boldsymbol{\Phi = \{\phi_1, \ldots, \phi_i, \ldots, \phi_n\} |^{n\geq 1} : \phi_i : S} \to \mathbb{C}.$$

Definition 4.13 ([19]). *Given a descriptive dynamical system (X, f, Φ), a subset A of X is*

(1) a descriptive fixed subset of f, provided $\Phi(f(A)) = \Phi(A)$ and

$$\Phi(f^j(A)) \neq \Phi(A), \textit{for the set of descriptive periodic sets } j = 1, 2, \ldots, m-1.$$

(2) a descriptive period-m set of f if $\Phi(f^m(A)) = \Phi(A)$.

When $A = \{a\}$ is a singleton set, we simply replace the word set with the word point. In that case, we say that a point $a \in X$ is

(1) a descriptive fixed point of f, if $\Phi(f(a)) = \Phi(a)$,
(2) a descriptive period-m point of f if $\Phi(f^m(a)) = \Phi(a)$, and $\Phi(f^j(a)) \neq \Phi(a)$ for $j = 1, 2, \ldots, m-1$.

Definition 4.14 (Descriptive periodic points). *Given a descriptive dynamical system (X, f, Φ),*

(1) the set of descriptive m-periodic points is given by

$$Per_m(f, \Phi) = \{a \in X : \Phi(f^m(a)) = \Phi(a)\}.$$

(2) the set of descriptive periodic points is given by

$$Per(f, \Phi) = \bigcup_{m\in\mathbb{Z}^+} Per_m(f, \Phi) = \left\{a \in X \ : \ \Phi(f^m(a)) = \Phi(a), m \in \mathbb{Z}^+\right\}.$$

Remark 4.3. *Here, one can understand from Definition 3.10 and Theorem 3.2 that every periodic m-set on a chaotic dynamical system is an indefinite descriptive periodic m-set. However, the converse may not be true. Even so, a periodic m-set is a descriptive periodic k-set for $k \leq m$.*

Definition 4.15. *The set-valued function $\bar{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ is defined by $\bar{f}(A) = \{f(a) : a \in A\}$ so that the complete description of A is*

$$\begin{aligned}\Phi(\bar{f}(A)) &= \{\Phi(f(a)) \ : \ a \in A\} \\ &= \big\{\big(\phi_1(f(a)), \phi_2(f(a)), \ldots, \phi_n(f(a))\big) : a \in K\big\} \subseteq \mathbb{C}^n\end{aligned}$$

and its descriptive orbit is a collection of sets

$$\Phi(\bar{f}(A)), \Phi(\bar{f}^2(A)), \ldots, \Phi(\bar{f}^n(A)).$$

Remark 4.4. *For a subset A of X, the extension of A to $\mathcal{K}(X)$ is defined by*

$$e(A) = \{K \in \mathcal{K}(X) \ : \ K \subset A\}.$$

Also the following lemma is given in [21].

Lemma 4.3. *For two subsets A and B of X, the following holds*

- *$e(A) \neq \emptyset$ if and only if $A \neq \emptyset$,*
- *$e(A)$ is open subset of $\mathcal{K}(X)$, if $A \subseteq X$ is open,*
- *$e(A \cap B) = e(A) \cap e(B)$,*
- *$\bar{f}(e(A)) \subseteq e(\bar{f}(A))$,*
- *$\bar{f}^n = \bar{f}^n$, for all $m \in \mathbb{Z}^+$.*

Here, we present a descriptive approach to Devaney's definition of chaos by introducing new definitions: a dense collection of descriptive periodic sets, descriptive transitivity and descriptively sensitive.

Definition 4.16. *For $(\mathcal{K}(X), d_H)$, a mapping $\bar{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ is said to be descriptively chaotic if*

- *the collection of Per$(\bar{f}, \Phi)$ descriptive periodic compact sets which is given by*
$$\bigcup_{m\in\mathbb{Z}^+} Per_m(\bar{f}, \Phi) = \big\{K \in \mathscr{K}(X) \ : \ \Phi(\bar{f}^m(K)) = \Phi(K), m \in \mathbb{Z}^+\big\}$$
is dense in $\mathscr{K}(X)$,
- *for every $e(U), e(V)$ which are respectively extensions of nonempty open pair $U, V \subseteq X$, there exists $n \in \mathbb{Z}^+$ such that $\bar{f}(e(U))$ and $e(V)$ are descriptively proximal, i.e., $\Phi(\bar{f}^n(e(U))) \cap \Phi(e(V)) \neq \emptyset$ ($\bar{f}$ is descriptively transitive), and*
- *there is a $\delta > 0$ (sensitive constant) such that for any elements A, B of $e(U)$ which is an extension of nonempty open subset U of X, there exists $n \in \mathbb{Z}^+$ so that $d^{\Phi}(\bar{f}^n(A), \bar{f}^n(B)) = d_H(\Phi(\bar{f}^n(A), \bar{f}^n(B))) > \delta$ ($\bar{f}$ is descriptively sensitive).*

Lemma 4.4. *If the collection of periodic sets in a Hausdorff metric space is dense, so is the collection of descriptively periodic sets.*

Proof. Immediately follows from the fact that any periodic set is also descriptively periodic set. □

Lemma 4.5. *If $\bar{f}$ is topologically transitive, so is descriptively transitive.*

Proof. Let $e(U), e(V)$ be extensions of two nonempty open subsets U, V of X respectively. Since $\bar{f}$ is topologically transitive, there exists a positive integer m such that $\bar{f}^m(e(U)) \cap e(V) \neq \emptyset$. In that case there exists a compact subset K in $\mathcal{K}(X)$ such that $K \in \bar{f}^m(e(U))$ and $K \in e(V)$. Since K is in the image of $\bar{f}^m(e(U))$, there is K' in $e(U)$ so that $\bar{f}^m(K') = K$. This implies $\Phi(\bar{f}^m(K')) = \Phi(, K)$ i.e., $\Phi(\bar{f}^m(e(U))) \cap \Phi(e(V)) \neq \emptyset$. □

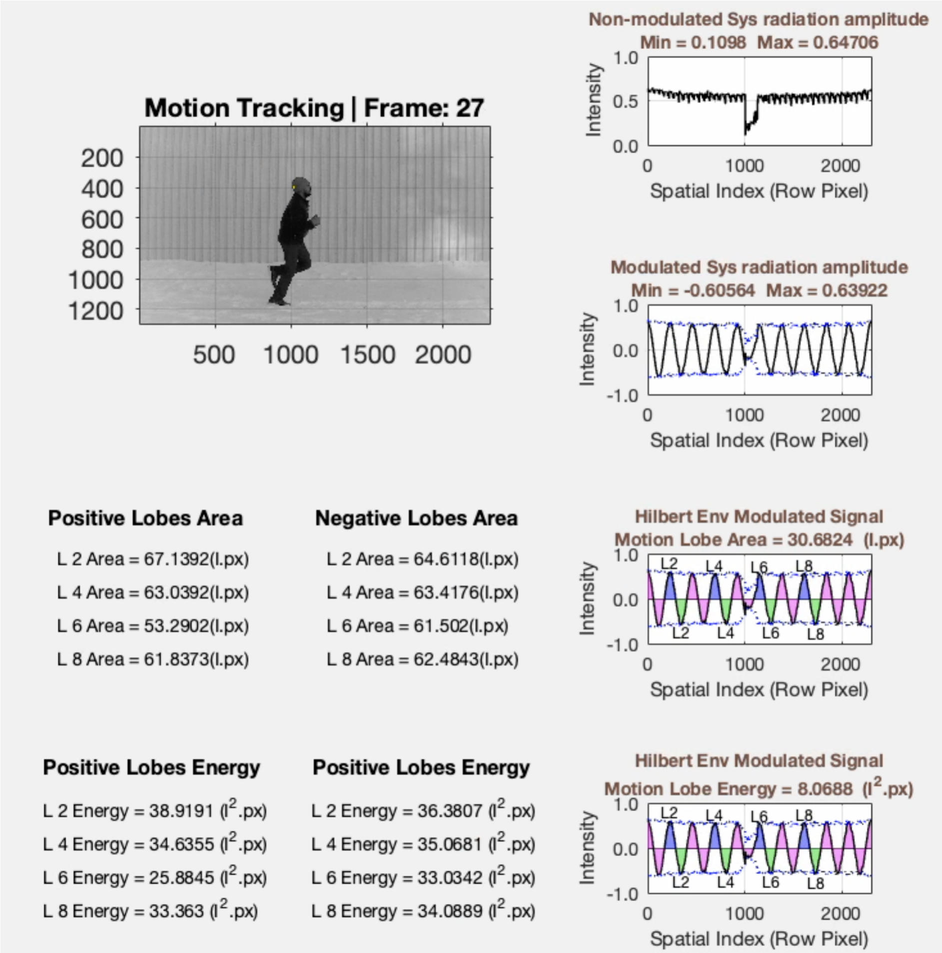

FIGURE 4. Relaxed proximities between Hilbert lobes.

5. Application: Relaxed Descriptive Proximities

This application introduces a relaxed descriptive proximity (denoted by δ_{Φ_o}), which is a refinement of the indefinite proximity relation $\delta_{\lim\Phi}$ in the proof of Lemma. 3.1. δ_{Φ_o} serves as means of pinpointing non-equal descriptively close segments of motion waveforms emanating from vibrating dynamical systems. Let A, B be measurable bounded regions in a typical motion waveform of a dynamical system. A practical outcome of $\delta_{\Phi_o}(A, B)$ is the detection those portions of system motion that are stable and have low energy dissipation.

A pair of nonempty sets A, B have relaxed descriptive proximity (denoted by $A\delta_{\Phi_o}B$), provided the characteristics of A and B are close enough.

Definition 5.17 (Relaxed descriptive proximity, [6]). *Let $\varepsilon \in [0, 1]$ and 2^X denote the collection of subsets of a nonempty set X. The relaxed descriptive proximity relation δ_{Φ_o} is defined by*

$$A\delta_{\Phi_o}B \Leftrightarrow |\Phi(A) - \Phi(B)| < \epsilon, A, B \subset X.$$

The focus here is on proximal Hilbert envelope lobes inherent in time-constrained dynamical systems. Encapsulated in a Hilbert envelope, there are comparable regions called lobes.

Definition 5.18 (Hilbert envelope lobe)**.** *A Hilbert envelope lobe (denoted by $H_{env_{lobe}}$) is a tiny bounded planar region attached to single waveform peak point on a waveform envelope, defined by* [2]

$$H_{env} = \sqrt{m(t)^2 + (-m(t))^2}.$$

Remark 5.5. *Hilbert envelope lobe regions provide a measure of motion waveform energy dissipation. The bigger a lobe region, the greater the motion energy. Our interest here is in identifying those motion waveforms having a high number of proximal Hilbert lobe energy regions that have a corresponding low energy dissipation (denoted by $E_{diss}(t)$ at time t).*

Expenditure of energy $E_{m(t)}$ by a dynamical system S_d is measured in terms of the area bounded by the motion $m(t)$ waveform emanating from S_d at time t, i.e.:

Definition 5.19 (Waveform energy)**.** *A measure of dynamical system energy is the area of a finite planar region bounded by system waveform $m(t)$ curve at time t, defined by*

$$\boldsymbol{E_{m(t)} = \int_{t_0}^{t_1} |m(t)|^2 \, dt.}$$

For simplicity, we limit our consideration to elapsed time to clock tick, which is measured by an observed of a local mechanical clock.

Definition 5.20 (Elapsed time measured in clock ticks)**.** *Let $p(t)$ be a characteristic function that models a cog in a rotating wheel in a mechanical clock with a velocity $\boldsymbol{\frac{dp}{dt}}$. A clock tick t_k is defined by*

$$\boldsymbol{t_k = \frac{dp(t)}{dt}|^{t=t_0}}.$$

N.B. 5.4 (Varying clock tick value)**.** *The value of $\boldsymbol{t_k}$ will vary, since it depends on a mechanical system, which is a collection of wearable parts subject to varying friction.*

Axiom 5.3 (Instants clock)**.** *Every system has its own instants clock, which is a cyclic mechanism that is a simple closed curve with an instant hand with one end of the instant hand at the centroid of the cycle and the other end tangent to a point indicating an elapsed time in the motion of a vibrating system. A clock tick occurs at every instant that a system changes its state.*

Axiom 5.4 (Waveform energy, [20])**.** *A measure of dynamical system energy is the area of a finite planar region bounded by system waveform $m(t)$ curve over a bounded temporal interval $[t_0, t_1]$., defined by*

$$\boldsymbol{E_m = \int_{t_0}^{t_1} |m(t)|^2 \, dt.}$$

Lemma 5.6 (Dynamical system energy, [20])**.** *Dynamical system energy is time-constrained and is always limited.*

Proof. Let E_m be the energy of a dynamical system, defined in Axiom 5.4. From Axiom 5.4, system energy always occurs in a bounded temporal interval $[t_0, t_1]$. Hence, E_m is time constrained. From Axiom 5.3, the length of a system waveform is finite, since, from Axiom 5.4, system duration is finite. From Axiom 5.4, system energy is derived from the area of a finite, bounded region. Consequently, system energy is always finite. □

From Def. 5.20 and Lemma 5.6, we obtain:

Theorem 5.4 (Time-constrained dynamical system energy [20]). *Let X be a dynamical system with waveform $m(t)$ at time t. The energy of X varies with every clock tick.*

Definition 5.21 (Energy dissipation [20]). *For a motion waveform $m(t)$ at time t, let loc identify a bounded region (called a Hilbert lobe) at location x in an enveloped waveform and let t be the time of occurrence of the lobe. A measure of motion dissipated energy is the mapping $E_{diss} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which is defined by*

$$E_{diss}(loc, t) = |E(x, t) - E(x', t')| \geq 0$$

for quantized dissipation. There are two different forms of motion energy dissipation to consider, namely,

1^o ***Energy dissipation within one frame:*** *Let x, x' identify lobes in two different locations at time t in an enveloped waveform in a single video frame at elapsed time t. Then*

$$E_{diss}(loc, t) = |E(x, t) - E(x', t)|\,.$$

2^o **Energy dissipation between different frames:** *Let $x = x'$ identify a lobe in the same location at times t, t' in a pair video frames. Then*

$$E_{diss}(loc, t) = |E(x, t) - E(x, t')|\,.$$

The motivation for considering two different forms of $E_{diss}(loc, t)$ stems from Lemma 5.7, which associates the relaxed descriptive proximity of a pair of energy regions with corresponding energy dissipation that is minimal.

Lemma 5.7. *Let $\boldsymbol{E_{m(t)}}$,*

$$\boldsymbol{E_{m(t)}\ \delta_{\Phi_o}\ E_{m(t')} \Leftrightarrow E_{diss}(loc, t) < \varepsilon, \varepsilon \in [0, 1].}$$

That is, $E_{diss}(loc, t)$ is minimal.

Proof. It is known that energy dissipation $E_{diss}(t')$ in a motion curve is minimal, when the difference between $E_{m(t)}, E_{m(t')}, t < t'$ is small [20]. This occurs when $E_{m(t)}\ \delta_{\Phi_o}\ E_{m(t')}$, i.e., $E_{diss}(t) < \varepsilon, \varepsilon \in [0, 1]$. This gives the desired result. □

Theorem 5.5. *Every Hilbert enveloped motion curves $m(t), m(t')$ with $E_{m(t)}\ \delta_{\Phi_o}\ E_{m(t')}$ at times $t \neq t'$ has low energy dissipation.*

Proof. Immediate from Lemma 5.7. □

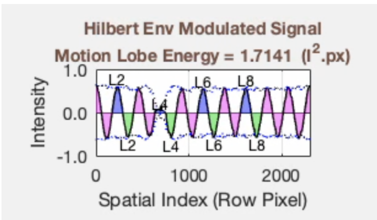

FIGURE 5. Source of Hilbert energy lobes in Table 2.

The sample lobe areas (a.k.a. motion energy regions) in Table 2 (see Fig. 5) lead to Example 5.4.

Lobe	+ve Lobe Areas (+ve lobe)	-ve Lobe Areas (-ve lobe)	Energy Dissipation $\boxed{E_{diss}}$
L2	38.952	36.2096	2.7424
L4	2.522	32.3247	29.8027
L6	33.1404	33.3126	0.1722
L8	33.6902	33.9613	0.2711

TABLE 2. Lobe energy for runner in IR video frame 19.

Example 5.4 (Relaxed proximities within a frame). *Let $\varepsilon = 0.03$. From Table 2, observe*

$$
\begin{aligned}
&1^o \;\; L2^{+ve} \;\; \not\delta_{\Phi_o} \; L2^{-ve},\\
&2^o \;\; L4^{+ve} \;\; \not\delta_{\Phi_o} \; L4^{-ve},\\
&3^o \;\; L6^{+ve} \; \delta_{\Phi_o} \; L6^{-ve},\\
&4^o \;\; L8^{+ve} \; \delta_{\Phi_o} \; L8^{-ve}.
\end{aligned}
$$

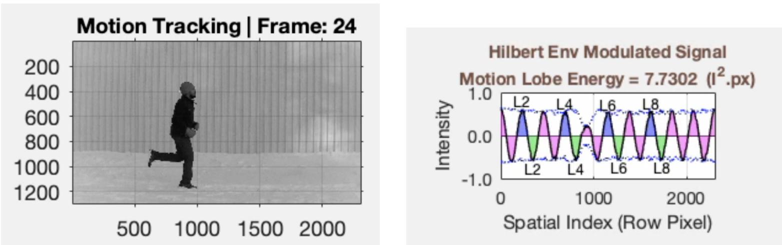

FIGURE 6. Source of Hilbert energy lobes in Table 3.

The sample lobe areas (a.k.a. motion energy regions) in Table 3 (see Fig. 6) and in Table 4 (see Fig. 7) are a source of important relaxed proximities between Hilbert lobes in terms of lobe energy. Lobe energy proximities between frames serve as an indication of the stability of system energy. Relaxed proximities between IR frames are considered in Example 5.5.

A check on relaxed proximities between Hilbert lobes in terms of lobe energy is considered next.

The sample lobe areas (a.k.a. motion energy regions) in Table 3 (see Fig. 6).

Lobe	+ve Lobe Areas (+ve lobe)	-ve Lobe Areas (-ve lobe)	Energy Dissipation $\boxed{E_{diss}}$
L2	38.7384	36.3319	2.4065
L4	34.5799	35.0868	0.5069
L6	32.8082	33.3491	0.5409
L8	33.0718	33.5572	0.4854

TABLE 3. Lobe energy for runner in IR video frame 24.

Example 5.5 (Relaxed proximities between frames). *Let $\varepsilon = 0.2$ for the dynamical system represented by motion $m(t)$ in Fig. 6 and motion $m(t')$ in Fig. 7. Also, let*

$$\textit{(lobe) } L_k^{Tablek}, \textit{(energy) } E_{\pm L_k}^{Tablek}, k \in \{19, 24, 26\}, n \in \{2, 4, 6, 8\}.$$

Observe

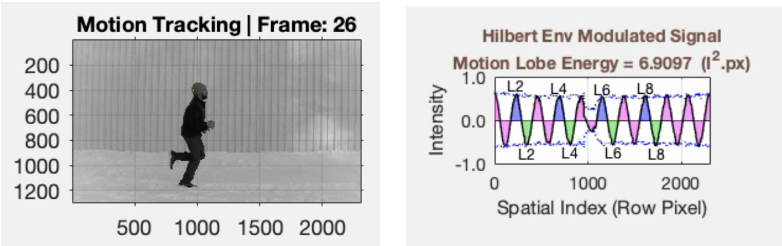

FIGURE 7. Source of Hilbert energy lobes in Table 4.

Lobe	+ve Lobe Areas (+ve lobe)	-ve Lobe Areas (-ve lobe)	Energy Dissipation $\boxed{E_{diss}}$
L2	39.0058	36.4091	2.5967
L4	34.3714	35.023	0.6516
L6	32.8635	33.424	0.5605
L8	33.1931	34.1811	0.988

TABLE 4. Lobe energy for runner in IR video frame 26.

1^o $+veL_4^{Table24}\ \delta_{\Phi_o}\ +veL_4^{Table26}$, *since*

$$\left|E_{+L_4}^{Table24} - E_{+L_4}^{Table26}\right| = |34.5799 - 34.3714| = 0.2085 < \epsilon.$$

2^o $-veL_4^{Table24}\ \delta_{\Phi_o}\ -veL_4^{Table26}$, *since*

$$\left|E_{+L_4}^{Table24} - E_{+L_4}^{Table26}\right| = |35.0868 - 35.023| = 0.0638 < \epsilon.$$

3^o $+veL_6^{Table24}\ \delta_{\Phi_o}\ +veL_6^{Table26}$, *since*

$$\left|E_{+L_6}^{Table24} - E_{+L_6}^{Table26}\right| = |32.8082 - 32.8635| = 0.0553 < \epsilon.$$

4^o $-veL_4^{Table24}\ \delta_{\Phi_o}\ -veL_4^{Table26}$, *since*

$$\left|E_{-L_6}^{Table24} - E_{-L_6}^{Table26}\right| = |33.3491 - 33.424| = 0.0749 < \epsilon.$$

Hence, $E_{m(t)}\delta_{\Phi_o}E_{m(t')}$. *From Theorem 5.5, system S at times t and t' has low energy dissipation.*

ACKNOWLEDGMENTS

The research has been supported by the Natural Sciences & Engineering Research Council of Canada (NSERC) discovery grant 185986 and Instituto Nazionale di Alta Matematica (IN-dAM) Francesco Severi, Gruppo Nazionale per le Strutture Algebriche, Geometriche e Loro Applicazioni grant 9 920160 000362, n.prot U 2016/000036 and Scientific and Technological Research Council of Turkey (TÜBİTAK) Scientific Human Resources Development (BIDEB) under grant no: 2221-1059B211301223.

REFERENCES

[1] J. Addison: *The works of Joseph Addison*, Oxford U. Press, Oxford (1891).
[2] A. Brandt: *Noise and vibration analysis. Signal analysis and experimental procedures*, Wiley, New York (2011).
[3] Á. Császár: *Proximities, screens, merotopies, uniformities. I.*, Acta Math. Hungar., **49** (3-4) (1987), 459–479.
[4] R. L. Devaney: *An introduction chaotic dynamical systems*, The Benjamin/Cummings Publishing, Menlo Park CA (1986).

[5] M. Edelman, E. E. N. Macau and M. A. F. Sanjuan: *Chaotic, fractional, and complex dynamics: New insights and perspectives*, Springer, Cham (2018).
[6] E. Erdag, J. F. Peters and Ö. Deveci: *The Jacobsthal-Padovan-Fibonacci p-sequence and its application in the concise representation of vibrating systems with dual proximal groups*, J. Supercomput., **81** (2025), Article ID:197.
[7] L. Gül, J. F. Peters and Ö. Deveci: *Dual proximal groups concisely representing complex Hosoya triangles*, Adv. Math. Phys., **2026** (2026), Article ID:1742124.
[8] M. S. Haider, J. F. Peters: *Temporal proximities: self-similar temporally close shapes*, Chaos solit. fractals, **151** (2021), Article ID:111237.
[9] W. R. Hamilton: *Essay on the theory of rays*, Trans. Royal Irish Academy, **15** (1) (1828), 69–174.
[10] F. Hausdorff: *Grundzüge der mengenlehre*, Veit and Company, Leipzig (1914).
[11] F. Hausdorff: *Set theory*, AMS Chelsea Publishing, Rhode Island (1957).
[12] D. Hilbert: *Ueber die theorie der algebraischen formen. (German) (On the Theory of Algebraic Forms)*, Math. Ann., **36** (4) (1890), 473–534.
[13] P. J. Hilton: *An introduction to homotopy theory*, Cambridge University Press, Cambridge (1953).
[14] S. A. Naimpally, J. F. Peters: *Topology with applications. Topological spaces via near and far*, World Scientific, Singapore (2013).
[15] E. Ozkan, B. Kuloglu and J. F. Peters: *k-Narayana sequence self-similarity. Flip graph views of k-Narayana self-similarity*, Chaos solit. fractals, **153** (2021), Article ID:111473.
[16] J. F. Peters, S. A. Naimpally: *Applications of near sets*, Notices Amer. Math. Soc., **59** (4) (2012), 536–542.
[17] J. F. Peters: *Computational proximity*, Intelligent Systems Reference Library, **102**, Springer, Cham (2016).
[18] J. F. Peters, T. Vergili: *Descriptive fixed set properties for ribbon complexes*, arXiv:2007.04394, (2020).
[19] J. F. Peters, T. Vergili: *Good coverings of proximal Alexandrov spaces. Path cycles in the extension of the Mitsuishi-Yamaguchi good covering and Jordan curve theorem*, arXiv:2104.05601, (2021).
[20] J. F. Peters, T. U. Liyanage: *Energy dissipation in Hilbert envelopes on motion waveforms detected in vibrating systems: An axiomatic approach*, Commun. Adv. Math. Sci., **7** (4) (2024), 178–186.
[21] H. Roman-Flores: *A note on transitivity in set-valued discrete systems*, Chaos solit. fractals, **17** (2003), 99–104.
[22] A. Schief: *Separation properties for self-similar sets*, Proc. Am. Math. Soc., **122** (1) (1994), 111–115.
[23] O. M. Sharkovsky: *Descriptive theory of deterministic chaos*, Ukrainian Math. J., **74** (12) (2022), 1950–1960.
[24] C. E. Springer: *Tensor and vector analysis. With applications to differential geometry*, The Ronald Press, New York (1962).
[25] E. M. Stein, R. Shakarchi: *Complex analysis. Princeton Lectures in Analysis*, Princeton University Press, Princeton (2003).
[26] Y. Wei, M. J. Wu: *A fractal measure of period-doubling leading to chaos-a computer-assisted quantitative analysis*, J. East China Norm. Univ. Natur. Sci. Ed., **1** (2) (1995), 27–45.
[27] R. Zhao, Y. Xu, Y. Jiao, Z. Li, Z. Chen and Z. Chen: *Nonlinear motion cascade to chaos in a rotor system based on energy transfer*, Nonlinear Dyn, **112** (2024), 10803–10821.

JAMES F. PETERS
UNIVERSITY OF MANITOBA
COMPUTATIONAL INTELLIGENCE LABORATORY
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
WPG, MB, R3T 5V6, CANADA
Email address: `james.peters3@umanitoba.ca`

TANE VERGILI
KARADENIZ TECHNICAL UNIVERSITY
DEPARTMENT OF MATHEMATICS
TRABZON, TÜRKIYE
Email address: `tane.vergili@ktu.edu.tr`

FATIH UÇAN
KARADENIZ TECHNICAL UNIVERSITY
DEPARTMENT OF MATHEMATICS
TRABZON, TÜRKIYE
Email address: `fatihucan@ktu.edu.tr`

Divagar Vakeesan
University of Manitoba
Computational Intelligence Laboratory
Department of Electrical and Computer Engineering
WPG, MB, R3T 5V6, Canada
Email address: divagarv@myumanitoba.ca