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Survey Article

A survey on the distance functions

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ABSTRACT. This survey aims to express different distance functions and consider their relationships, if any. Indeed, in the literature, numerous different and interesting distance functions with distinct properties have been introduced. Presenting and discussing distance functions with their original motivations can open a new window for researchers working in various disciplines. The distance functions mentioned here and the corresponding abstract spaces may offer alternative solutions for the existing problems.

Keywords: Abstract metric space, fixed point, metric spaces, perturbed metric spaces, self-mappings.

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1. INTRODUCTION AND PRELIMINARIES

Throughout the history of mathematics, the idea of distance is among the first concepts that have been recognized. In contemporary mathematics, the idea of distance is generalized through the concept of a metric [37, 38]. It would not be an overstatement to claim that the idea of metric serves as the essential driving force of contemporary mathematics. In reality, it has been applied not just in mathematics, but also to address issues in other sciences and fields that can be described mathematically; for instance, the idea of metrics has been widely utilized in imaging challenges and computer science. Consequently, the idea of metrics has captured the interest of numerous researchers. This idea has been examined through various methods and viewpoints, with efforts made to broaden, enhance, and refine it.

In academic publications, various types of metrics can be discovered. It is not an easy task to mention all these abstract structures since there are several mixed version of the following abstract distance notions: 2-metric [40], D-metric [29], G-metric [66], S-metric [84], A-metric [1], a quasi-metric [64, 67, 72, 73, 74, 75, 83], ultra-metric, symmetric metric, bipolar metric [59], modular metric [24, 80], fuzzy metric, b-metric [13, 15, 27, 71, 77], strong b-metric [63], partial metric [42, 65], cone metric (Banach-valued metric) [44], b-cone metric [43], TVS-valued metric [35], complex-valued metric [8, 12], C^*-algebra valued metric space [10], quaternion-valued metric [5], generalized metric [62], Branciari distance function [20], supra-metric [17, 18], super metric [53], and interpolative metric [63], along with numerous others, as seen in [16, 55, 56]. In these mentioned structures, it is possible to put some of them in the same class. More precisely, we can consider three-point structures and put the following abstract spaces together: 2-metric [40], D-metric [29], G-metric [66], S-metric [84], and even A-metric [1]. In the classical distance notion, for any two points, the definition of the distance function assigns a non-negative real number. See also, e.g. [7, 9, 21, 22, 23, 25, 51, 54, 57, 60, 68, 69,

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76, 78, 79, 81, 88, 89]. On the other hand, in three-point structures, the considered distance function needs three points to assign a non-negative real number. Another class of the distance function can be considered as extended range metrics: cone metric (Banach-valued metric) [44], b-cone metric [52], TVS-valued metric [35], complex-valued metric [8, 12], C^*-algebra valued metric space [10], quaternion-valued metric [5] are the straightforward examples of this class. In the classical metric structure, for any two points, the metric function assigns a non-negative real number. In other words, the range of the metric function is the set of non-negative reals. For an extended range of metric functions, the range, non-negative real numbers, has been changed by a positive cone of the given Banach space or TVS or quaternion space or C^*-algebra space. Necessarily, in this class, considered new distance function assigns not a nonnegative real number but, instead, a Banach value, complex value, TVS-value, quaternion value, or C^*-algebra value, and so on. On the other hand, there are some other abstract structures that can not be comparable or embedded in another structure. For instance, Branciari distance is not comparable with the standard metric or any other metric mentioned above. There are also some equivalent structure with different disguises. For example, a multiplicative metric space can be considered as an equivalent structure to the standard metric space.

This short survey aims to fill one of the literature's gaps by compiling all significant abstract distance functions as much as possible. In the upcoming sections, we shall consider significant abstract structures with some examples. We start this section with one of the most interesting structures, so-called, ultra metric spaces.

2. Quasi-Metric Spaces

The definition of a quasi-metric is given as follows:

Definition 2.1. *Let X be a non-empty and let $d : X \times X \to [0, \infty)$ be a function which satisfies:*
(d1) $d(x, y) = 0$ if and only if $x = y$,
(d2) $d(x, y) \leq d(x, z) + d(z, y)$.
Then d is called a quasi-metric and the pair (X, d) is called a quasi-metric space.

Remark 2.1. *Any metric space is a quasi-metric space, but the converse is not true, in general.*

Now, we give convergence, completeness, and continuity on quasi-metric spaces.

Definition 2.2. *Let (X, d) be a quasi-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if*

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0. \tag{2.1}$$

Example 2.1 ([6]). *Let X be a subset of $\mathbb{R}$ containing $[0, 1]$ and define, for all $x, y \in X$,*

$$q(x,y) = \begin{cases} x - y, & \text{if } x \geq y, \\ 1, & \text{otherwise.} \end{cases}$$

Then (X, q) is a quasi-metric space. Notice that $\{q(1/n, 0)\} \to 0$ but $\{q(0, 1/n)\} \to 1$. Therefore, $\{1/n\}$ converges to 0 on the right but does not converge from the left. We also point out that this quasi-metric verifies the following property: if a sequence $\{x_n\}$ has a right limit x, then it is unique.

Remark 2.2. *A quasi-metric space is Hausdorff, that is, we have the uniqueness of limit of a convergent sequence.*

Definition 2.3. *Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is left-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n \geq m > N$.*

Definition 2.4. *Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is right-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n > N$.*

Definition 2.5. *Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > N$.*

Remark 2.3. *A sequence $\{x_n\}$ in a quasi-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.*

Definition 2.6. *Let (X, d) be a quasi-metric space. We say that*
(1) (X, d) is left-complete if and only if each left-Cauchy sequence in X is convergent.
(2) (X, d) is right-complete if and only if each right-Cauchy sequence in X is convergent.
(3) (X, d) is complete if and only if each Cauchy sequence in X is convergent.

Definition 2.7. *Let (X, d) be a quasi-metric space. The map $f : X \to X$ is continuous if for each sequence $\{x_n\}$ in X converging to $x \in X$, the sequence $\{fx_n\}$ converges to fx, that is,*

$$\lim_{n\to\infty} d(fx_n, fx) = \lim_{n\to\infty} d(fx, fx_n) = 0. \tag{2.2}$$

3. Ultrametric Spaces

In this section, we shall recall one of the most interesting abstract space structures, so-called, ultrametric spaces.

Definition 3.8 (Ultrametric)**.** *Let X be a non-empty set. A distance function*

$$d : X \times X \to [0, \infty)$$

is called an ultrametric on X if for all $x, y, z \in X$, the following conditions hold:
(U_1) $d(x, y) \geq 0$ (Non-negativity),
(U_2) $d(x, y) = 0 \iff x = y$ (Identity of indiscernible),
(U_3) $d(x, y) = d(y, x)$ (Symmetry),
(U_4) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ (Ultrametric inequality, or strong triangle inequality).

Example 3.2 (p-adic metric on $\mathbb{Q}_p$)**.** *Let p be a prime number. The p-adic absolute value $|\cdot|_p$ on the field of rational numbers $\mathbb{Q}$ is defined as follows: for any non-zero rational numbers $x = \frac{a}{b}p^n$, where a, b are integers not divisible by p, define*

$$|x|_p = p^{-n}, \quad |0|_p = 0.$$

The corresponding metric is

$$d_p(x, y) = |x - y|_p.$$

This metric satisfies the ultrametric inequality

$$d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z)\},$$

and hence defines an ultrametric on $\mathbb{Q}$. Its completion gives the field $\mathbb{Q}_p$ of p-adic numbers.

Example 3.3 (Space of infinite sequences over a finite alphabet)**.** *Let $A = \{a_1, a_2, \ldots, a_k\}$ be a finite alphabet. Let $X = A^{\mathbb{N}}$ be the set of infinite sequences over A. Define a metric on X by:*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-n} & \text{if } x \neq y \text{ and } n \text{ is the first index where } x_n \neq y_n. \end{cases}$$

This metric satisfies the ultrametric inequality, because if x and y agree up to position n, y and z agree up to position m, then x and z must agree at least up to $\min(n, m)$, so

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

Thus, (X, d) is an ultrametric space.

Example 3.4 (Hierarchical clustering/dendrograms). *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of objects (e.g., data points). Suppose we are given a hierarchical clustering of X, represented by a dendrogram (tree structure). Define a distance between two points as the height at which the two points merge in the dendrogram*

$$d(x_i, x_j) = \text{height at which clusters containing } x_i \text{ and } x_j \text{ merge.}$$

This distance satisfies the ultrametric inequality and gives rise to an ultrametric on X. Such metrics are used in phylogenetics, taxonomy, and cluster analysis.

Remark 3.4. *In general, every ultrametric space is a standard metric space, but not every standard metric space is an ultrametric space.*

(1) *Every ultrametric space is a standard metric space:*

If (X, d) is an ultrametric space, then (X, d) is a metric space.

(2) *Not every standard metric space is an ultrametric space:*

There exist standard metric spaces that are not ultrametric spaces.

Example 3.5. *Consider the standard Euclidean metric on $\mathbb{R}^2$:*

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

This metric does not satisfy the ultrametric condition. For example, with points $x = (0, 0)$, $y = (1, 0)$, and $z = (0, 1)$

$$d(x, z) = \sqrt{(0 - 0)^2 + (0 - 1)^2} = 1, \quad d(x, y) = 1, \quad d(y, z) = \sqrt{2}.$$

Here, $d(x, z) \not\leq \max(d(x, y), d(y, z))$, since $1 \not\leq \max(1, \sqrt{2})$.

Theorem 3.1 (All triangles are isosceles or equilateral). *In an ultrametric space, every triangle is either isosceles or equilateral.*

Proof. Let $x, y, z \in X$. Without loss of generality, assume $d(x, y) \leq d(x, z)$. By the ultrametric inequality

$$d(y, z) \leq \max\{d(y, x), d(x, z)\} = \max\{d(x, y), d(x, z)\} = d(x, z).$$

Also applying the inequality in reverse:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

Since $d(x, y) \leq d(x, z)$, this implies

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} \leq \max\{d(x, z), d(y, z)\} = d(x, z),$$

so equality must hold throughout. Thus, at least two sides have equal length: either $d(x, y) = d(x, z)$ or $d(y, z) = d(x, z)$. Therefore, all triangles are isosceles or equilateral. □

Theorem 3.2 (Every interior point is a center). *If $y \in B(x, r)$, then $B(x, r) = B(y, r)$, where $B(x, r) = \{z \in X : d(x, z) < r\}$.*

Proof. Let $y \in B(x,r)$, so $d(x,y) < r$. We show that $B(x,r) = B(y,r)$. First, take any $z \in B(x,r)$, i.e., $d(x,z) < r$. Then

$$d(y,z) \leq \max\{d(y,x), d(x,z)\} < \max\{r,r\} = r,$$

so $z \in B(y,r)$. Hence, $B(x,r) \subseteq B(y,r)$. Now take any $z \in B(y,r)$, so $d(y,z) < r$. Then

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} < \max\{r,r\} = r,$$

so $z \in B(x,r)$. Thus, $B(y,r) \subseteq B(x,r)$, and therefore $B(x,r) = B(y,r)$. This shows that every point inside a ball is also a center of the ball. □

Theorem 3.3 (Balls are clopen)**.** *In an ultrametric space, all balls $B(x,r) = \{y \in X : d(x,y) < r\}$ are both open and closed (clopen).*

Proof. We already know from general metric space theory that open balls are open sets. To show that they are also closed, we will prove that their complement is open.

Suppose $y \notin B(x,r)$, so $d(x,y) \geq r$. Consider the ball $B(y,r')$ with radius $r' = d(x,y)$. Take any $z \in B(y,r')$, so $d(y,z) < r' = d(x,y)$. Then by the ultrametric inequality:

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} = d(x,y),$$

and since $d(y,z) < d(x,y)$, we get

$$d(x,z) \leq d(x,y).$$

But also

$$d(x,y) \leq \max\{d(x,z), d(z,y)\} = \max\{d(x,z), d(y,z)\} = d(x,z),$$

since $d(y,z) < d(x,y) \leq d(x,z)$. So $d(x,z) = d(x,y) \geq r$, which means $z \notin B(x,r)$. Hence, $B(y,r') \cap B(x,r) = \emptyset$, and so the complement of $B(x,r)$ is open. Therefore, $B(x,r)$ is closed. Thus, all open balls are clopen. □

4. Partial Metric Spaces

Definition 4.9 (Partial metric space [65])**.** *Let X be a nonempty set. A function $p : X \times X \to [0,\infty)$ is called a partial metric if it satisfies the following conditions for all $x, y, z \in X$:*

- *(P1)* $x = y \iff p(x,x) = p(x,y) = p(y,y)$
- *(P2)* $p(x,x) \leq p(x,y)$
- *(P3)* $p(x,y) = p(y,x)$
- *(P4)* $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$

The pair (X,p) is called a partial metric space.

Example 4.6. *If $X := \{[a,b] |\ a,b \in \mathbb{R},\ a \leq b\}$ then $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,b\}$ defines a partial metric p on X.*

Example 4.7. *Let $X := \mathbb{R}^{\mathbb{N}_0} \cup \bigcup_{n\geq 1} \mathbb{R}^{\{0,1,\ldots,n-1\}}$, where $\mathbb{N}_0$ is the set of nonnegative integers. By $L(x)$ denote the set $\{0,1,\ldots,n\}$ if $x \in \mathbb{R}^{\{0,1,\ldots,n-1\}}$ for some $n \in \mathbb{N}$, and the set $\mathbb{N}_0$ if $x \in \mathbb{R}^{\mathbb{N}_0}$. Then a partial metric is defined on X by*

$$p(x,y) = \inf\left\{\frac{1}{2^i} | i \in L(x) \cap L(y) \text{ and } \forall j \in \mathbb{N}_0\ (j < i \implies x(j) = y(j))\right\}.$$

Let (X, p) be a partial metric space. Then, the functions $d_p, d_m : X \times X \to [0, \infty)$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

and

$$d_m(x, y) = \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\}$$

are (usual) metrics on X. It is easy to check that d_p and d_m are equivalent. Note that each partial metric p on X generates a T_0-topology τ_p with a base of the family of open $p-$balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$.

Definition 4.10 ([2, 45]). *Let (X, p) be a partial metric space.*

(1) A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and finite).

(3) (X, p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$.

(4) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

Referring to [70], we say that a sequence $\{x_n\}$ in (X, p) is called the sequence *0-Cauchy* if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Also, we say that (X, p) is *0-complete* if every 0-Cauchy sequence in X converges, with respect to the partial metric p, to a point $x \in X$ such that $p(x, x) = 0$. Notice that if (X, p) is complete, then it is 0-complete, but the converse does not hold. Moreover, every 0-Cauchy sequence in (X, p) is Cauchy in (X, d_p).

Example 4.8 ([65, 70]). *(1) Let $X = [0, +\infty)$ and define $p(x, y) = \max\{x, y\}$, for all $x, y \in X$. Then (X, p) is a complete partial metric space. It is clear that p is not a (usual) metric.*

(2) Let $X = [0, +\infty) \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. Define $p(x, y) = \max\{x, y\}$, for all $x, y \in X$. Then (X, p) is a 0-complete partial metric space which is not complete.

Proposition 4.1 ([2, 45]). *Let (X, p) be a partial metric space.*

(1) A sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if $\{x_n\}$ is a Cauchy sequence in (X, d_p).

(2) (X, p) is complete if and only if (X, d_p) complete. Moreover,

$$\lim_{n\to\infty} d_p(x_n, x) = 0 \Leftrightarrow \lim_{n\to\infty} p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_m, x_n).$$

The following lemmas play a crucial role in the proof of the theorems.

Lemma 4.1 ([2, 50]). *Assume $x_n \to z$ as $n \to \infty$ in a partial metric space (X, p) such that $p(z, z) = 0$. Then, $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

Lemma 4.2 ([2, 50]). *Let (X, p) be a complete partial metric space. Then*

(1) If $p(x, y) = 0$ then $x = y$.

(2) If $x \neq y$, then $p(x, y) > 0$.

Lemma 4.3 ([2, 50]). *Let (X, p) be a partial metric spaces. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ for all $x_n, y_n, x, y \in X$ then $p(x_n, y_n) \to p(x, y)$ as $n \to \infty$.*

5. Interpolative Metric Spaces

In what follows, we shall state the definition of (α, c)-interpolative metric.

Definition 5.11. *Let X be a nonempty set. We say that $d : X \times X \to [0, +\infty)$ is (α, c)-interpolative metric if*

$(m1)$ $d(x,y)=0$, *if and only if*, $x=y$ *for all* $x,y\in X$,
$(m2)$ $d(x,y)=d(y,x)$, *for all* $x,y\in X$,
$(m3)$ *there exist an* $\alpha\in(0,1)$ *and* $c\geq 0$ *such that*

$$d(x,y)\leq d(x,z)+d(z,y)+c\left[(d(x,z))^{\alpha}\,(d(z,y))^{1-\alpha}\right]$$

for all $(x,y,z)\in X\times X\times X$.

Then, we call (X,d) *an* (α,c)*-interpolative metric space.*

Note that each metric space can be considered an (α,c)-interpolative metric space with $c=0$. In the following example, we shall clarify that the converse is invalid.

Example 5.9. *Let* (X,ρ) *be a standard metric space. Define a function* $d:X\times X\to[0,\infty)$ *as follows*

$$d(x,y):=\rho(x,y)(\rho(x,y)+A),$$

where $A>0$. *Since* ρ *is a metric on* X, *the conditions* $(m1)$ *and* $(m2)$ *are straightforward. For* $(m3)$, *it is enough to consider* $c\geq 2$ *for any* $\alpha\in(0,1)$. *Thus,* (X,d) *is* $(\frac{1}{2},2)$*-interpolative metric space. Indeed, we have*

$$\begin{aligned}
d(x,y)&=\rho(x,y)(\rho(x,y)+A)\\
&\leq(\rho(x,z)+\rho(z,y))(\rho(x,z)+\rho(z,y)+A)\\
&\leq(\rho(x,z)+\rho(z,y))(\rho(x,z)+\rho(z,y)+A)\\
&\leq[\rho(x,z)(\rho(x,z)+A)+\rho(x,z)\rho(z,y)]+[\rho(z,y)(\rho(z,y)+A)+\rho(z,y)\rho(x,z)]\\
&\leq[\rho(x,z)(\rho(x,z)+A)]+[\rho(z,y)(\rho(z,y)+A)]+2\rho(x,z)\rho(z,y)\\
&\leq d(x,z)+d(z,y)+2\,(\rho(x,z))^{\frac{1}{2}}\,(\rho(x,z))^{\frac{1}{2}}\,(\rho(z,y))^{\frac{1}{2}}\,(\rho(z,y))^{\frac{1}{2}}\\
&\leq d(x,z)+d(z,y)+2\,(\rho(x,z))^{\frac{1}{2}}\,[\rho(x,z)+A]^{\frac{1}{2}}\,(\rho(z,y))^{\frac{1}{2}}\,[\rho(z,y)+A]^{\frac{1}{2}}\\
&\leq d(x,z)+d(z,y)+2(d(x,z))^{\frac{1}{2}}(d(z,y))^{\frac{1}{2}}.
\end{aligned}$$

The function $d(x,y)$ *does not form a metric. Note also that the above estimation for the pair* $(\frac{1}{2},2)$ *is very rough and can be improved in several ways.*

Example 5.10. *Let* X *be a non-empty set and define a function* $d:X\times X\to[0,\infty)$ *as follow*

$$d(x,y):=|x-y|^{3},\ \text{for all } x,y\in X.$$

Regarding the notion of the absolute value function, we conclude that the conditions $(m1)$ *and* $(m2)$ *are satisfied trivially. For* $(m3)$, *it is enough to consider* $c\geq 6$ *for any* $\alpha=\frac{1}{3}\in(0,1)$. *Then,* (X,d) *is* $(\frac{1}{3},6)$*-interpolative metric space. More precisely, by a simple calculation and manipulation, we derive that*

$$\begin{aligned}
d(x,y)=|x-y|^{3}&=|x-z+z-y|^{3}\\
&=|x-z|^{3}+|z-y|^{3}+3|x-z|^{2}|z-y|+3|x-z||z-y|^{2}\\
&\leq d(x,z)+d(z,y)+3\left[(d(x,z))^{\frac{2}{3}}\,(d(z,y))^{\frac{1}{3}}\right]+3\left[(d(x,z))^{\frac{1}{3}}\,(d(z,y))^{\frac{2}{3}}\right]
\end{aligned}\tag{5.3}$$

without loss of generality, we assume $d(x,z)\geq d(z,y)$,

$$d(x,y)\leq d(x,z)+d(z,y)+6\left[(d(x,z))^{\frac{2}{3}}\,(d(z,y))^{\frac{1}{3}}\right].$$

Consequently, we conclude that $(m3)$ *is fulfilled. Hence,* (X,d) *is* $(\frac{1}{3},6)$*-interpolative metric space.*

Lemma 5.4 ([58]). *For each* $p,q\in[0,\infty)$ *and each* $\alpha\in(0,1)$, $p^{\alpha}q^{1-\alpha}\leq p+q$.

Proof. If $p = 0$ or $q = 0$, then the inequality follows trivially. Thus, we assume that $p > 0$ and $q > 0$. In this case, we find

$$p^\alpha q^{1-\alpha} \leq (\max\{p,q\})^\alpha (\max\{p,q\})^{1-\alpha} = \max\{p,q\} \leq p + q.$$

□

Employing the technical lemma above, we can state the following theorem:

Theorem 5.4. *Let (X, d) an (α, c)-interpolative metric space. Then, it lies between the standard metric and the b-metric.*

Proof. Trivially, we have the following,

$(m1)$ $d(x, y) = 0$, if and only if, $x = y$ for all $x, y \in X$,
$(m2)$ $d(x, y) = d(y, x)$, for all $x, y \in X$.

To prove the assertion of the theorem above, we shall use Lemma 5.4. Indeed, for any $\alpha \in (0, 1)$,

$$p^\alpha q^{1-\alpha} \leq (p + q),$$

is equivalent to

$$[d(x,z)]^\alpha [d(z,y)]^{1-\alpha} \leq d(x,z) + d(z,y),$$

by letting $p = d(x, z)$ and $q = d(z, y)$, for all $x, y, z \in X$. Since d is an (α, c)-interpolative metric space, for any $x, y, z \in X$ there is $c \in X$ there exist an $\alpha \in (0, 1)$ and $c \geq 0$ such that

$$\begin{aligned} d(x,y) &\leq d(x,z) + d(z,y) + c\left[(d(x,z))^\alpha (d(z,y))^{1-\alpha}\right] \\ &\leq d(x,z) + d(z,y) + c[d(x,z) + d(z,y)] \\ &\leq (c+1)[d(x,z) + d(z,y)] \\ &\leq s[d(x,z) + d(z,y)] \end{aligned}$$

where $s = c + 1$. Indeed, it is clear to see that

$$d(x,z) + d(z,y) \leq d(x,z) + d(z,y) + c\left[(d(x,z))^\alpha (d(z,y))^{1-\alpha}\right] \leq s[d(x,z) + d(z,y)].$$

□

6. b-METRIC

Definition 6.12 (b-metric)**.** *Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ be a function satisfying:*

(b_1) $d(x, y) = 0$ if and only if $x = y$,
(b_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(b_3) There exists a constant $s \geq 1$ such that

$$d(x,y) \leq s[d(x,z) + d(z,y)], \quad \textit{for all } x, y, z \in X.$$

Then (X, d) is called a b-metric space with coefficient s.

Example 6.11. *Let $X = \mathbb{R}$ and define a function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by:*

$$d(x,y) = |x - y|^2.$$

Then (X, d) is a b-metric space with coefficient $s = 2$. This satisfies the inequality:

$$d(x,y) \leq 2[d(x,z) + d(z,y)] \quad \textit{for all } x, y, z \in X.$$

Example 6.12. *Let $X = \{0, 1, 2\}$ and define $d : X \times X \to \mathbb{R}_+$ as follows*

$$d(0,1) = d(1,0) = d(0,2) = d(2,0) = 1,$$
$$d(1,2) = d(2,1) = \alpha \geq 2,$$
$$d(0,0) = d(1,1) = d(2,2) = 0.$$

Then (X, d) is a b-metric space with coefficient $s = \frac{\alpha}{2}$.

Example 6.13. *Let $X = \mathbb{R}^m$ be the space of all ordered m-tuples of real numbers. For any two points $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$, define*

$$d_p(x, y) = \left(\sum_{k=1}^{m} |y_k - x_k|^p \right)^{1/p}, \quad p \geq 1.$$

Then (X, d_p) is a b-metric space with constant $s = 2^{1/p}$.

Example 6.14. *Let $X = \ell^\infty$, the space of all bounded sequences $x = \{x_k\}_{k=1}^\infty$, with the metric*

$$d_\infty(x, y) = \sup_{k \in \mathbb{N}} |x_k - y_k|.$$

Then (X, d_∞) is a complete b-metric space.

Example 6.15. *Let $X = L^2([-1, 1])$, the space of square-integrable functions on $[-1, 1]$, with the metric*

$$d(f, g) = \left(\int_{-1}^{1} (f(t) - g(t))^2 dt \right)^{1/2}.$$

Then (X, d) is not a standard metric space but is a b-metric space with $s = \sqrt{2}$.

Example 6.16. *Let $X = C([0, 1], \mathbb{R})$, the space of continuous real-valued functions on $[0, 1]$. Define*

$$d(f, g) = \left(\int_{0}^{1} |f(t) - g(t)|^2 dt \right)^{1/2}.$$

Then (X, d) is a b-metric space with $s = \sqrt{2}$.

Example 6.17. *Let $X = \{a, b, c\}$ and $d : X \times X \to [0, \infty)$ be defined by*

$$d(a, b) = 1, \quad d(a, c) = \frac{1}{2} \quad \text{and} \quad d(b, c) = 2,$$

with $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$. Notice that d is not a metric since $d(b, c) > d(b, a) + d(a, c)$. However, it is easy to see that d is a b-metric space with $s \geq \frac{4}{3}$.

7. Strong b-Metric Space

Kirk and Shahzad [63] defined the strong b-metric space as follows:

Definition 7.13 (Strong b-metric). *Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ be a function satisfying:*

- *(b_1) $d(x, y) = 0$ if and only if $x = y$,*
- *(b_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,*
- *(b_3) There exists a constant $K \geq 1$ such that*

$$d(x, y) \leq d(x, z) + K d(z, y), \quad \text{for all } x, y, z \in X.$$

Then $(X, d,)$ is called a strong b-metric space with coefficient K.

Example 7.18. *Let* $X = \{x_1, x_2, x_3\}$, *the function* $d : X \times X \to [0, \infty)$ *be defined by* $d(x_i, x_i) = 0$, for each $i = 1, 2, 3$.

$$d(x_1, x_2) = d(x_2, x_1) = 2,$$
$$d(x_2, x_3) = d(x_3, x_2) = 1,$$
$$d(x_1, x_3) = d(x_3, x_1) = 6.$$

It is clear that (X, d) *is a strong b-metric with* $K = 4$.

8. Extended b-Metric Space

Definition 8.14 (Extended b-metric space). *A function* $d_\theta : X \times X \to [0, \infty)$ *is called an extended b-metric if there exists a function* $\theta : X \times X \to [1, \infty)$ *such that:*

(1) $d_\theta(x, y) = 0 \iff x = y$,
(2) $d_\theta(x, y) = d_\theta(y, x)$ *for all* $x, y \in X$,
(3) $d_\theta(x, y) \le \theta(x, z) d_\theta(x, z) + \theta(z, y) d_\theta(z, y)$, *for all* $x, y, z \in X$.

Then (X, d_θ) *is called an extended b-metric space.*

Example 8.19. *Let* $X = [0, 1]$, *define* $d_\theta(x, y) = |x - y|^3$, *and let* $\theta(x, y) = 2$ *for all* x, y. *Then* (X, d_θ) *is an extended b-metric space.*

9. 2-Metric Spaces

During the 1960s, Gahler [39, 40] introduced the notation of 2-metric spaces as an extension of the standard metric space.

Definition 9.15. *Let* X *be a nonempty set. A function* $d : X \times X \times X \to [0, \infty)$, *satisfying the following properties:*

(d1) For distinct $x, y \in X$, *there exists* $z \in X$ *such that* $d(x, y, z) \neq 0$,
(d2) $d(x, y, z) = 0$ *if two of the triple* $x, y, z \in X$ *are equal,*
(d3) $d(x, y, z) = d(x, z, y) = \cdots$ *(symmetry in all three variables),*
(d4) $d(x, y, z) \le d(x, y, a) + d(x, a, z) + d(a, y, z)$, *for all* $x, y, z, a \in X$ *(rectangle inequality)*

is called a 2*- metric on* X. *The set* X *together with such a* 2*-metric,* d, *is called a* 2*-metric space and denoted by* (X, d).

Gahler asserted that a 2-metric extends the standard concept of a metric, yet it is not. It was demonstrated that 2-metric is not a continuous function, while a typical metric is. Gahler noted that geometrically $d(x, y, z)$ signifies the area of a triangle constructed by the points x, y, and z in X, although not required. For more details, see e.g. [41, 87].

10. D-Metric Spaces

In 1992, Dhage [29, 30, 31, 32, 33] attempted to develop a 2-metric by introducing a new concept of a generalized metric:

Definition 10.16. *Let* X *be a nonempty set, a function* $D : X \times X \times X \to \mathbf{R}^+$ *satisfying the following axioms:*

(D1) $D(x, y, z) \ge 0$ *for all* $x, y, z \in X$,
(D2) $D(x, y, z) = 0$ *if and only if* $x = y = z$,
(D3) $D(x, y, z) = D(x, z, y) = \cdots$ *(symmetry in all three variables),*
(D4) $D(x, y, z) \le D(x, y, a) + D(x, a, z) + D(a, y, z)$, *for all* $x, y, z, a \in X$ *(rectangle inequality)*

is called a generalized metric or a D-metric on X. *An additional property was proposed on a D-metric (see [30]) is*

(D5) $D(x,y,y) \leq D(x,z,z) + D(z,y,y)$ *for all* $x,y,z \in X$.

The set X together with such a generalized metric, D, is called a generalized metric space or D-metric space, and denoted by (X,D). *It is called symmetric if* $D(x,x,y) = D(x,y,y)$ *for all* $x,y \in X$.

In a D–metric space (X,D), three possible notions for the convergence of a sequence (x_n) to a point x present themselves:

$(C1)$ $x_n \to x$ if $D(x_n,x,x) \to 0$ as $n \to \infty$,
$(C2)$ $x_n \to x$ if $D(x_n,x_n,x) \to 0$ as $n \to \infty$,
$(C3)$ $x_n \to x$ if $D(x_m,x_n,x) \to 0$ as $m,n \to \infty$.

Clearly, $(C3) \Rightarrow (C2)$ and if D is symmetric then $(C1) \Leftrightarrow (C2)$.

In generalized metric spaces, the demonstrations for many fixed point theorems proposed by Dhage and others depended, either explicitly or implicitly, on the persistence of D regarding convergence in the sense of $(C3)$ or - regarding the convergence in the context of $(C2)$. Nonetheless, there are opposing examples provided by Mustafa and Sims [66].

Example 10.20 ([66]). *Let* $A = \{\frac{1}{n} : n = 1,2,3,\ldots\}$ *and* $X = A \cup \{0\}$ *and let* p *be the semi-metric on* X *defined by:*

$$p(x,x) := 0, \text{ for all } x \in X,$$
$$p(0,\frac{1}{n}) := p(\frac{1}{n},0) = \frac{1}{n}, \text{ for } n = 2,3,\ldots,$$
$$p(x,y) := 1, \text{ for } x,y \in A \text{ with } x \neq y.$$

Then, for $D(x,y,z) := \max\{p(x,y),p(x,z),p(y,z)\}$ *we have* $D(x_n,x_m,0) = 1$ *for all* n,m *with* $n \neq m$, *thus the sequence* $(\frac{1}{n})$ *does not converge in the sense of* $(C3)$.

Example 10.21 ([66]). *For* X *as above, define* D *by:*

$$D(x,y,z) := \begin{cases} 0, & \text{if } x = y = z \\ \frac{1}{n}, & \text{if one of } x,y,z \text{ is equal to } 0 \text{ and the other two are equal to } \frac{1}{n} \\ 1, & \text{otherwise.} \end{cases}$$

Then it is readily seen that D *is a generalized metric which satisfies* $C2(D5)$. *Further,* $(\frac{1}{n})$ *does not converge in the sense of* $(C1)$ *or* $(C3)$.

Example 10.22 ([66]). *For* X *as above, define* D *by:*

$$D(x,y,z) := \begin{cases} 0, & \text{if } x = y = z \\ \frac{1}{n}, & \text{if two of } x,y,z \text{ are equal to } 0 \text{ and the other is equal to } \frac{1}{n} \\ 1, & \text{otherwise.} \end{cases}$$

Then it is readily seen that D *is a generalized metric satisfying* $(D5)$. *The sequence* $(\frac{1}{n})$ *does not converge in the sense of* $(C2)$ *or* $(C3)$.

Example 10.23 ([66]). *For* X *again as in Example 10.20, but with semi-metric* p *defined by:*

$$p(0,1) := p(1,0) = 1,$$
$$p(1,\frac{1}{n}) := p(\frac{1}{n},1) = \frac{1}{2}, \text{ for } n = 2,3,\ldots,$$
$$p(1,1) := 0,$$
$$p(x,y) := |x-y|, \text{ for } x,y \in X \setminus \{1\}.$$

Then $D(x,y,z) := p(x,y) + p(x,z) + p(y,z)$ *converges to* 0 *in each of the senses* $(C1),(C2),(C3)$.

11. G-Metric Spaces

Definition 11.17 ([66]). *A generalized metric (or a G-metric) on X is a mapping $G : X \times X \times X \to [0, \infty)$ verifying, for all $x, y, z \in X$:*

(G_1) $G(x, x, x) = 0$,
(G_2) $G(x, x, y) > 0$ *if* $x \neq y$,
(G_3) $G(x, x, y) \leq G(x, y, z)$ *if* $y \neq z$,
(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ *(symmetry in all three variables),*
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *(rectangle inequality).*

Let (X, G) be a G-metric space, let $\{x_n\} \subseteq X$ be a sequence and let $x \in X$. Then the following conditions are equivalent.

(a) $\{x_n\}$ G-converges to x.
(b) $\lim_{n\to\infty} G(x_n, x_n, x) = 0 \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : x_n \in B_G(x, \varepsilon)\, \forall n \geq n_0$.
(c) $\lim_{n\to\infty} G(x_n, x, x) = 0$.
(d) $\lim_{n,m\to\infty,\ m\geq n} G(x_n, x_m, x) = 0$.
(e) $\lim_{n\to\infty} G(x_n, x_n, x) = 0$ and $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0$.
(f) $\lim_{n\to\infty} G(x_n, x, x) = 0$ and $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0$.
(g) $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$ and $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0$.
(h) $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$ & $\lim_{n,m\to\infty,\ m>n} G(x_n, x_m, x) = 0$.
(i) $\lim_{n,m\to\infty,\ m>n} G(x_n, x_m, x) = 0$.

Every G-metric on X defines a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \text{ for all } x, y \in X. \tag{11.4}$$

Example 11.24. *Let (X, d) be a metric space. The functions $G_m(d), G_s(d) : X \times X \times X \to [0, +\infty)$, defined by*

$$G_m(d)(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}, \tag{11.5}$$

$$G_s(d)(x, y, z) = \frac{1}{3}[d(x, y) + d(y, z) + d(z, x)], \tag{11.6}$$

for all $x, y, z \in X$, are standard G-metrics on X.

Definition 11.18. *Let (X, G) be a G-metric space, and let $x_0 \in X$, given $\varepsilon > 0$, define the sets*

$$B_G(x_0, \varepsilon) := \{y \in X; G(x_0, y, y) < \varepsilon\}$$

and

$$\overline{B}_G(x_0, \varepsilon) := \{y \in X, G(x_0, y, y) \leq \varepsilon\}$$

Then, $B_G(x_0, \varepsilon)$ and $\overline{B}_G(x_0, \varepsilon)$ are called the open and closed balls, with centers x_0 and radius ϵ, respectively.

Each G-metric G on X generates a topology τ_G on X whose base is a family of open G-balls $\{B_G(x, \varepsilon) : x \in X, \varepsilon > 0\}$. A nonempty set A in the $G-$metric space (X, G) is G-closed if $\overline{A} = A$. Moreover,

$$x \in \overline{A} \Leftrightarrow B_G(x, \varepsilon) \cap A \neq \emptyset, \quad \text{for all } \varepsilon > 0.$$

Definition 11.19. *A sequence* $\{x_n\}$ *in a G-metric space* X *is said to converge if there exists* $x \in X$ *such that* $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$*, and one say that the sequence* (x_n) *is G-convergent to* x*. We call* x *the limit of the sequence* $\{x_n\}$ *and write* $x_n \to x$ *or* $\lim_{n\to\infty} x_n = x$*.*

Theorem 11.5. *Every G-convergent sequence in a G-metric space* (X, G) *has a unique limit.*

Definition 11.20. *In a G-metric space* X*, a sequence* (x_n) *is said to be G-Cauchy if given* $\epsilon > 0$*, there is* $N \in \mathbf{N}$ *such that* $G(x_n, x_m, x_l) < \epsilon$*, for all* $n, m, l \geq N$*, that is* $G(x_n, x_m, x_l) \longrightarrow 0$ *as* $n, m, l \longrightarrow \infty$*.*

Definition 11.21. *A G-metric space* X *is said to be complete if every G-Cauchy sequence in* X *is G-convergent in* X*.*

Theorem 11.6. *Let* (X, G) *be a G-metric space. Let* $d : X \times X \to [0, \infty)$ *be the function defined by* $d(x, y) = G(x, y, y)$*. Then*

(1) (X, d) *is a quasi-metric space,*
(2) $\{x_n\} \subset X$ *is G-convergent to* $x \in X$ *if and only if* $\{x_n\}$ *is convergent to* x *in* (X, d)*,*
(3) $\{x_n\} \subset X$ *is G-Cauchy if and only if* $\{x_n\}$ *is Cauchy in* (X, d)*,*
(4) (X, G) *is G-complete if and only if* (X, d) *is complete.*

Every quasi-metric induces a metric, that is, if (X, d) is a quasi-metric space, then the function $\delta : X \times X \to [0, \infty)$ defined by $\delta(x, y) = \max\{d(x, y), d(y, x)\}$ is a metric on X.

12. S-Metric

Another such generalization of three point was given by S. Sedghi, N. Shobe and A. Aliouche [84] in 2012 as follows:

Definition 12.22. *Let* X *be a nonempty set, a function* $S : X \times X \times X \to \mathbf{R}^+$ *satisfying the following axioms:*

(S1) $S(x, y, z) = 0$ *if and only if* $x = y = z$*,*
(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$*, for all* $x, y, z, a \in X$ *(rectangle inequality),*

is called a S-metric *on* X*. The pair* (X, S) *is called as S-metric space.*

Definition 12.23. *Let* (X, S) *be an S-metric space. For* $r > 0$ *and* $x \in X$*, we define the open ball* $B_S(x, r)$ *and the closed ball* $B_S[x, r]$ *with center* x *and radius* r *as follows:*

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\},$$
$$B_S[x, r] = \{y \in X : S(y, y, x) \leq r\}.$$

The topology induced by the S-metric is the topology generated by the base of all open balls in X*.*

Definition 12.24. *Let* (X, S) *be an S-metric space.*

(1) A sequence $\{x_n\} \subset X$ *converges to* $x \in X$ *if* $S(x_n, x_n, x) \to 0$ *as* $n \to \infty$*. That is, for each* $\epsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *we have* $S(x_n, x_n, x) < \epsilon$*. We write* $x_n \to x$ *for brevity.*
(2) A sequence $\{x_n\} \subset X$ *is a Cauchy sequence if* $S(x_n, x_n, x_m) \to 0$ *as* $n, m \to \infty$*. That is, for each* $\epsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $n, m \geq n_0$ *we have* $S(x_n, x_n, x_m) < \epsilon$*.*
(3) The S-metric space (X, S) *is complete if every Cauchy sequence is a convergent sequence.*

Lemma 12.5. *In an S-metric space, we have* $S(x, x, y) = S(y, y, x)$ *for all* $x, y \in X$*.*

Lemma 12.6. *Let* (X, S) *be an S-metric space. If* $x_n \to x$ *and* $y_n \to y$ *then* $S(x_n, x_n, y_n) \to S(x, x, y)$*.*

Example 12.25 ([84]). *Let $X = \mathbb{R}^n$ and $\|\cdot\|$ be a norm on X. Then, $S(x,y,z) = \|y+x-2z\|+\|y-z\|$ is an S-metric on X. In general, if X is a vector space over $\mathbb{R}$ and $\|\cdot\|$ is a norm on X, then it is easy to see that*

$$S(x,y,z) = \|y-z\| + \|x-z\|$$

where $x + \lambda y = z$ for every $\lambda \geq 1$, is an S-metric on X.

Theorem 12.7 ([86]). *Let (X,S) be a S-metric space. Let $d : X \times X \to [0,\infty)$ be the function defined by $d(x,y) = S(x,x,y)$. Then*

(1) (X,d) is a b-metric space,
(2) $\{x_n\} \subset X$ is S-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X,d),
(3) $\{x_n\} \subset X$ is S-Cauchy if and only if $\{x_n\}$ is Cauchy in (X,d).

13. Cone Metric Space

Let E be a real Banach space. A subset P of E is called a cone if and only if the following hold:

(a_1) P is closed, nonempty, and $P \neq \{0\}$,
(a_2) $a,b \in \mathbb{R}$, $a,b \geq 0$, and $x,y \in P$ imply that $ax + by \in P$,
(a_3) $x \in P$ and $-x \in P$ imply that $x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$, if and only if, $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal, if there exist a number $K > 1$ such that, $0 \leq x \leq y$ implies $||x|| \leq K||y||$, for all $x,y \in E$. The least positive number satisfying this, called the normal constant. E denotes a real Banach space, P denotes a cone in E with $intP \neq \emptyset$, and $\leq$ denotes partial ordering with respect to P.

Definition 13.25 ([44]). *Let X be a nonempty set. A function $d : X \times X \to E$ is called a cone metric on X, if it satisfies the following conditions:*

(b_1) *$d(x,y) \geq 0,\ \forall x,y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$,*
(b_2) *$d(x,y) = d(y,x)$, for all $x,y \in X$,*
(b_3) *$d(x,y) \leq d(x,z) + d(y,z)$, for all $x,y,z \in X$.*

Then, (X,d) is called a cone metric space.

Definition 13.26. *Let (X,d) be a cone metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. If for all $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n,x_0) \ll c$, then $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent and $\{x_n\}_{n\in\mathbb{N}}$ converges to x and x is the limit of $\{x_n\}_{n\in\mathbb{N}}$.*

Definition 13.27. *Let (X,d) be a cone metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. If for all $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $m,n > n_0$, $d(x_n,x_m) \ll c$, then $\{x_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence in X.*

Definition 13.28. *Let (X,d) be a cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.*

Definition 13.29. *Let (X,d) be a cone metric space. A self-map T on X is said to be continuous, if $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} T(x_n) = T(x)$ for all sequence $\{x_n\}_{n\in\mathbb{N}}$ in X.*

Lemma 13.7. *Let (X,d) be a cone metric space and P be a cone. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then, $\{x_n\}_{n\in\mathbb{N}}$ converges to x, if and only if,*

$$\lim_{n\to\infty} d(x_n,x) = 0. \tag{13.7}$$

Lemma 13.8. *Let (X, d) be a cone metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. If $\{x_n\}_{n\in\mathbb{N}}$ is convergent, then it is a Cauchy sequence.*

Lemma 13.9. *Let (X, d) be a cone metric space and P be a cone in E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, if and only if, $\lim_{m,n\to\infty} d(x_m, x_n) = 0$.*

Example 13.26 ([44]). *Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E \mid y > 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that*

$$d(x, y) = |(x - y), \alpha|x - y||,$$

where $\alpha > 0$ is a constant. Then (X, d) is a cone metric space.

The cone P is called normal if there exists a constant $K > 0$ such that for all $a, b \in E$, $0 \leq a \leq b$ implies $|a| \leq K|b|$. The cone $[0, \infty) \subset \mathbb{R}^2$ is a normal cone with constant $K = 1$. However, there are examples of non-normal cones.

Example 13.27 ([28]). *Let $E = \mathbb{R}^2$ with the norm $||\cdot||$ and consider the cone $P = \{(x, y) \in E \mid y > 0\}$. For each $k \geq 1$, let $f(x) = x$ and $g(x) = 2^k$. Then $0 \leq g(x) \leq 2k$. Since $||f|| = 2$ and $||g|| = 2k + 1$. There are no normal cones with normal constant $K < 1$. Indeed, if P were a normal cone and if $|x| < K|y|$, then $0 < K < 1$. For each k, consider the real vector space E with the supremum norm and the cone $P = \{x = a + b : a, b \geq 0\}$. Since P is regular and normal, it can be shown that the normal constant for this cone is greater than one. This shows that we can construct cones with different normal constants $K > 1$.*

14. Cone b-Metric Space

Definition 14.30 ([43]). *Let X be a nonempty set and $p > 1$ be a given real number. A mapping $d : X \times X \to E$ is said to be a cone b-metric if, for all $x, y, z \in X$, the following conditions hold:*

(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a cone b-metric space.

Remark 14.5. *The class of cone b-metric spaces is larger than the class of cone metric spaces since any cone metric space must be a cone b-metric space. However, the converse is not always true.*

Example 14.28 ([43]). *Let $E = \mathbb{R}^2$, $p > 1$, $X = R$ and $d(x, y) = |y - x|^p$. We see that the mapping d satisfies the conditions for a cone b-metric. Let $x, y \in X$ and $x \neq y$. From the inequality*

$$|y - x|^p \geq 0,$$

which implies that $d(x, y) \geq 0$. It is impossible for all x, y to be equal. Indeed, taking account of the inequality

$$|y - x|^p > 0,$$

for all $x \neq y$. Thus, (d_3) in Definition 14 is not satisfied, i.e., (X, d) is not a cone metric space.

Example 14.29 ([43]). *Let $X = [0, 1]$, $E = \mathbb{R}^2$ and p be a constant. Take*

$$P = \{(x, y) \in E : x, y > 0\}.$$

We define $d : X \times X \to E$ as

$$d(x, y) = |(x - y)^p|^{\frac{1}{p}}.$$

Then (X, d) is a complete cone b-metric space.

15. Complex Valued Metric Spaces

The concept of complex valued metric space which is given by Azam et al. [12]. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

It follows that

$$z_1 \precsim z_2$$

if one of the following conditions is satisfied

(h_1) $Re(z_1) = Re(z_2); Im(z_1) < Im(z_2)$
(h_2) $Re(z_1) < Re(z_2); Im(z_1) = Im(z_2)$
(h_3) $Re(z_1) < Re(z_2); Im(z_1) < Im(z_2)$
(h_4) $Re(z_1) = Re(z_2); Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(h_1), (h_2)$ and (h_3) is satisfied and we will write $z_1 \prec z_2$ if only (h_3) is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \implies |z_1| < |z_2|$$

where $|\cdot|$ represent modulus or magnitude of z, and

$$z_1 \precsim z_2, \ z_2 \prec z_3 \implies z_1 \prec z_3.$$

Definition 15.31. *Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X, if it satisfies the following conditions:*

- (b_1) $0 \precsim d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$, *if and only if,* $x = y$,
- (b_2) $d(x, y) = d(y, x)$, *for all* $x, y \in X$,
- (b_3) $d(x, y) \precsim d(x, z) + d(y, z)$, *for all* $x, y, z \in X$.

Here, the pair (X, d) is called a complex valued metric space.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space.

Lemma 15.10. *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 15.11. *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

Let $(X, d_{\mathbb{C}})$ be a complex-valued metric space where $\mathbb{C}$ is the skew field of complex number z, i.e.,

$$\mathbb{C} = \{x + yi : (x, y) \in \mathbb{R}^2\}.$$

Define

$$\mathcal{P}_{\mathbb{C}} = \{x + yi : x \geq 0, y \geq 0\}.$$

It is apparent that $\mathcal{P}_{\mathbb{C}} \subset \mathbb{C}$. Assume $\mathbf{0}_{\mathbb{C}}$ be the zero of $\mathbb{C}$ from now on. Note that $(\mathbb{C}, |.|)$ is a real Banach space.

Lemma 15.12. *$\mathcal{P}_{\mathbb{C}}$ is a normal cone in real Banach space $(\mathbb{C}, |.|)$.*

Lemma 15.13. *Any complex-valued metric space $(X, d_{\mathbb{C}})$ is a cone metric space.*

16. Quaternion-Valued Metric Space

The skew field of quaternion, denoted by $\mathbb{H}$ means to write each element $q \in \mathbb{H}$ in the form

$$q = x_0 + x_1 i + x_2 j + x_3 k,$$

$x_n \in \mathbb{R}$; where $1, i, j, k$ are the basis elements of $\mathbb{H}$ and $n = 1, 2, 3$. For these elements, we have the multiplication rules

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, \\ &ij = -ji = k, \\ &kj = -jk = -i, \\ &ki = -ik = j. \end{aligned}$$

The conjugate element $\bar{q}$ is given by

$$\bar{q} = x_0 - x_1 i - x_2 j - x_3 k.$$

The quaternion modulus has the form of

$$|q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

A quaternion q may be viewed as a four-dimensional vector (x_0, x_1, x_2, x_3). Define a partial order $\precsim$ on $\mathbb{H}$ as follows:

$$q_1 \precsim q_2 \iff \begin{cases} Re(q1) \le Re(q2) \\ Im_s(q_1) \le Im_s(q_2), \; q_1, q_2 \in \mathbb{H}, s = i, j, k. \end{cases}$$

where $Im_i = x_1, Im_j = x_2$ and $Im_k = x_3$. It follows that $q_1 \precsim q_2$ if one of the following conditions hold:

$$\begin{array}{llll}
(I) & Re(q_1) = Re(q_2); & Im_{s_1}(q_1) = Im_{s_1}(q_2) & where\ s_1 = j, k; Im_i(q_1) < Im_i(q_2) \\
(II) & Re(q_1) = Re(q_2); & Im_{s_2}(q_1) = Im_{s_2}(q_2) & where\ s_2 = i, k; Im_j(q_1) < Im_j(q_2) \\
(III) & Re(q_1) = Re(q_2); & Im_{s_3}(q_1) = Im_{s_3}(q_2) & where\ s_3 = i, j; Im_k(q_1) < Im_k(q_2) \\
(IV) & Re(q_1) = Re(q_2); & Im_{s_1}(q_1) < Im_{s_1}(q_2); & Im_i(q_1) = Im_i(q_2) \\
(V) & Re(q_1) = Re(q_2); & Im_{s_2}(q_1) < Im_{s_2}(q_2); & Im_j(q_1) = Im_j(q_2) \\
(VI) & Re(q_1) = Re(q_2); & Im_{s_3}(q_1) < Im_{s_3}(q_2); & Im_k(q_1) = Im_k(q_2) \\
(VII) & Re(q_1) = Re(q_2); & Im_s(q_1) < Im_s(q_2) & \\
(VIII) & Re(q_1) < Re(q_2); & Im_s(q_1) = Im_s(q_2) & \\
(IX) & Re(q_1) < Re(q_2); & Im_{s_1}(q_1) = Im_{s_1}(q_2); & Im_i(q_1) < Im_i(q_2) \\
(X) & Re(q_1) < Re(q_2); & Im_{s_2}(q_1) = Im_{s_2}(q_2); & Im_j(q_1) < Im_j(q_2) \\
(XI) & Re(q_1) < Re(q_2); & Im_{s_3}(q_1) = Im_{s_3}(q_2); & Im_k(q_1) < Im_k(q_2) \\
(XII) & Re(q_1) < Re(q_2); & Im_{s_1}(q_1) < Im_{s_1}(q_2); & Im_i(q_1) = Im_i(q_2) \\
(XIII) & Re(q_1) < Re(q_2); & Im_{s_2}(q_1) < Im_{s_2}(q_2); & Im_i(q_1) = Im_i(q_2) \\
(XIV) & Re(q_1) < Re(q_2); & Im_{s_3}(q_1) < Im_{s_3}(q_2); & Im_i(q_1) = Im_i(q_2) \\
(XV) & Re(q_1) < Re(q_2); & Im_s(q_1) < Im_s(q_2) & \\
(XVI) & Re(q_1) = Re(q_2); & Im_s(q_1) = Im_s(q_2). &
\end{array}$$

Remark 16.6. *In particular, we write $q_1 \precnsim q_2$ if $q_1 \neq q_2$ and one from (I) to (XVI) is satisfied. Also, we will write $q_1 < q_2$ if only (XV) is satisfied. It should be remarked that*

$$q_1 \precsim q_2 \;\Rightarrow\; |q_1| \le |q_2|.$$

Ahmed et al. [4], introduced the definition of the quaternion-valued metric space as follows:

Definition 16.32. *Let X be a nonempty set. A function $d_{\mathbb{H}} : X \times X \to \mathbb{H}$ is called a quaternion valued metric on X, if it satisfies the following conditions:*

(d_1) $0 \precsim d_{\mathbb{H}}(x,y)$ *for all* $x, y \in X$ *and* $d_{\mathbb{H}}(x,y) = 0$, *if and only if*, $x = y$,
(d_2) $d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(y,x)$, *for all* $x, y \in X$,
(d_3) $d_{\mathbb{H}}(x,y) \precsim d_{\mathbb{H}}(x,z) + d_{\mathbb{H}}(y,z)$, *for all* $x, y, z \in X$.

Then, $(X, d_{\mathbb{H}})$ *is called a quaternion valued metric space.*

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 \precsim c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d_{\mathbb{H}}(x_n, x) \precsim c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \precsim c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d_{\mathbb{H}}(x_n, x_{n+m}) \precsim c$, then $\{x_n\}$ is called a Cauchy sequence in $(X, d_{\mathbb{H}})$. If every Cauchy sequence is convergent in $(X, d_{\mathbb{H}})$, then $(X, d_{\mathbb{H}})$ is called a complete quaternion valued metric space.

Lemma 16.14. *Let* (X, d) *be a quaternion valued metric space and let* $\{x_n\}$ *be a sequence in* X *. Then* $\{x_n\}$ *converges to* x *if and only if* $|d(x_n, x)| \to 0$ *as* $n \to \infty$.

Lemma 16.15. *Let* $(X, d_{\mathbb{H}})$ *be a quaternion valued metric space and let* $\{x_n\}$ *be a sequence in* X *. Then* $\{x_n\}$ *is a Cauchy sequence if and only if* $|d_{\mathbb{H}}(x_n, x_{n+m})| \to 0$ *as* $n \to \infty$.

Let $(X, d_{\mathbb{H}})$ be a quaternion-valued metric space where $\mathbb{H}$ is the skew field of quaternion number q, i.e.,

$$\mathbb{H} = \{x_0 + x_1 i + x_2 j + x_3 k : (x_0, x_1, x_2, x_3) \in \mathbb{R}^4\}.$$

Define

$$\mathcal{P}_{\mathbb{H}} = \{x_0 + x_1 i + x_2 j + x_3 k : x_0 \geq 0, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}.$$

It is apparent that $\mathcal{P}_{\mathbb{H}} \subset \mathbb{H}$. Assume $\mathbf{0}_{\mathbb{H}}$ be the zero of $\mathbb{H}$ from now on. Note that $(\mathbb{H}, |.|)$ is a real Banach space.

Lemma 16.16. $\mathcal{P}_{\mathbb{H}}$ *is a normal cone in real Banach space* $(\mathbb{H}, |.|)$.

Lemma 16.17. *Any quaternion valued metric space* $(X, d_{\mathbb{H}})$ *is a cone metric space.*

Lemma 16.18. *A sequence* $\{x_n\}$ *in* $(X, d_{\mathbb{H}})$ *be convergent in the context of quaternion valued metric space if and only if* $\{x_n\}$ *be convergent in the setting of cone metric space.*

17. Multiplicative Metric Spaces

Definition 17.33 ([14]). *Let* X *be a nonempty set. An operator* $d* : X \times X \to [1, \infty)$ *is a multiplicative metric (MM for short) on* X, *if it satisfies:*

(m_1*) $d*(x,y) \geq 1\ \forall x, y \in X$, *and*
$d*(x,y) = 1$ *if and only if* $x = y$,
(m_2*) $d*(x,y) = d*(y,x)$ *for all* $x, y \in X$,
(m_3*) $d*(x,z) \leq d*(x,y) \cdot d*(y,z)$ *for all* $x, y, z \in X$, *(multiplicative triangle inequality).*

If the operator $d*$ *satisfies* $(m_1*) - (m_3*)$ *then the pair* $(X, d*)$ *is called a multiplicative metric space (MMS).*

$$\ln(\max\{a, b\}) = \max\{\ln a, \ln b\}$$

for all $a, b > 0$ as well as

$$e^{\max\{a,b\}} = \max\{e^a, e^b\}$$

for all $a, b \in \mathbb{R}$.

Theorem 17.8 ([34]). $(X, d*)$ *is an MMS if and only if* $(X, \ln d*)$ *is an S-MS, that is,* (X, d) *is an S-MS if and only if* (X, e^d) *is an MMS.*

18. Perturbed Metric Space

Definition 18.34 ([48]). *For a nonempty set X, a function $D : X \times X \to [0, \infty)$ is called perturbed metric (PM) with respect to $R : X \times X \to [0, \infty)$ if*

$$d = D - R : X \times X \quad \text{such that} \quad d(x, u) \mapsto D(x, u) - R(x, u)$$

forms a usual metric over X. More precisely, for any $u, z \in X$, the upcoming statements are provided:

(i) $(D - R)(q, u) \geq 0$,
(ii) $(D - R)(q, u) = 0$ *if and only if* $q = u$,
(iii) $(D - R)(q, u) = (D - R)(u, q)$,
(iv) $(D - R)(q, u) \leq (D - R)(q, z) + (D - R)(z, u)$.

In addition, $R : X \times X$ shall be named perturbed mapping, where $d = D - R$ is a standard metric. The triple (X, D, R) is reserved for perturbed metric space.

The topological basics of perturbed metric space are given in the next definition.

Definition 18.35 ([48]). *Let $\{x_n\}$ be a sequence in perturbed metric space (X, D, R) T be a self-mapping on perturbed metric space (X, D, R).*

(i) A sequence $\{x_n\}$ is a perturbed convergent or p-convergent in perturbed metric space (X, D, R) if $\{x_n\}$ converges in the corresponding standard metric space (SMS) (U, d), with $d = D - R$.
(ii) A sequence $\{x_n\}$ in (X, D, R) is called perturbed Cauchy or p-Cauchy if $\{x_n\}$ forms a Cauchy sequence with respect to standard metric space (X, d).
(iii) A triple (X, D, R) is called a complete perturbed metric space (in short, Cperturbed metric space) if the corresponding MS (X, d) is complete.
(iv) A map T is called perturbed continuous or p-continuous with respect to perturbed metric space (X, D, R) if the same mapping T is continuous within the standard metric space (X, d).

Example 18.30 ([48]). *Let $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by*

$$D(x, y) = |x - y| + \frac{x^2 y^4}{2}, \quad x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping

$$P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x, y) = \frac{x^2 y^4}{2}, \quad x, y \in \mathbb{R}.$$

In this case, the exact metric is the mapping $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by

$$d(x, y) = |x - y|, \quad x, y \in \mathbb{R}.$$

Remark that D is not a metric on X. This can be easily seen observing that $D(1, 1) = 1 \neq 0$.

Example 18.31 ([48]). *Let $D : C([0, 1]) \times C([0, 1]) \to [0, \infty)$ be the mapping defined by*

$$D(f, g) = \int_0^1 |f(t) - g(t)| \, dt + (f(0) - g(0))^2, \quad f, g \in C([0, 1]),$$

where $C([0, 1]) = \{f : [0, 1] \to \mathbb{R} \mid f \text{ is continuous on } [0, 1]\}$. Then D is a perturbed metric on $C([0, 1])$ with respect to the perturbed mapping

$$P : C([0, 1]) \times C([0, 1]) \to [0, \infty)$$

given by

$$P(f, g) = (f(0) - g(0))^2, \quad f, g \in C([0, 1]).$$

In this case, the exact metric is the mapping $d : C([0,1]) \times C([0,1]) \to [0,\infty)$ *defined by*

$$d(f,g) = \int_0^1 |f(t) - g(t)|\, dt, \quad f,g \in C([0,1]).$$

Remark that D *is symmetric and* $D(f,g) = 0$ *if and only if* $f = g$*. However,* D *is not a metric on* $C([0,1])$*. Namely, consider three constant functions* $f_1 \equiv C_1$, $f_2 \equiv C_2$, $f_3 \equiv C_3$*. Then*

$$D(f_1,f_3) = |C_1 - C_3| + (C_1 - C_3)^2,$$

$$D(f_1,f_2) = |C_1 - C_2| + (C_1 - C_2)^2,$$

and

$$D(f_2,f_3) = |C_2 - C_3| + (C_2 - C_3)^2.$$

In particular, for $(C_1, C_2, C_3) = (0, \frac{1}{2}, 1)$*, we get*

$$D(f_1,f_3) = 2, \quad D(f_1,f_2) = \frac{3}{4}, \quad D(f_2,f_3) = \frac{3}{4},$$

which yields

$$D(f_1,f_3) > D(f_1,f_2) + D(f_2,f_3).$$

Example 18.32 ([48])**.** *Let* $D : \mathbb{N} \times \mathbb{N} \to [0,\infty)$ *be the mapping defined by*

$$D(n,m) = (n-m)^2, \quad n,m \in \mathbb{N}.$$

Then D *is a perturbed metric on* $\mathbb{N}$*, where the perturbed mapping*

$$P : \mathbb{N} \times \mathbb{N} \to [0,\infty)$$

is given by

$$P(n,m) = (n-m)^2 - |n-m|, \quad n,m \in \mathbb{N},$$

and the exact metric $d : \mathbb{N} \times \mathbb{N} \to [0,\infty)$ *is given by*

$$d(n,m) = |n-m|, \quad n,m \in \mathbb{N}.$$

Remark that D *is not a metric on* $\mathbb{N}$*, but it is a* b*-metric on* $\mathbb{N}$*.*

19. Suprametric

Very recently, another new distance notion, so-called suprametric, was defined by Berzig [17, 18].

Definition 19.36. *Let* X *be a nonempty set. A function* $d : X \times X \to [0,\infty)$ *is called suprametric if for all* $x,y,z \in X$ *the following properties hold:*

- (d_1^*) $d(x,y) = 0$ *if and only if* $x = y$,
- (d_2^*) $d(x,y) = d(y,x)$,
- (d_3^*) $d(x,y) \le d(x,z) + d(z,y) + \rho d(x,z) d(z,y)$ *for some constant* $\rho \in \mathbb{R}^+$.

A suprametric space is a pair (X,d)*, where* X *is a nonempty set and* d *is a suprametric.*

Example 19.33 ([26])**.** *Let* $X = \{1,2,3\}$ *and* $d : X \times X \mapsto [0,\infty)$ *fulfilling* (d_1) *and* (d_2) *such that* $d(1,2) = 1$, $d(1,3) = 3$ *and* $d(2,3) = 1$*. Then* (X,d) *is not a metric space since* $d(1,3) > d(1,2) + d(2,3)$*, but is a suprametric space for* $\rho = 1$*.*

Clearly, every metric serves as a suprametric; however, there are various methods for constructing a suprametric from a metric without including the triangle inequality overall.

Example 19.34 ([17, 26]). *If (X, d) is a metric space, then the functions*

$$d_\alpha(x, y) = d(x, y)(d(x, y) + \alpha),$$
$$d_\beta(x, y) = \beta(e^{d(x,y)} - 1)$$

for any $x, y \in X$ are suprametrics on X with $\rho = \frac{2}{\alpha}$ and $\rho = \frac{1}{\beta}$, respectively. While the function

$$d_\gamma(x, y) = e^{-\gamma d(x,y)^2} - 1$$

for any $x, y \in X$ is a suprametric with a constant $\rho = 1$.

Observe that if (d_3^*) is satisfied for a particular $\rho > 0$, it will also be true for any larger value of ρ. The concepts of convergent and Cauchy sequences, as well as the continuity of mappings, are presented similarly to their corresponding terms in metric space

Definition 19.37 ([17, 18]). *Let (X, d) be a suprametric space and (x_n) be a sequence in X. The sequence (x_n) converges to $x \in X$ if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_0) < \varepsilon$ for all $n \geq n_0$. Furthermore, if for all $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n \geq n_0$, then (x_n) is a Cauchy sequence in X. If any Cauchy sequence is convergent in a suprametric space (X, d), then (X, d) is a complete suprametric space.*

Definition 19.38 ([17, 18]). *Let (X, d) be a suprametric space. A mapping $T : X \mapsto X$ is continuous at a point $x \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(Tx, Ty) < \varepsilon$ whenever $d(x, y) < \delta$. If T is continuous at every point of X, then it is a continuous mapping on a suprametric space (X, d).*

Definition 19.39 ([19]). *Let X be a nonempty set, $d\colon X \times X \to [0, +\infty)$ be a function and $b \geq 1$ and $c \geq 0$ be real constants. Then d is called strong b-suprametric if for all $x, y, z \in X$ the following properties hold:*

- $(s1)$ *$d(x, y) = 0$ if and only if $x = y$,*
- $(s2)$ *$d(x, y) = d(y, x)$,*
- $(s3)$ *$d(x, y) \leq d(x, z) + bd(z, y) + cd(x, z)d(z, y)$.*

A pair (X, d) is called strong b-suprametric (or sb-suprametric) space.

Proposition 19.2 ([17]). *The topology τ is Hausdorff.*

Proof. Let $x, y \in X$ with $x \neq y$ and $r = d(x, y)$. Denote $U = B(x, \frac{r}{2})$ and $V = B(y, \frac{r}{2+\rho r})$. We shall show that $U \cap V = \emptyset$. If not, there exists $z \in U \cap V$, so from $d(x, z) < \frac{r}{2}$ and $d(y, z) < \frac{r}{2+\rho r}$, we obtain

$$r = d(x, y) \leq d(x, z) + d(z, y) + \rho d(x, z)d(z, y) < \frac{r}{2} + \frac{r}{2 + \rho r} + \rho \frac{r}{2} \frac{r}{2 + \rho r} = r,$$

which is absurd, so $U \cap V = \emptyset$ and therefore X is Hausdorff. □

20. Conclusion

Listing all abstract distance structures in this survey is quite challenging. Why? The extent of our understanding is determined by the creativity of the researchers. There have consistently been fresh methods and innovative concepts to enhance, refine, and broaden the current ones. Conversely, there are numerous unique hybrid structures. For example, a b-metric may be regarded as a partial b-metric, cone b-metric, quasi b-metric, G_b-metric, etc. This method generates numerous different combinations, and it is not necessary to list all since we can deduce how one can obtain such hybrid spaces.

The evolution of the 2-metric occurred as follows: First, the D-metric was introduced to make the 2-metric function continuous. Later, the G-metric was defined by criticizing the lack of some technical topological properties in the D-metric concept. As a result, the notion of G-metric became the most advanced version of 2-metric. Indeed, G-metric was the perfect version of this trend. But, it was understood that under certain condition it is equivalent to quasi-metric [47, 82]. After these discussion, the notion of S-metric was defined but the destiny of it was not different from G-metric [85].

The historical advances of cone metric space or Banach-valued spaces is in the following way. First of all, TVS-valued metric was defined as a natural generalization [35]. It was realized that its composition with a scalarization function can be equivalent to the standard metric [3, 11, 35, 36, 46, 49, 61]. As a continuation of this trend, the notion of complex-valued metric, quaternion valued metric, b-cone metric, C^*-algebra valued metric were defined. Naturally, the novelty of these notions were discussed in [5, 8, 10, 52] and some related references therein. For the multiplicative metric, the equivalence of it with the standard metric with a simple transformation was given in [34]. Another discussion for the novelty of bipolar metric was given in [59]. Very clearly, quasi-metric space is contained by standard metric space and standard metric space lies in an strong b-metric space and it is covered by an interpolative metric space. It was clear from [58] that interpolative metric space was covered by b-metric space. Clearly, b-metric space lies in extended b-metric space. One can easily show that extended metric space and supra-metric space lie in perturbed metric space. It is also clear that ultra-metric space lies in the standard metric space. The relation of ultra-metric space and quasi-metric space is needed to be investigated. It can be considered as an open problem.

As a result, there are a number of abstract distance structures. We aim to put some significant examples of them with known relations. This may helps to researchers to use the proper one for solving their own problems.

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