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Research Article

The Navier-Stokes problem. Solution of a millennium problem related to the Navier-Stokes equations

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ABSTRACT. The goal of this paper is to present the author's results concerning the Navier-Stokes problem (NSP) in $\mathbb{R}^3$ without boundaries. It is proved that the NSP is contradictory in the following sense:

Assume (for simplicity only) that the exterior force $f = f(x,t) = 0$. If one assumes that the initial data $v(x,0) \not\equiv 0$, $v(x,0)$ is a smooth and rapidly decaying at infinity vector function, $\nabla \cdot v(x,0) = 0$, and the solution to the NSP exists for all $t \geq 0$, then one proves that the solution $v(x,t)$ to the NSP has the property $v(x,0) = 0$.

This paradox (the NSP paradox) shows that the NSP is not a correct description of the fluid mechanics problem and the NSP does not have a solution defined for all times $t > 0$. This solves the millennium problem concerning the Navier-Stokes equations: the solution does not exist for all $t > 0$ if $v(x,0) \not\equiv 0$, $v(x,0)$ is a smooth and rapidly decaying at infinity vector function, $\nabla \cdot v(x,0) = 0$. In the exceptional case, when the data are equal to zero, the solution $v(x,t)$ to the NSP exists for all $t \geq 0$ and is equal to zero, $v(x,t) \equiv 0$.

Keywords: the Navier-Stokes problem, the paradox, the solution to the millennium problem related to the Navier-Stokes equations.

2020 Mathematics Subject Classification: 35Q30, 76D05.

1. INTRODUCTION

The author's theory of the Navier-Stokes problem (NSP) is presented in detail in monograph [13]. Our goal in this relatively short paper to outline the basic steps of this theory in a self-contained way accessible for broad audience. The logical structure of this work is simple: we use an integral equation equivalent to the Navier-Stokes problem, see equation (1.12); we derive from this equation an integral inequality (1.19) and the corresponding integral equation (1.23) with hyper-singular kernels; we give a theory of such equations; we derive the NSP paradox.

The problem we deal with consists of solving the Navier-Stokes problem (NSP) in $\mathbb{R}^3$ without boundaries:

$$v' + (v,\nabla)v = -\nabla p + \nu\Delta v + f, \quad x \in \mathbb{R}^3, \; t \geq 0, \tag{1.1}$$

$$\nabla \cdot v = 0, \tag{1.2}$$

$$v(x,0) = v_0(x), \tag{1.3}$$

see, for example, books [3, 4]. Here $v = v(x,t)$ is the velocity of incompressible viscous fluid, a vector function, $v' := v_t$, $p = p(x,t)$ is the pressure, a scalar function, $f = f(x,t)$ is the exterior force, $\nu = const > 0$ is the viscoucity coefficient, $v_0 = v_0(x) = v(x,0)$ is the initial velocity,

$$\nabla \cdot v_0 = 0. \tag{1.4}$$

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The data v_0 and f are given, the v and p are to be found. The fluid's density ρ is assumed to be constant, namely $\rho = 1$.

We assume, for simplicity only, that $f(x,t) = 0$ and $v(x,0) \not\equiv 0$ is a smooth rapidly decaying function. We prove that the solution to problem (1.1)–(1.4) does not exist for $t \in [0,\infty)$. In the literature, for example, in [2, Theorem 2, p. 472], there is a statement that, for $f = 0$ and $v_0(x)$ sufficiently small, the solution to problem (1.1)-(1.4) exists for all $t > 0$ if $m \le q$, where m is the dimension of the space and the solution belongs to $L^q(\mathbb{R}^3)$. In our case $m = 3$, $q = 2$, so the claim in [2, p. 472] is not applicable. There is a very large literature on fluid dynamics, both mathematical and physical. We mention only books [3] and [4]. Our results, see references in this paper, have no intersections with the published results of other authors. We mention below several results from monograph [13].

From 1822, when C-L. Navier published the Navier-Stokes equations, until 2021, when monograph [12] and paper [14] have appeared, it was not known whether the solution to the Navier-Stokes problem (1.1)-(1.4) exists for all times $t > 0$. This problem was known as the millennium problem related to the NSP. The NSP paradox, see below, yields a negative answer to this millennium problem.

Our presentation in this paper is simple. First, we prove an a priori estimate

$$\sup_{t>0} \|v(x,t)\| \le c, \tag{1.5}$$

where the norm is $L^2(\mathbb{R}^3)$ norm throughout this paper and $c > 0$ are various estimation constants. Then, we prove that problem (1.1)-(1.4) is equivalent to the integral equation

$$v(x,t) = F - \int_0^t ds \int_{\mathbb{R}^3} G(x-y,t-s)(v,\nabla)v dy, \tag{1.6}$$

where $F = F(x,t)$ depends only on the data $f(x,t)$ and $v_0(x)$,

$$F(x,t) := \int_0^t ds \int_{\mathbb{R}^3} G(x-y,t-s) f(y,s) dy + \int_{\mathbb{R}^3} g(x-y,t) v_0(y) dy, \tag{1.7}$$

see [13, Theorem 3.2, p.18], there. We assume (for simplicity only and without loss of generality) that $f = f(x,t) = 0$. Under this assumption one has

$$F(x,t) := \int_{\mathbb{R}^3} g(x-y,t) v_0(y) dy, \tag{1.8}$$

where

$$g(x,t) = \frac{e^{-\frac{|x|^2}{4\nu t}}}{(4\nu\pi t)^{3/2}}, \quad t > 0; \quad g(x,t) = 0, \quad t \le 0; \quad \tilde{g} = e^{-|\xi|^2 \nu t}. \tag{1.9}$$

The $\tilde{g}$ is the Fourier transform of $g(x,t)$ with respect to x-variable, see [13, Formula (3.50), p.17], there. The Fourier transform is defined by formula (1.11) below. The function $G = G_{jm}(x,t)$ is calculated in [13, p. 15]:

$$\begin{aligned} G(x,t) &= (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\xi\cdot x} \Big(\delta_{jm} - \frac{\xi_p \xi_m}{\xi^2}\Big) e^{-\nu|\xi|^2 t} d\xi \\ \tilde{G} &= (2\pi)^{-3} \Big(\delta_{pm} - \frac{\xi_p \xi_m}{\xi^2}\Big) e^{-\nu|\xi|^2 t}, \end{aligned} \tag{1.10}$$

where δ_{jm} is the Kronecker delta. Let us define the Fourier transform $\mathcal{F}v = \tilde{v}$:

$$\mathcal{F}(v) := \tilde{v}(\xi,t) := (2\pi)^{-3} \int_{\mathbb{R}^3} v(x,t) e^{-i\xi\cdot x} dx. \tag{1.11}$$

Taking the Fourier transform of equation (1.6), we get an equivalent equation:

$$\tilde{v} = \tilde{F} - (2\pi)^3 \int_0^t ds \tilde{G}(\xi, t-s)\tilde{v}\bigstar(i\xi\tilde{v})ds, \tag{1.12}$$

where $\bigstar$ denotes the convolution in $\mathbb{R}^3$. By the Cauchy inequality, one has

$$|\tilde{v}\bigstar(i\xi\tilde{v})| \leq \|\tilde{v}\|\||\xi|\tilde{v}\|. \tag{1.13}$$

It is known that

$$\mathcal{F}(v\bigstar w) = (2\pi)^3\mathcal{F}(v)\mathcal{F}(w), \quad (2\pi)^3\|\mathcal{F}(v)\|^2 = \|v\|^2. \tag{1.14}$$

Theorem 1.1. *The following a priori estimate holds:*

$$\sup_{t\geq 0} \|\tilde{v}\| \leq c. \tag{1.15}$$

We assume that $v = v(x,t)$ and other functions in the NSP are real-valued and the following inequality holds for the data:

$$\|v_0\| + \int_0^\infty \|f(x,t)\|dt < c. \tag{1.16}$$

Proof. Multiply equation (1.1) by v, integrate over $\mathbb{R}^3$ and get

$$\frac{1}{2}(\|v\|^2)_{,t} \leq |(f,v)| \leq \|f\|\|v\|.$$

In deriving this inequality, we have used integration by parts:

$$-\int p_{,j}v_j dx = \int p v_{j,j} dx = 0, \quad \int \nu v_{,jj} v_j dx = -\nu \int v_{,j} v_{,j} dx \leq 0,$$

and

$$\int v_m v_{j,m} v_j dx = -\frac{1}{2}\int v_{m,m} v_j v_j dx = 0.$$

It follows that

$$\|v\|_{,t} \leq \|f\|.$$

Consequently,

$$\|v\| \leq \|v_0\| + \int_0^\infty \|f\|dt \leq c.$$

This and our assumption (1.16) imply the estimate $\sup_{t\geq 0}\|v\| \leq c$. By the Parseval equality the estimate $\sup_{t>0}\|\tilde{v}\| \leq c$ follows. Theorem 1.1 is proved. □

Inequalities (1.13) and (1.15) imply

$$|\tilde{v}\bigstar(i\xi\tilde{v})| \leq c\||\xi|\tilde{v}\|. \tag{1.17}$$

This inequality is important, because it allows one to estimate the nonlinear term in equation (1.12) by the linear term on the right side of (1.17). From formula (1.10) it follows that

$$|\tilde{G}(\xi, t-s)| \leq ce^{-\nu(t-s)\xi^2}, \tag{1.18}$$

because

$$|\Big(\delta_{pm} - \frac{\xi_p\xi_m}{\xi^2}\Big)| \leq c.$$

Estimate (1.18) will be used more than once in this paper. Next, we prove that any solution to equation (1.12) satisfies the following integral inequality

$$b(t) \le b_0(t) + c\int_0^t (t-s)^{-\frac{5}{4}} b(s)ds, \quad b(t) = \||\xi|\tilde{v}\| \ge 0, \tag{1.19}$$

where

$$b_0(t) := \||\xi|\tilde{F}(\xi,t)\|, \quad b(t) := \||\xi|\tilde{v}(\xi,t)\| \ge 0. \tag{1.20}$$

Equation (1.19) has hyper-singular kernel. The integral in this equation diverges classically (that is, from the classical analysis point of view). We define such integrals in Section 2. We prove the estimate

$$\sup_{t>0} |b(t)| \le c. \tag{1.21}$$

This, Theorem 1.1 and Parseval's identity imply the apriori estimate

$$\sup_{t>0} \Big(\|v\| + \|\nabla v\|\Big) \le c. \tag{1.22}$$

Together with inequality (1.19), we study the integral equation

$$q(t) = b_0(t) + c\int_0^t (t-s)^{-\frac{5}{4}} q(s)ds. \tag{1.23}$$

We solve this equation analytically and prove the inequality:

$$0 \le b(t) \le q(t). \tag{1.24}$$

One can check that

$$\|e^{-\nu t|\xi|^2}\| = c_1(t-s)^{-\frac{3}{4}}, \quad \||\xi|e^{-\nu t|\xi|^2}\| = c_{11}(t-s)^{-\frac{5}{4}}, \tag{1.25}$$

where

$$c_1 = \Big(\frac{4\pi\int_0^\infty e^{-s^2}s^2ds}{(2\nu)^{3/2}}\Big)^{1/2}, \quad c_{11} = \Big(\frac{4\pi\int_0^\infty e^{-s^2}s^4ds}{(2\nu)^{5/2}}\Big)^{1/2}. \tag{1.26}$$

To derive inequality (1.19), we start with the inequality

$$|\tilde{v}| \le |\tilde{F}| + \int_0^t ds|\tilde{G}(\xi,t-s)||\tilde{v}\bigstar(i\xi\tilde{v}|ds, \tag{1.27}$$

which follows from (1.12). We use inequality (1.15), multiply (1.27) by $|\xi|$, take the norm of both sides and use formulas (1.26) to get (1.19).

If $f=0$, then $F(x,t) = \int_{\mathbb{R}^3} g(x-y)v_0(y)dy$, so $\tilde{F} = (2\pi)^3 e^{-\nu t|\xi|^2}\tilde{v}_0$. Let us define the integral in (1.23). First, let us define $\Phi_\lambda = \Phi_\lambda(t) = \frac{t^{\lambda-1}}{\Gamma(\lambda)}$, where $\Gamma(\lambda)$ is the gamma-function, $\lambda \in \mathbb{C}$ is a complex number, $t = 0$ for $t \le 0$, $t = t$ for $t > 0$. Let

$$Lh := \int_0^\infty e^{-pt}h(t)dt,$$

be the Laplace transform, Re$p > 0$. One can check that

$$L\Phi_\lambda = \frac{1}{p^\lambda}, \quad L(t^{\lambda-1}) = \frac{\Gamma(\lambda)}{p^\lambda}, \quad \lambda \in \mathbb{C}. \tag{1.28}$$

For Re$\lambda > 0$ the integral $L\Phi_\lambda$ is defined classically. For Re$\lambda \le 0$ the $L\Phi_\lambda$ is defined by analytic continuation with respect to λ. Let us rewrite equations (1.19) and (1.23) as

$$b(t) \le b_0(t) - cc_1 \Phi_{-\frac{1}{4}} \star b, \quad b(t) = \| |\xi| \tilde{v} \| \ge 0, \quad c_1 := |\Gamma(-\frac{1}{4})| > 0, \tag{1.29}$$

and

$$q(t) = b_0(t) - cc_1 \Phi_{-\frac{1}{4}} \star q, \tag{1.30}$$

where $\star$ stands for the convolution in $\mathbb{R}^1$. One has

$$L(\Phi_\lambda \star q) = p^{-\lambda} Lq,$$

where formula (1.28) and the known formula $L(h \star u) = Lh \cdot Lu$ were used. The right side of the expression $p^{-\lambda} Lq$ admits analytic continuation with respect to λ to the whole complex plane $\mathbb{C}$ because Lq does not depend on λ and $p^{-\lambda}$ is an entire function of λ for Re$p > 0$. We define $\Phi_\lambda \star q$ as $L^{-1}(\frac{Lq}{p^\lambda})$. Let us find the solution to equation (1.30). Take the Laplace transform of this equation and get $Lq = Lb_0 - cc_1 p^{\frac{1}{4}} Lq$, so

$$Lq = \frac{Lb_0}{1 + cc_1 p^{1/4}}, \tag{1.31}$$

and the solution $q = q(t)$ is:

$$q = L^{-1}\Big(\frac{Lb_0}{1 + cc_1 p^{1/4}}\Big). \tag{1.32}$$

One can easily prove (see [13, p. 28]), that

$$\Phi_\lambda \star \Phi_\mu = \Phi_{\lambda+\mu}, \quad \Phi_0 = \delta(t), \tag{1.33}$$

where $\delta(t)$ is the delta-function. Indeed, using formula (1.28) one has

$$L(\Phi_\lambda \star \Phi_\mu) = \frac{1}{p^\lambda} \frac{1}{p^\mu} = \frac{1}{p^{\lambda+\mu}} = L(\Phi_{\lambda+\mu}). \tag{1.34}$$

By the injectivity of the Laplace transform, the first formula (1.33) follows. If $\lambda + \mu = 0$, then $L(\Phi_0) = 1$ by formula (2.36), so $\Phi_0(t) = \delta(t)$. The second formula (1.33) is proved. We prove in Section 2 that the solution given by formula (1.32) is a bounded funtion such that

$$\sup_{t>0} |q(t)| \le c, \quad q(0) = 0. \tag{1.35}$$

From this we derive that $v_0(x) = 0$. Since we assumed originally that $v_0(x) \not\equiv 0$, we obtain a paradox.

The NSP Paradox. *If one assumes that the initial data $v(x,0) \not\equiv 0$, $\nabla \cdot v(x,0) = 0$ and the solution to the NSP exists for all $t \ge 0$, then one proves that the solution $v(x,t)$ to the NSP has the property $v(x,0) = 0$.*

This paradox (**the NSP paradox**) shows that:

The NSP is not a correct description of the fluid mechanics problem and the NSP does not have a solution defined on all $t \ge 0$. This solves the millennium problem related to the NSP.

2. Proofs

The equivalence of the problem (1.1)–(1.4) to integral equation (1.6) is proved in [13, p. 18]. Formula (1.10) is derived in [13, p. 15].

From equation (1.6) we have derived the equivalent equation (1.12). From this equation and inequality (1.19) we derived inequality (1.27), multipied it by $|\xi|$, took the norm of both sides of the resulting inequality, used formulas (1.26) and got inequalities (1.19) and (1.29).

Theorem 2.2. *Inequality (1.24) holds.*

From Theorem 2.2, Theorem 1.1, estimate (1.35) and Parseval's identity the apriori estimate (1.22) follows immediately. Proof of Theorem 2.2 requires some preparations. Note that if $h \geq w$, then $\|h\| \geq \|w\|$, $Lh \geq Lw$ for $p > 0$ and $\Phi_\lambda \star h \geq c$ for $\lambda > 0$. Define a linear operator

$$Aq := \int_0^t (t-s)^a q(s)ds.$$

Lemma 2.1. *If $a > -1$, then the spectral radius $r(A) = 0$ in the space $X = C(0,T)$ for any $T > 0$. The equation*

$$q = Aq + h$$

is uniquely solvable in X and its solution is

$$q = \sum_{j=0}^{\infty} A^j h.$$

The iterative process

$$q_{n+1} = Aq_n + h$$

converges in X and

$$\lim_{n\to\infty} q_n = q.$$

Proof. Recall that

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

By the mathematical induction one gets

$$\|A^n h\| \leq T^{n(a+1)} \frac{\Gamma^n(a+1)}{\Gamma(1+n(a+1))}\|h\|_X, \quad n \geq 1.$$

Using the known asymptotic

$$\Gamma(z) = e^{(z-0.5)\ln z - z + 0.5\ln(2\pi)}[1 + O(|z|^{-1})], \quad |z| \gg 1, \quad |argz| \leq \pi - \delta, \ \delta > 0,$$

see [5], one derives that $r(A) = 0$. If $r(A)$ =0, then the other two statements of Lemma 2.1 are easy to prove. Lemma 2.1 is proved. □

Apply the operator $\Phi_{1/4}\star$ to equation (1.30), use formulas (1.33) and get

$$q = c_2(\Phi_{1/4} \star b_0 - \Phi_{1/4} q), \quad c_2 := (cc_1)^{-1}. \tag{2.36}$$

Proof of Theorem 2.2. By Lemma 2.1, the unique solution to equation (2.36) can be written as

$$q = \sum_{j=0}^{\infty} (-c_2 \Phi_{1/4}\star)^j c_2 \Phi_{1/4} \star b_0. \tag{2.37}$$

Apply the operator $\Phi_{1/4}\star$ to equation (1.29), use formulas (1.33) and get

$$b \leq c_2(\Phi_{1/4} \star b_0 - \Phi_{1/4} b), \quad c_2 := (cc_1)^{-1}. \tag{2.38}$$

Since $\lambda = 1/4 > 0$, the sign in (1.29) is preserved by the application of the operator $\Phi_{1/4}\star$ to equation (1.29). Applying the iterations to inequality (2.38) one gets

$$b \le \sum_{j=0}^{\infty} (-c_2\Phi_{1/4}\star)^j c_2\Phi_{1/4} \star b_0. \tag{2.39}$$

Comparing equations (2.37) and (2.39), one obtains inequality (1.24). Theorem 2.2 is proved. □

Let us derive from (1.32) the following result.

Theorem 2.3. *One has $b(0) = 0$.*

Proof. It is sufficient to prove that $q(0) = 0$. Indeed, $b(t) \ge 0$ and by inequality (1.24) it follows that $b(0) = 0$ if $q(0) = 0$. To prove that $q(0) = 0$ we use formula (1.32) and the following result.

Theorem 2.4. *Let $F(p)$, $p = \sigma + is$, $s \in \mathbb{R}$, be analytic in the region $\sigma > 0$, $\lim_{\sigma\to 0} F(\sigma + is) = F(is) \in L^1_{loc}(\mathbb{R})$ exists for almost all s, and*

$$|F(p)| \le c(1 + |p|)^{-b}, \ \ |p| \gg 1, \, b > 1.$$

Then $F(p) = L(h)$, $h = h(t) = 0$ for $t < 0$, $\sup_{t\ge 0} |h(t)| \le c$, $h \in C(\mathbb{R})$, and $h(0) = 0$.

Proof. From our assumption on $F(p)$ it follows that the function $h(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ist}F(is)ds$ is well defined and is a continuous uniformly bounded function. By the analyticity of F one has $\int_{C_n} F(p)dp = 0$, where C_n is a closed contour, a union of $K_n := -in, in$ and the semi-circle $\gamma_n = ne^{i\phi}$, $-\pi/2 \le \phi \le \pi/2$. Since $b > 1$, it follows that $\lim_{n\to\infty}\int_{\gamma_n} F(p)dp = 0$. Therefore $\lim_{n\to\infty}\int_{K_n} F(p)dp = \int_{-\infty}^{\infty} F(is)ds = 0$. So, $h(0) = 0$.

Let us calculate

$$Lh = \int_0^{\infty} e^{-pt}h(t)dt = \int_0^{\infty} e^{-pt}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ist}F(is)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{is - p}F(is)ds.$$

Analyticity of F implies $F(p) = \frac{1}{2\pi i}\int_{C_n}\frac{F(p')dp'}{p'-p}$. Let $n \to \infty$. The result is

$$F(p) = \frac{1}{2\pi}\int_K \frac{F(is)ds}{is - p}.$$

Therefore, $Lh = F(p)$. We have used the fact $lim_{n\to\infty}\int_{\gamma_n}\frac{F(p')dp'}{p'-p} = 0$. Theorem 2.4 is proved. □

Let us finish the proof of Theorem 2.3. The right side of formula (1.32) is analytic for $\sigma > 0$, it is $O(|p|^{-b})$ for $|p| \gg 1$ and $b = 5/4$. Indeed, for smooth and rapidly decaying $v_0(x)$ one has $Lb_0 = O(\frac{1}{1+|p|})$, the function $\frac{1}{1+cp^{1/4}}$ is analytic in the half-plane $\sigma > 0$ and $\frac{1}{1+cp^{1/4}} = O(|p|^{-1/4})$ for $|p| \gg 1$. By Theorem 2.4, the conclusion $b(0) = 0$ of Theorem 2.3 follows. □

Theorem 2.5. *From $b(0) = 0$, it follows that $v_0(x) = 0$.*

Proof. If $b(0) = 0$, then $\||\xi|\tilde{v}\| = 0$. Therefore, the $\tilde{v} = 0$ for $|\xi| > 0$. By estimate (1.5) and the Parseval identity it follows that $\|v_0(x) = 0\|$, so $v_0(x) = 0$. Theorem 2.5 is proved. □

As was explained at the end of Section 1, Theorem 2.5 implies the NSP paradox and the resulting consequences: It shows the contradictory nature of the Navier-Stokes equations. It also proves that the solution to the NSP does not exists on the semi-axis $t \in \mathbb{R}_+$.

Let us prove the following a priori estimate for the solution v to problem (1.1)–(1.4).

Theorem 2.6. *The inequality*

$$\sup_{t\geq 0}(\|v(x,t)\| + \|\nabla v(x,t)\|) \leq c \tag{2.40}$$

holds, where v solves the NSP problem and c depends on the data.

Proof. We have proved in Theorem 1.1 inequality (1.5). By the Parseval identity, one has $b(t) = \|\nabla v(x,t)\|$. We have proved earlier that $\sup_{t\geq 0} b(t) \leq c$. Therefore, Theorem 2.6 is proved. □

Remark 2.1. *If the data are equal to zero, that is, $f(x,t) = 0$ and $v_0(x) = 0$, then the solution $v(x,t)$ to the NSP exists for all $t \geq 0$ and $v(x,t) \equiv 0$.*

This follows from the a priori estimate (2.40). Indeed, if the data are zeros, then $c = 0$ and formula (2.40) implies that $v(x,t) = 0$.

3. Conclusions

The basic result of this paper is a proof of the NSP paradox and its consequences: the contradictory nature of the Navier-Stokes equations. The NSP paradox proves that the millennium problem related to the Navier-Stokes equations in $\mathbb{R}^3$ without boundaries does not have a solution. If one assumes that the data are smooth, rapidly decaying and not identically equal to zero, then one proves that these data are equal to zero.

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