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Research Article

Approximation by symmetrized and perturbed hyperbolic tangent activated convolutions as positive linear operators

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ABSTRACT. In this work, we studied further the univariate symmetrized and perturbed hyperbolic tangent activated convolution type operators of three kinds. Here, this is done with the method of positive linear operators. Their new approximation properties are established by the quantitative convergence to the unit operator using the modulus of continuity. It is also studied the related simultaneous approximation, as well as the iterated approximation.

Keywords: Symmetrized and perturbed hyperbolic tangent, convolution operator, quantitative approximation, simultaneous approximation, iterated approximation.

2020 Mathematics Subject Classification: 41A17, 41A25, 41A35.

1. INTRODUCTION

The author studied extensively the quantitative approximation of positive linear operators to the unit since 1985, see for example [1]-[3], [8]. He originated from the quantitative weak convergence of finite positive measures to the unit Dirac measure, having as a method the geometric moment theory, see [2], and he produced best upper bounds, leading to attained (i.e. sharp Jackson type inequalities), e.g. see [1], [2]. These studies have been gone to all possible directions, univariate and multivariate, though in this work, we stay only on the univariate approach over an infinite domain.

Our convolution operators here have as a kernel the symmetrized and perturbed hyperbolic tangent activation function, which is used very commonly in the study of neural networks, and they can be interpreted as positive linear operators. So here our proving methods come from the theory of positive linear operators. Thus, in Section 2, we discuss about the symmetrized and perturbed hyperbolic tangent activation function. In Section 3, we describe our activated convolution type operators and we present their properties, such as differentiation and iteration, along with positive linear operators results to be applied. In Section 4, we derive some auxiliary results which are estimates to our operators, when applied to polynomial type functions and to be used into our main results. In Section 5, we present our main explicit results under the lens of positive linear operators theory. We treat also the simultaneous and iterated approximation cases under the same spirit. We are greatly inspired by our earlier works [4, 5, 7]. Furthermore, general motivation comes from the great works [12] and [13].

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2. About q-Deformed and λ-Parametrized Hyperbolic Tangent Function $g_{q,\lambda}$

Here, all this initial background comes from [6, Chapter 18]. We use $g_{q,\lambda}$, see (2.1), and exhibit that it is a sigmoid function and we will present several of its properties related to the approximation by neural network operators. So, let us consider the hyperbolic tangent activation function

$$g_{q,\lambda}(x) := \frac{e^{\lambda x} - qe^{-\lambda x}}{e^{\lambda x} + qe^{-\lambda x}}, \quad \lambda, q > 0, \ x \in \mathbb{R}. \tag{2.1}$$

We have that

$$g_{q,\lambda}(0) = \frac{1-q}{1+q}.$$

We notice also that

$$g_{q,\lambda}(-x) = \frac{e^{-\lambda x} - qe^{\lambda x}}{e^{-\lambda x} + qe^{\lambda x}} = \frac{\frac{1}{q}e^{-\lambda x} - e^{\lambda x}}{\frac{1}{q}e^{-\lambda x} + e^{\lambda x}} = -\frac{\left(e^{\lambda x} - \frac{1}{q}e^{-\lambda x}\right)}{e^{\lambda x} + \frac{1}{q}e^{-\lambda x}} = -g_{\frac{1}{q},\lambda}(x). \tag{2.2}$$

That is

$$g_{q,\lambda}(-x) = -g_{\frac{1}{q},\lambda}(x), \ \forall\, x \in \mathbb{R}, \tag{2.3}$$

and

$$g_{\frac{1}{q},\lambda}(x) = -g_{q,\lambda}(-x),$$

hence

$$g'_{\frac{1}{q},\lambda}(x) = g'_{q,\lambda}(-x). \tag{2.4}$$

It is

$$g_{q,\lambda}(x) = \frac{e^{2\lambda x} - q}{e^{2\lambda x} + q} = \frac{1 - \frac{q}{e^{2lx}}}{1 + \frac{q}{e^{2\lambda x}}} \underset{(x \to +\infty)}{\to} 1,$$

i.e.

$$g_{q,\lambda}(+\infty) = 1. \tag{2.5}$$

Furthermore

$$g_{q,\lambda}(x) = \frac{e^{2\lambda x} - q}{e^{2\lambda x} + q} \underset{(x \to -\infty)}{\to} \frac{-q}{q} = -1,$$

i.e.

$$g_{q,\lambda}(-\infty) = -1. \tag{2.6}$$

We find that

$$g'_{q,\lambda}(x) = \frac{4q\lambda e^{2\lambda x}}{\left(e^{2\lambda x} + q\right)^2} > 0, \tag{2.7}$$

therefore $g_{q,\lambda}$ is strictly increasing. Next, we obtain ($x \in \mathbb{R}$)

$$g''_{q,\lambda}(x) = 8q\lambda^2 e^{2\lambda x}\left(\frac{q - e^{2\lambda x}}{\left(e^{2\lambda x} + q\right)^3}\right) \in C(\mathbb{R}). \tag{2.8}$$

We observe that

$$q - e^{2\lambda x} \gtrless 0 \Leftrightarrow q \gtrless e^{2\lambda x} \Leftrightarrow \ln q \gtrless 2\lambda x \Leftrightarrow x \lessgtr \frac{\ln q}{2\lambda}.$$

So, in case of $x < \frac{\ln q}{2\lambda}$, we have that $g_{q,\lambda}$ is strictly concave up, with $g''_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = 0$. In case of $x > \frac{\ln q}{2\lambda}$, we have that $g_{q,\lambda}$ is strictly concave down. Clearly, $g_{q,\lambda}$ is a shifted sigmoid function with $g_{q,\lambda}(0) = \frac{1-q}{1+q}$, and $g_{q,\lambda}(-x) = -g_{q^{-1},\lambda}(x)$, (a semi-odd function). By $1 > -1$, $x+1 > x-1$, we consider the function

$$M_{q.\lambda}(x) := \frac{1}{4}\left(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right) > 0, \tag{2.9}$$

$\forall x \in \mathbb{R}$; $q, \lambda > 0$. Notice that $M_{q,\lambda}(\pm\infty) = 0$, so the x-axis is horizontal asymptote. We have that

$$\begin{aligned} M_{q,\lambda}(-x) &= \frac{1}{4}\left(g_{q,\lambda}(-x+1) - g_{q,\lambda}(-x-1)\right) = \frac{1}{4}\left(g_{q,\lambda}(-(x-1)) - g_{q,\lambda}(-(x+1))\right) \\ &= \frac{1}{4}\left(-g_{\frac{1}{q},\lambda}(x-1) + g_{\frac{1}{q},\lambda}(x+1)\right) = \frac{1}{4}\left(g_{\frac{1}{q},\lambda}(x+1) - g_{\frac{1}{q},\lambda}(x-1)\right) \\ &= M_{\frac{1}{q},\lambda}(x), \quad \forall\, x \in \mathbb{R}. \end{aligned} \tag{2.10}$$

Thus

$$M_{q,\lambda}(-x) = M_{\frac{1}{q},\lambda}(x), \ \forall\, x \in \mathbb{R};\ q, \lambda > 0, \tag{2.11}$$

a deformed symmetry. Next, we have that

$$M'_{q,\lambda}(x) = \frac{1}{4}\left(g'_{q,\lambda}(x+1) - g'_{q,\lambda}(x-1)\right), \ \ \forall\, x \in \mathbb{R}. \tag{2.12}$$

Let $x < \frac{\ln q}{2\lambda} - 1$, then $x - 1 < x + 1 < \frac{\ln q}{2\lambda}$ and $g'_{q,\lambda}(x+1) > g'_{q,\lambda}(x-1)$ (by $g_{q,\lambda}$ being strictly concave up for $x < \frac{\ln q}{2\lambda}$), that is $M'_{q,\lambda}(x) > 0$. Hence $M_{q,\lambda}$ is strictly increasing over $\left(-\infty, \frac{\ln q}{2\lambda} - 1\right)$. Let now $x - 1 > \frac{\ln q}{2\lambda}$, then $x+1 > x-1 > \frac{\ln q}{2\lambda}$, and $g'_{q,\lambda}(x+1) < g'_{q,\lambda}(x-1)$, that is $M'_{q,\lambda}(x) < 0$. Therefore $M_{q,\lambda}$ is strictly decreasing over $\left(\frac{\ln q}{2\lambda} + 1, +\infty\right)$. Let us next consider, $\frac{\ln q}{2\lambda} - 1 \le x \le \frac{\ln q}{2\lambda} + 1$. We have that

$$\begin{aligned} M''_{q,\lambda}(x) &= \frac{1}{4}\left(g''_{q,\lambda}(x+1) - g''_{q,\lambda}(x-1)\right) \\ &= 2q\lambda^2\left[e^{2\lambda(x+1)}\left(\frac{q - e^{2\lambda(x+1)}}{\left(e^{2\lambda(x+1)} + q\right)^3}\right) - e^{2\lambda(x-1)}\left(\frac{q - e^{2\lambda(x-1)}}{\left(e^{2\lambda(x-1)} + q\right)^3}\right)\right]. \end{aligned} \tag{2.13}$$

By $\frac{\ln q}{2\lambda} - 1 \le x \Leftrightarrow \frac{\ln q}{2\lambda} \le x + 1 \Leftrightarrow \ln q \le 2\lambda(x+1) \Leftrightarrow q \le e^{2\lambda(x+1)} \Leftrightarrow q - e^{2\lambda(x+1)} \le 0$. By $x \le \frac{\ln q}{2\lambda} + 1 \Leftrightarrow x - 1 \le \frac{\ln q}{2\lambda} \Leftrightarrow 2\lambda(x-1) \le \ln q \Leftrightarrow e^{2\lambda(x-1)} \le q \Leftrightarrow q - e^{2\lambda\beta(x-1)} \ge 0$. Clearly by (2.13), we get that $M''_{q,\lambda}(x) \le 0$, for $x \in \left[\frac{\ln q}{2\lambda} - 1, \frac{\ln q}{2\lambda} + 1\right]$. More precisely $M_{q,\lambda}$ is concave down over $\left[\frac{\ln q}{2\lambda} - 1, \frac{\ln q}{2\lambda} + 1\right]$, and strictly concave down over $\left(\frac{\ln q}{2\lambda} - 1, \frac{\ln q}{2\lambda} + 1\right)$.

Consequently, $M_{q,\lambda}$ has a bell-type shape over $\mathbb{R}$. Of course it holds $M''_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) < 0$. At $x = \frac{\ln q}{2\lambda}$, we have

$$M'_{q,\lambda}(x) = \frac{1}{4}\left(g'_{q,\lambda}(x+1) - g'_{q,\lambda}(x-1)\right) = q\lambda\left(\frac{e^{2\lambda(x+1)}}{\left(e^{2\lambda(x+1)} + q\right)^2} - \frac{e^{2\lambda(x-1)}}{\left(e^{2\lambda(x-1)} + q\right)^2}\right). \tag{2.14}$$

Thus

$$M'_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = q\lambda\left(\frac{e^{2\lambda\left(\frac{\ln q}{2\lambda}+1\right)}}{\left(e^{2\lambda\left(\frac{\ln q}{2\lambda}+1\right)} + q\right)^2} - \frac{e^{2\lambda\left(\frac{\ln q}{2\lambda}-1\right)}}{\left(e^{2\lambda\left(\frac{\ln q}{2\lambda}-1\right)} + q\right)^2}\right)$$

$$= \lambda \left(\frac{e^{2\lambda} \left(e^{-2\lambda}+1\right)^2 - e^{-2\lambda} \left(e^{2\lambda}+1\right)^2}{\left(e^{2\lambda}+1\right)^2 \left(e^{-2\lambda}+1\right)^2} \right) = 0. \tag{2.15}$$

That is, $\frac{\ln q}{2\lambda}$ is the only critical number of $M_{q,\lambda}$ over $\mathbb{R}$. Hence at $x = \frac{\ln q}{2\lambda}$, $M_{q,\lambda}$ achieves its global maximum, which is

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = \frac{1}{4}\left[g_{q,\lambda}\left(\frac{\ln q}{2\lambda}+1\right) - g_{q,\lambda}\left(\frac{\ln q}{2\lambda}-1\right)\right] = \frac{1}{4}\left[\left(\frac{e^{\lambda}-e^{-\lambda}}{e^{\lambda}+e^{-\lambda}}\right) - \left(\frac{e^{-\lambda}-e^{\lambda}}{e^{-\lambda}+e^{\lambda}}\right)\right]$$

$$= \frac{1}{4}\left[\frac{2\left(e^{\lambda}-e^{-\lambda}\right)}{e^{\lambda}+e^{-\lambda}}\right] = \frac{1}{2}\left(\frac{e^{\lambda}-e^{-\lambda}}{e^{\lambda}+e^{-\lambda}}\right) = \frac{\tanh(\lambda)}{2}. \tag{2.16}$$

Conclusion 2.1. *The maximum value of $M_{q,\lambda}$ is*

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = \frac{\tanh(\lambda)}{2}, \quad \lambda > 0. \tag{2.17}$$

We mention the following:

Theorem 2.1 ([6, Ch. 18, p. 458])**.** *We have that*

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(x-i) = 1, \ \forall\, x \in \mathbb{R}, \forall\, \lambda, q > 0. \tag{2.18}$$

Also, it holds the following:

Theorem 2.2 ([6, Ch. 18, p. 459])**.** *It holds*

$$\int_{-\infty}^{\infty} M_{q,\lambda}(x)\,dx = 1, \quad \lambda, q > 0. \tag{2.19}$$

So that $M_{q,\lambda}$ is a density function on $\mathbb{R}$; $\lambda, q > 0$. Similarly, we get that

$$\int_{-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x)\,dx = 1, \quad \lambda, q > 0, \tag{2.20}$$

so that $M_{\frac{1}{q},\lambda}$ is a density function. Furthermore, we observe the symmetry

$$\left(M_{q,\lambda} + M_{\frac{1}{q},\lambda}\right)(-x) = \left(M_{q,\lambda} + M_{\frac{1}{q},\lambda}\right)(x), \quad \forall\, x \in \mathbb{R}. \tag{2.21}$$

Furthermore

$$\varphi = \frac{M_{q,\lambda} + M_{\frac{1}{q},\lambda}}{2} > 0 \tag{2.22}$$

is a new density function over $\mathbb{R}$, i.e.

$$\int_{-\infty}^{\infty} \varphi(x)\,dx = 1,$$

and φ is an even function. Clearly, then

$$\int_{-\infty}^{\infty} \varphi(nx-u)\,du = 1, \ \forall\, n \in \mathbb{N}, x \in \mathbb{R}. \tag{2.23}$$

3. Basics

We give the following:

Definition 3.1. *Let $f \in C_B(\mathbb{R})$ (continuous and bounded functions on $\mathbb{R}$), $n \in \mathbb{N}$. We define the following basic activated hyperbolic tangent perturbed convolution type operators*

$$A_n(f)(x) := \int_{-\infty}^{\infty} f\left(\frac{u}{n}\right) \varphi(nx-u)\, du, \quad \forall\, x \in \mathbb{R}. \tag{3.24}$$

In this work, we examine the quantitative convergence of A_n to the unit operator. We study similarly the activated Kantorovich type operators,

$$A_n^*(f)(x) := n \int_{-\infty}^{\infty} \left(\int_{\frac{u}{n}}^{\frac{u+1}{n}} f(t)\, dt \right) \varphi(nx-u)\, du$$

$$= n \int_{-\infty}^{\infty} \left(\int_{0}^{\frac{1}{n}} f\left(t + \frac{u}{n}\right) dt \right) \varphi(nx-u)\, du, \tag{3.25}$$

where $f \in C_B(\mathbb{R})$, $n \in \mathbb{N}$, $x \in \mathbb{R}$, and the activated quadrature operators

$$\overline{A_n}(f)(x) := \int_{-\infty}^{\infty} \left(\sum_{i=1}^{r} w_i f\left(\frac{u}{n} + \frac{i}{nr}\right) \right) \varphi(nx-u)\, du, \tag{3.26}$$

where $w_i \geq 0$, $\sum_{i=1}^{r} w_i = 1$; $f \in C_B(\mathbb{R})$, $n \in \mathbb{N}$, $x \in \mathbb{R}$. An essential property follows:

Theorem 3.3 ([7]). *Let $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. Then*

$$\int_{\{u \in \mathbb{R} : |nx-u| \geq n^{1-\alpha}\}} \varphi(nx-u)\, du < \frac{\left(q + \frac{1}{q}\right)}{e^{2\lambda(n^{1-\alpha}-1)}}, \quad q, \lambda > 0. \tag{3.27}$$

The first modulus of continuity here is

$$\omega_1(f, \delta) := \sup_{x, y \in \mathbb{R}, |x-y| \leq \delta} |f(x) - f(y)|, \quad \delta > 0. \tag{3.28}$$

We need the following:

Proposition 3.1 ([7]). *It holds ($k \in \mathbb{N}$)*

$$\int_{-\infty}^{\infty} |z|^k \varphi(z)\, dz \leq \left[\frac{\tanh(\lambda)}{(k+1)} + \frac{\left(q + \frac{1}{q}\right) e^{2\lambda} k!}{(2\lambda)^k} \right] < \infty. \tag{3.29}$$

We make the following:

Remark 3.1. *Given $f \in C_B(\mathbb{R})$, by [7], we obtain that $A_n(f)$, $A_n^*(f)$, $\overline{A}_n(f) \in C_B(\mathbb{R})$. Clearly, here A_n, A_n^*, $\overline{A}_n$ are positive linear operators from $C_B(\mathbb{R})$ into itself, with the property $A_n(1) = A_n^*(1) = \overline{A}_n(1) = 1$, $n \in \mathbb{N}$. Let $i \in \mathbb{N}$ be fixed. Assume that $f \in C^{(i)}(\mathbb{R})$, with $f^{(j)} \in C_B(\mathbb{R})$, for $j = 0, 1, ..., i$. We derive from [7] that*

$$(A_n(f))^{(j)}(x) = A_n\left(f^{(j)}\right)(x),$$

$$(A_n^*(f))^{(j)}(x) = A_n^*\left(f^{(j)}\right)(x), \tag{3.30}$$

$$\left(\overline{A_n}(f)\right)^{(j)}(x) = \overline{A_n}\left(f^{(j)}\right)(x), \quad \forall\, x \in \mathbb{R},$$

for all $j = 1, ..., i$.

Call Ψ_n any of the A_n, A_n^* and $\overline{A}_n$, $n \in \mathbb{N}$. We also make the following:

Remark 3.2. *Furthermore, it holds*

$$|A_n (f)(x)| \leq \|f\|_\infty \int_{-\infty}^{\infty} \varphi(z)\, dz = \|f\|_\infty ,$$

i.e.

$$\|A_n (f)\|_\infty \leq \|f\|_\infty \tag{3.31}$$

so A_n *is a bounded positive linear operator. Clearly, it holds*

$$\left\|A_n^2 (f)\right\|_\infty = \|A_n (A_n (f))\|_\infty \leq \|A_n (f)\|_\infty \leq \|f\|_\infty . \tag{3.32}$$

And for $k \in \mathbb{N}$ *we obtain*

$$\left\|A_n^k (f)\right\|_\infty \leq \left\|A_n^{k-1} (f)\right\|_\infty \leq \left\|A_n^{k-2} (f)\right\|_\infty \leq ... \leq \|f\|_\infty , \tag{3.33}$$

so the contraction property valid and A_n^k *is a bounded linear operator. Let* $r \in \mathbb{N}$*. We observe that*

$$\begin{aligned} A_n^r f - f = &\left(A_n^r f - A_n^{r-1} f\right) + \left(A_n^{r-1} f - A_n^{r-2} f\right) + \left(A_n^{r-2} f - A_n^{r-3} f\right) \\ &+ ... + \left(A_n^2 f - A_n f\right) + (A_n f - f) . \end{aligned} \tag{3.34}$$

Then

$$\begin{aligned} \|A_n^r f - f\|_\infty &\leq \left\|A_n^r f - A_n^{r-1} f\right\|_\infty + \left\|A_n^{r-1} f - A_n^{r-2} f\right\|_\infty + \left\|A_n^{r-2} f - A_n^{r-3} f\right\|_\infty \\ &+ ... + \left\|A_n^2 f - A_n f\right\|_\infty + \|A_n f - f\|_\infty \\ &= \left\|A_n^{r-1} (A_n f - f)\right\|_\infty + \left\|A_n^{r-2} (A_n f - f)\right\|_\infty + ... + \|A_n (A_n f - f)\|_\infty \\ &+ \|A_n f - f\|_\infty \leq r \|A_n f - f\|_\infty . \end{aligned}$$

Therefore

$$\|A_n^r f - f\|_\infty \leq r \|A_n f - f\|_\infty . \tag{3.35}$$

Let now $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$*, and* A_{m_i} *as above, then*

$$\begin{aligned} &A_{m_r} \left(A_{m_{r-1}} (...A_{m_2} (A_{m_1} f))\right) - f \\ &= ... = A_{m_r} \left(A_{m_{r-1}} (...A_{m_2})\right) (A_{m_1} f - f) + A_{m_r} \left(A_{m_{r-1}} (...A_{m_3})\right) (A_{m_2} f - f) \\ &+ A_{m_r} \left(A_{m_{r-1}} (...A_{m_4})\right) (A_{m_3} f - f) + ... + A_{m_r} \left(A_{m_{r-1}} f - f\right) + A_{m_r} f - f. \end{aligned} \tag{3.36}$$

Consequently it holds, as in [6, Chapter 2],

$$\left\|A_{m_r} \left(A_{m_{r-1}} (...A_{m_2} (A_{m_1} f))\right) - f\right\|_\infty \leq \sum_{i=1}^{r} \|A_{m_i} f - f\|_\infty . \tag{3.37}$$

All of (3.31)-(3.37) are also true for A_n^* *and* $\overline{A}_n$*, and* $n \in \mathbb{N}$.

We need the following Hölder's type inequality for positive linear operators.

Theorem 3.4 ([9]). *Let* L *be a positive linear operator from* $C(\mathbb{R})$ *into* $C_B(\mathbb{R})$*, and* $f, g \in C(\mathbb{R})$*, furthermore let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*. Assume that* $L((|f(\cdot)|^p))(s_*)$*,* $L((|g(\cdot)|^q))(s_*) > 0$ *for some* $s_* \in \mathbb{R}$*. Then*

$$L(|f(\cdot) g(\cdot)|)(s_*) \leq (L((|f(\cdot)^p|))(s_*))^{\frac{1}{p}} (L((|g(\cdot)^q|))(s_*))^{\frac{1}{q}} . \tag{3.38}$$

We also need the following

Theorem 3.5 ([9]). *Let $N \in \mathbb{N}$ and $f, f^{(N)} \in C_B(\mathbb{R})$, $x \in \mathbb{R}$. Consider L_n a sequence of positive linear operators from $C_B(\mathbb{R})$ into itself, $n \in \mathbb{N}$, such that $L_n(1) = 1$. Assume $L_n\left(|\cdot - x|^{N+1}\right)(x) > 0$, and $f^{(i)}(x) = 0$, for $i = 1, ..., N$. Then*

$$|L_n(f)(x) - f(x)| \leq \omega_1\left(f^{(N)}, \left(\left(L_n\left(|\cdot - x|^{N+1}\right)\right)(x)\right)^{\frac{1}{N+1}}\right)$$

$$\times\left[L_n\left(|\cdot - x|^N\right)(x) + \frac{\left(L_n\left(|\cdot - x|^{N+1}\right)(x)\right)^{\frac{N}{N+1}}}{(N+1)}\right] < +\infty, \quad \forall\, n \in \mathbb{N}. \tag{3.39}$$

Given that $\lim_{N\to+\infty} L_n\left(|\cdot - x|^{N+1}\right)(x) = 0$, then $\lim_{n\to+\infty} L_n(f)(x) = f(x)$. If $N = 1$, we derive

$$|L_n(f)(x) - f(x)| \leq \omega_1\left(f', \left(\left(L_n\left((\cdot - x)^2\right)\right)(x)\right)^{\frac{1}{2}}\right)$$

$$\times\left[L_n(|\cdot - x|)(x) + \frac{\left(L_n\left((\cdot - x)^2\right)(x)\right)^{\frac{1}{2}}}{2}\right] < +\infty, \quad \forall\, n \in \mathbb{N}. \tag{3.40}$$

Given that $\lim_{n\to+\infty} L_n\left((\cdot - x)^2\right)(x) = 0$, then $\lim_{n\to+\infty} L_n(f)(x) = f(x)$.

Note 3.1. *Assuming $L_n\left(|\cdot - x|^{N+1}\right)(x) > 0$, by Theorem 3.4, for $g = 1$, and L_n such that $L_n(1) = 1$, we obtain*

$$L_n\left(|\cdot - x|^N\right)(x) \leq \left(L_n\left(|\cdot - x|^{N+1}\right)(x)\right)^{\frac{N}{N+1}}. \tag{3.41}$$

Proof. In case of $N = 1$, we derive

$$L_n(|\cdot - x|)(x) \leq \sqrt{\left(L_n\left((\cdot - x)^2\right)(x)\right)}. \tag{3.42}$$

∎

We also need the following:

Theorem 3.6 ([11]). *Let $f \in C_B(\mathbb{R})$, $x \in \mathbb{R}$. Consider L_n a sequence of positive linear operators from $C_B(\mathbb{R})$ into itself, $n \in \mathbb{N}$, such that $L_n(1) = 1$. Assume that $L_n(|\cdot - x|)(x) > 0$. Then*

$$|L_n(f)(x) - f(x)| \leq 2\omega_1(f, L_n(|\cdot - x|)(x)), \quad \forall\, n \in \mathbb{N}. \tag{3.43}$$

Given that $\lim_{n\to+\infty} L_n(|\cdot - x|)(x) = 0$, and f is also uniformly continuous, we obtain $\lim_{n\to+\infty} L_n(f)(x) = f(x)$.

4. Auxiliary Results

We have the following result.

Lemma 4.1. *Let $m \in \mathbb{N}$. Then*

$$0 < A_N\left(|\cdot - x|^m\right)(x) \leq \frac{1}{n^m}\left[\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right](< +\infty) \to 0, \text{ as } n \to +\infty, \tag{4.44}$$

$\forall\, x \in \mathbb{R}$. *And, it is*

$$0 < \left\|A_N\left(|\cdot - x|^m\right)(x)\right\|_\infty \leq \frac{1}{n^m}\left[\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right](< +\infty) \to 0, \tag{4.45}$$

as $n \to +\infty$.

Proof. We have that ($m \in \mathbb{N}$, $x \in \mathbb{R}$)

$$\begin{aligned} 0 < A_N\left(|\cdot - x|^m\right)(x) &= \int_{-\infty}^{\infty}\left|\frac{u}{n} - x\right|^m \varphi(nx-u)\,du \\ &= \frac{1}{n^m}\int_{-\infty}^{\infty}|nx-u|^m\,\varphi(nx-u)\,du \\ &\overset{(z:=nx-u)}{=} \frac{1}{n^m}\int_{-\infty}^{\infty}|z|^m\,\varphi(z)\,dz \\ &\leq \frac{1}{n^m}\left[\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right](< +\infty) \to 0, \text{ as } n \to +\infty. \end{aligned} \tag{4.46}$$

□

It follows:

Lemma 4.2. *Let* $m \in \mathbb{N}$. *Then*

$$0 < A_N^*\left(|\cdot - x|^m\right)(x) \leq \frac{2^{m-1}}{n^m}\left[1 + \left(\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right)\right](< +\infty) \to 0, \tag{4.47}$$

as $n \to +\infty$ *and* $\forall\, x \in \mathbb{R}$. *And, it is*

$$0 < \left\|A_N^*\left(|\cdot - x|^m\right)(x)\right\|_\infty \leq \frac{2^{m-1}}{n^m}\left[1 + \left(\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right)\right](< +\infty) \to 0, \tag{4.48}$$

as $n \to +\infty$.

Proof. We have that ($m \in \mathbb{N}$, $x \in \mathbb{R}$)

$$\begin{aligned} 0 < A_N^*\left(|\cdot - x|^m\right)(x) &= n\int_{-\infty}^{\infty}\left(\int_0^{\frac{1}{n}}\left|t + \frac{u}{n} - x\right|^m dt\right)\varphi(nx-u)\,du \\ &\leq n\int_{-\infty}^{\infty}\left(\int_0^{\frac{1}{n}}\left(|t| + \left|\frac{u}{n} - x\right|\right)^m dt\right)\varphi(nx-u)\,du \\ &\leq \int_{-\infty}^{\infty}\left(\frac{1}{n} + \left|\frac{u}{n} - x\right|\right)^m \varphi(nx-u)\,du \\ &= \frac{1}{n^m}\int_{-\infty}^{\infty}(1 + |nx-u|)^m\,\varphi(nx-u)\,du \end{aligned} \tag{4.49}$$

$$\leq \frac{2^{m-1}}{n^m}\left[1+\int_{-\infty}^{\infty}|nx-u|^m \varphi(nx-u)\,du\right]$$
$$=\frac{2^{m-1}}{n^m}\left[1+\int_{-\infty}^{\infty}|z|^m \varphi(z)\,dz\right]$$
$$\leq \frac{2^{m-1}}{n^m}\left[1+\left(\frac{\tanh(\lambda)}{(m+1)}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right)\right]\to 0, \text{ as } n\to+\infty,$$

and it is finite. □

At last, we obtain the following:

Lemma 4.3. *Let $m \in \mathbb{N}$. Then*

$$0<\overline{A}_N\left(|\cdot-x|^m\right)(x)\leq \frac{2^{m-1}}{n^m}\left[1+\left(\frac{\tanh(\lambda)}{(m+1)}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right)\right](<+\infty)\to 0, \tag{4.50}$$

as $n\to+\infty$ and $\forall\, x\in\mathbb{R}$. And, it is

$$0<\left\|\overline{A}_N\left(|\cdot-x|^m\right)(x)\right\|_\infty \leq \frac{2^{m-1}}{n^m}\left[1+\left(\frac{\tanh(\lambda)}{(m+1)}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right)\right](<+\infty)\to 0, \tag{4.51}$$

as $n\to+\infty$.

Proof. We observe that

$$\begin{aligned} 0<\overline{A}_N\left(|\cdot-x|^m\right)(x) &= \int_{-\infty}^{\infty}\left(\sum_{i=1}^{r} w_i\left|\frac{u}{n}+\frac{i}{nr}-x\right|^m\right)\varphi(nx-u)\,du \\ &\leq \int_{-\infty}^{\infty}\left(\sum_{i=1}^{r} w_i\left(\left|\frac{u}{n}-x\right|+\frac{i}{nr}\right)^m\right)\varphi(nx-u)\,du \\ &\leq \int_{-\infty}^{\infty}\left(\frac{1}{n}+\left|\frac{u}{n}-x\right|\right)^m\varphi(nx-u)\,du \\ &= \frac{1}{n^m}\int_{-\infty}^{\infty}\left(1+|nx-u|\right)^m\varphi(nx-u)\,du. \end{aligned} \tag{4.52}$$

The proof finishes as in the proof of Lemma 4.2. □

5. Main Results

We present the following results.

Theorem 5.7. *Let $N\in\mathbb{N}$ and $f, f^{(N)}\in C_B(\mathbb{R})$, $x\in\mathbb{R}$. Assume $f^{(i)}(x)=0$, $i=1,...,N$. Then*

$$\begin{aligned} |\Psi_n(f)(x)-f(x)| &\leq \omega_1\left(f^{(N)},\left(\left(\Psi_n\left(|\cdot-x|^{N+1}\right)\right)(x)\right)^{\frac{1}{N+1}}\right) \\ &\times\left[\Psi_n\left(|\cdot-x|^N\right)(x)+\frac{\left(\Psi_n\left(|\cdot-x|^{N+1}\right)(x)\right)^{\frac{N}{N+1}}}{(N+1)}\right]<+\infty,\quad \forall\, n\in\mathbb{N}. \end{aligned} \tag{5.53}$$

Hence, $\lim_{n\to+\infty} \Psi_n(f)(x) = f(x)$. *If* $N=1$, *we obtain*

$$|\Psi_n(f)(x) - f(x)| \leq \omega_1\left(f', \left(\left(\Psi_n\left((\cdot - x)^2\right)\right)(x)\right)^{\frac{1}{2}}\right)$$

$$\times\left[\Psi_n(|\cdot - x|)(x) + \frac{\left(\Psi_n\left((\cdot - x)^2\right)(x)\right)^{\frac{1}{2}}}{2}\right] < +\infty, \quad \forall\, n \in \mathbb{N}. \tag{5.54}$$

Again, it holds $\lim_{n\to+\infty} \Psi_n(f)(x) = f(x)$.

Proof. By Theorem 3.5 and Lemmas 4.1-4.3. □

Theorem 5.8. *Let* $f \in C_B(\mathbb{R})$, $x \in \mathbb{R}$. *Then*

$$|\Psi_n(f)(x) - f(x)| \leq 2\omega_1(f, \Psi_n(|\cdot - x|)(x)) < +\infty, \quad \forall\, n \in \mathbb{N}. \tag{5.55}$$

If f *is also uniformly continuous, we derive* $\lim_{n\to+\infty} \Psi_n(f)(x) = f(x)$.

Proof. Direct application of Theorem 3.6 and Lemmas 4.1-4.3. □

We make the following:

Remark 5.3. *By* (3.41), (3.42), *we derive that*

$$\Psi_n\left(|\cdot - x|^N\right)(x) \leq \left(\Psi_n\left(|\cdot - x|^{N+1}\right)(x)\right)^{\frac{N}{N+1}}, \tag{5.56}$$

and

$$\Psi_n(|\cdot - x|)(x) \leq \sqrt{\left(\Psi_n\left((\cdot - x)^2\right)(x)\right)}, \quad \forall\, n \in \mathbb{N}. \tag{5.57}$$

Notation 5.1. *Let* $m, n \in \mathbb{N}$. *Denote by*

$$\rho_{1n}(m) := \frac{1}{n^m}\left[\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q + \frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right](< +\infty) \to 0, \quad as\ n \to +\infty, \tag{5.58}$$

and

$$\rho_{2n}(m) := \frac{2^{m-1}}{n^m}\left[1 + \left(\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q + \frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right)\right](< +\infty) \to 0, \quad as\ n \to +\infty. \tag{5.59}$$

We give the following explicit results.

Corollary 5.1. *Let* $N \in \mathbb{N}$ *and* $f, f^{(N)} \in C_B(\mathbb{R})$, $x \in \mathbb{R}$. *Assume* $f^{(i)}(x) = 0$, $i = 1, ..., N$. *Then*

$$|A_n(f)(x) - f(x)| \leq \omega_1\left(f^{(N)}, (\rho_{1n}(N+1))^{\frac{1}{N+1}}\right)$$

$$\times\left[\rho_{1n}(N) + \frac{(\rho_{1n}(N+1))^{\frac{N}{N+1}}}{(N+1)}\right] < +\infty, \quad \forall\, n \in \mathbb{N}. \tag{5.60}$$

Hence $\lim_{n\to+\infty} A_n(f)(x) = f(x)$. *If* $N = 1$, *we obtain*

$$|A_n(f)(x) - f(x)| \leq \omega_1\left(f', (\rho_{1n}(2))^{\frac{1}{2}}\right)$$

$$\times\left[\rho_{1n}(1)+\frac{(\rho_{1n}(2))^{\frac{1}{2}}}{2}\right]<+\infty,\ \ \forall\, n\in\mathbb{N}. \tag{5.61}$$

Again, it holds $\lim_{n\to+\infty} A_n(f)(x) = f(x)$.

Proof. By Theorem 5.7 and Lemma 4.1, see also (5.58). □

Corollary 5.2. *Let* $N\in\mathbb{N}$ *and* $f, f^{(N)}\in C_B(\mathbb{R})$, $x\in\mathbb{R}$. *Assume* $f^{(i)}(x)=0$, $i=1,...,N$. *Then*

$$\left\{\begin{array}{c}\left|A_n^*(f)(x)-f(x)\right|\\ \left|\overline{A}_n(f)(x)-f(x)\right|\end{array}\right\}\leq\omega_1\left(f^{(N)},(\rho_{2n}(N+1))^{\frac{1}{N+1}}\right)$$

$$\times\left[\rho_{2n}(N)+\frac{(\rho_{2n}(N+1))^{\frac{N}{N+1}}}{(N+1)}\right]<+\infty,\ \ \forall\, n\in\mathbb{N}. \tag{5.62}$$

Hence $\lim_{n\to+\infty}A_n^*(f)(x)=\lim_{n\to+\infty}\overline{A}_n(f)(x)=f(x)$. *If* $N=1$, *we obtain*

$$\left\{\begin{array}{c}\left|A_n^*(f)(x)-f(x)\right|\\ \left|\overline{A}_n(f)(x)-f(x)\right|\end{array}\right\}\leq\omega_1\left(f',(\rho_{2n}(2))^{\frac{1}{2}}\right)$$

$$\times\left[\rho_{2n}(1)+\frac{(\rho_{2n}(2))^{\frac{1}{2}}}{2}\right]<+\infty,\ \ \forall\, n\in\mathbb{N}. \tag{5.63}$$

Again, it holds $\lim_{n\to+\infty}A_n^*(f)(x)=\lim_{n\to+\infty}\overline{A}_n(f)(x)=f(x)$.

Proof. By Theorem 5.7 and Lemmas 4.2, 4.3, see also (5.59). □

Corollary 5.3. *Let* $f\in C_B(\mathbb{R})$. *Then*

$$\|A_n(f)-f\|_\infty\leq 2\omega_1(f,\rho_{1n}(1))<+\infty,\ \ \forall\, n\in\mathbb{N}. \tag{5.64}$$

If f *is also uniformly continuous, we get that* $\lim_{n\to+\infty}A_n(f)=f$, *pointwise and uniformly.*

Proof. By Theorem 5.8, Lemma 4.1, see also (5.58). □

Corollary 5.4. *Let* $f\in C_B(\mathbb{R})$. *Then*

$$\left\{\begin{array}{c}\|A_n^*(f)-f\|_\infty\\ \left\|\overline{A}_n(f)-f\right\|_\infty\end{array}\right\}\leq 2\omega_1(f,\rho_{2n}(1))<+\infty,\ \ \forall\, n\in\mathbb{N}. \tag{5.65}$$

If f *is also uniformly continuous, we obtain that* $\lim_{n\to+\infty}A_n^*(f)=\lim_{n\to+\infty}\overline{A}_n(f)=f$, *pointwise and uniformly.*

Proof. By Theorem 5.8, Lemmas 4.2, 4.3 see also (5.59). □

We continue with simultaneous approximations.

Theorem 5.9. *Let* $f^{(j)}\in C_B(\mathbb{R})$, *for* $j=0,1,...,N\in\mathbb{N}$. *Then*

$$\left\|(A_n(f))^{(j)}-f^{(j)}\right\|_\infty\leq 2\omega_1\left(f^{(j)},\rho_{1n}(1)\right)<+\infty,\ \ \forall\, n\in\mathbb{N}. \tag{5.66}$$

If $f^{(j)}$ *is uniformly continuous, we get that* $\lim_{n\to+\infty}(A_n(f))^{(j)}=f^{(j)}$, *pointwise and uniformly.*

Proof. By (3.30) and Corollary 5.3. □

Theorem 5.10. *Let $f^{(j)} \in C_B(\mathbb{R})$, for $j = 0, 1, ..., N \in \mathbb{N}$. Then*

$$\left\{ \begin{array}{c} \left\| \left(A_n^*(f)\right)^{(j)} - f^{(j)} \right\|_\infty \\ \left\| \left(\overline{A}_n(f)\right)^{(j)} - f^{(j)} \right\|_\infty \end{array} \right\} \leq 2\omega_1 \left(f^{(j)}, \rho_{2n}(1) \right) < +\infty, \quad \forall\, n \in \mathbb{N}. \tag{5.67}$$

If $f^{(j)}$ is uniformly continuous, we obtain that $\lim_{n\to+\infty} \left(A_n^*(f)\right)^{(j)} = \lim_{n\to+\infty} \left(\overline{A}_n(f)\right)^{(j)} = f^{(j)}$, pointwise and uniformly.

Proof. By (3.30) and Corollary 5.4. □

Next, we talk about iterated approximation.

Remark 5.4. *Let $f \in C_B(\mathbb{R})$, $r \in \mathbb{N}$. Here Ψ_n is any of A_n, A_n^*, $\overline{A}_n$, $\forall\, n \in \mathbb{N}$. By Remark 3.2, see also (3.35), we have that*

$$\left\| \Psi_n^r(f) - f \right\|_\infty \leq r \left\| \Psi_n(f) - f \right\|_\infty, \quad \forall\, n \in \mathbb{N}. \tag{5.68}$$

Corollary 5.5. *Let $f \in C_B(\mathbb{R})$, $r \in \mathbb{N}$. Then*

$$\left\| A_n^r(f) - f \right\|_\infty \leq r \left\| A_n(f) - f \right\|_\infty \leq 2r\omega_1(f, \rho_{1n}(1)) < +\infty, \quad \forall\, n \in \mathbb{N}. \tag{5.69}$$

If f is also uniformly continuous, we get that $\lim_{n\to+\infty} A_n^r(f) = f$, pointwise and uniformly. And the speed of convergence of A_n^r to the unit operator is not worse than of A_n to the unit.

Proof. By (5.64) and (5.68). □

Corollary 5.6. *Let $f \in C_B(\mathbb{R})$, $r \in \mathbb{N}$. Then*

$$\begin{aligned} \left\{ \begin{array}{c} \left\| A_n^{*^r}(f) - f \right\|_\infty \\ \left\| \overline{A}_n^r(f) - f \right\|_\infty \end{array} \right\} &\leq r \left\{ \begin{array}{c} \left\| A_n^*(f) - f \right\|_\infty \\ \left\| \overline{A}_n(f) - f \right\|_\infty \end{array} \right\} \\ &\leq 2r\omega_1(f, \rho_{2n}(1)) < +\infty, \quad \forall\, n \in \mathbb{N}. \end{aligned} \tag{5.70}$$

If f is also uniformly continuous, we get that $\lim_{n\to+\infty} A_n^{*^r}(f) = \lim_{n\to+\infty} \overline{A}_n^r(f) = f$, pointwise and uniformly. And the speed of convergence of $A_n^{*^r}$, $\overline{A}_n^r$ to the unit operator is not worse than of A_n^*, $\overline{A}_n$ to the unit, respectively.

Proof. By (5.65) and (5.68). □

We finish with more general iterated approximation results.

Remark 5.5. *Let $f \in C_B(\mathbb{R})$, $r \in \mathbb{N}$, and $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$. Here Ψ_{m_i} is any of A_{m_i}, $A_{m_i}^*$, $\overline{A}_{m_i}$. By Remark 3.2, see also (3.37), we have that*

$$\left\| \Psi_{m_r}\left(\Psi_{m_{r-1}}\left(...\Psi_{m_2}\left(\Psi_{m_1} f \right) \right) \right) - f \right\|_\infty \leq \sum_{i=1}^{r} \left\| \Psi_{m_i} f - f \right\|_\infty. \tag{5.71}$$

Corollary 5.7. *Let $f \in C_B(\mathbb{R})$, $r \in \mathbb{N}$, and $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$. Then*

$$\begin{aligned} \left\| A_{m_r}\left(A_{m_{r-1}}\left(...A_{m_2}\left(A_{m_1} f \right) \right) \right) - f \right\|_\infty &\leq \sum_{i=1}^{r} \left\| A_{m_i} f - f \right\|_\infty \leq 2 \left(\sum_{i=1}^{r} \omega_1\left(f, \rho_{1m_i}(1)\right) \right) \\ &\leq 2r\omega_1\left(f, \rho_{1m_1}(1)\right). \end{aligned}$$

The speed of convergence to the unit operator of the above activated multiply iterated operators is not worse than the speed of convergence to the unit of A_{m_1}.

Proof. By (5.71), (5.64) and (5.58). ☐

Corollary 5.8. *Let $f \in C_B(\mathbb{R})$, $r \in \mathbb{N}$, and $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$. Then*

$$\left\| A^*_{m_r}\left(A^*_{m_{r-1}}\left(...A^*_{m_2}\left(A^*_{m_1}f\right)\right)\right) - f\right\|_\infty \leq \sum_{i=1}^{r} \left\|A^*_{m_i}f - f\right\|_\infty \leq 2\left(\sum_{i=1}^{r}\omega_1\left(f, \rho_{2m_i}(1)\right)\right) \leq 2r\omega_1\left(f, \rho_{2m_1}(1)\right),$$

and

$$\left\|\overline{A}_{m_r}\left(\overline{A}_{m_{r-1}}\left(...\overline{A}_{m_2}\left(\overline{A}_{m_1}f\right)\right)\right) - f\right\|_\infty \leq \sum_{i=1}^{r}\left\|\overline{A}_{m_i}f - f\right\|_\infty \leq 2\left(\sum_{i=1}^{r}\omega_1\left(f, \rho_{2m_i}(1)\right)\right) \leq 2r\omega_1\left(f, \rho_{2m_1}(1)\right).$$

The speed of convergence to the unit operator of the above activated multiply iterated operators is not worse than the speed of convergence to the unit of $A^*_{m_1}$, $\overline{A}_{m_1}$, respectively.

Proof. By (5.71), (5.70) and (5.59). ☐

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