MODERN MATHEMATICAL METHODS
3 (2025), No. 2, pp. 85-91
https://modernmathmeth.com/
ISSN 3023 - 5294

Research Article

On the general solution and stability of the functional equation $f(x-y) - f(x) f(y) = d \sin x \sin y$

JIRAPORN KUNRATTANAWORAWONG AND WUTIPHOL SINTUNAVARAT*

ABSTRACT. This paper is concerned with the investigation of the general solution to the following functional equation:

$$f(x-y) - f(x) f(y) = d \sin x \sin y$$

for all $x, y \in \mathbb{R}$, where $f : \mathbb{R} \to \mathbb{R}$ is an unknown function and $d \in \mathbb{R} \setminus \{0\}$ satisfying $d < 1$. This nonlinear functional equation establishes an intriguing interplay between multiplicative and additive behaviors of the function f, perturbed by a bounded trigonometric term. We provide a complete characterization of all real-valued functions satisfying the equation, under minimal regularity assumptions. In addition, we analyze the Hyers–Ulam stability and the so-called superstability of the equation in the sense of functional equations, establishing that approximate solutions under certain bounded perturbations necessarily converge to exact solutions.

Keywords: Functional equations, stability, trigonometric identity.

2020 Mathematics Subject Classification: 65Q20, 97I70.

1. INTRODUCTION

Functional equations constitute an important branch of mathematics. The theory of functional equations is typically divided into two main areas. The first concerns the determination of exact solutions to a given functional equation, while the second focuses on the study of the stability of functional equations. The concept of stability arises from the following classical question: Under what conditions does a function that approximately satisfies a functional equation ρ necessarily remain close to an exact solution of ρ? If the problem admits a solution, we say that the equation ρ is stable.

The aforementioned question has given rise to various stability concepts, among which Ulam [1] formulated the first notable problem in 1940 in the context of group homomorphisms. Formally, Ulam's problem can be stated as follows: Let G_1 be a group and let G_2 be a metric group with a metric d. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x) h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$?

In 1941, Hyers [2] provided a partial solution to Ulam's problem for the additive Cauchy functional equation in Banach spaces, stated as follows:

Received: 31.05.2025; Accepted: 13.08.2025; Published Online: 20.08.2025
*Corresponding author: Wutiphol Sintunavarat; wutiphol@mathstat.sci.tu.ac.th
DOI: 10.64700/mmm.67

Theorem 1.1 ([2]). *Let E_1 and E_2 be two Banach space and suppose that the mapping $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E_1$, where $\delta > 0$. Then there is a unique additive mapping $T : E_1 \to E_2$ satisfying

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E_1$.

The stability result of Hyers [2] was subsequently refined by Aoki [3], who established the case involving the existence of a unique additive Cauchy mapping. Building upon this foundation, Rassias [4] significantly broadened Hyers' theorem by introducing the concept of an unbounded Cauchy difference endowed with a p-order norm for $0 < p < 1$. In turn, the Rassias theorem was further generalized by Gajda [5] for the case $p > 1$, and by Rassias [6] for the case $p < 1$.

Over the past several decades, considerable effort has been devoted to investigating solutions and stability phenomena for a wide range of functional equations. Among these contributions, Butler [7] in 2003 posed the following intriguing problem concerning the solution of a specific functional equation: prove that, for $d < -1$, there exist exactly two functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) - f(x)f(y) = d \sin x \sin y \tag{1.1}$$

for all $x, y \in \mathbb{R}$. This problem was subsequently resolved by Rassias [8] in 2004, who proved that the general solution is given by

$$f(x) = \pm c \sin x + \cos x$$

for all $x \in \mathbb{R}$, where $c = \sqrt{-d-1}$. More recently, Jung et al. [9] have obtained notable results on the superstability and stability of the functional equation (1.1).

Motivated by the aforementioned perspective, this work introduces a new functional equation, closely related to (1.1), given by

$$f(x-y) - f(x)f(y) = d \sin x \sin y \tag{1.2}$$

for all $x, y \in \mathbb{R}$, where $f : \mathbb{R} \to \mathbb{R}$ is an unknown function and d is a nonzero real constant with $d < 1$. The general solution of (1.2) is derived in the next section, followed by a detailed analysis of its superstability and stability properties in Section 3.

2. General Solutions of Functional Equation (1.2)

In this section, we establish the complete characterization of the solutions to the functional equation (1.2), as formalized in the theorem below.

Theorem 2.2. *Let d be a nonzero constant with $d < 1$. The functional equation (1.2) has exactly two solutions in the class of functions $f : \mathbb{R} \to \mathbb{R}$. More precisely, if a function $f : \mathbb{R} \to \mathbb{R}$ is a solution of functional equation (1.2), then f has one of the forms*

$$f(x) = c \sin x + \cos x \quad \textit{and} \quad f(x) = -c \sin x + \cos x,$$

where $c = \sqrt{1-d}$.

Proof. It follows from (1.2) that

$$\begin{aligned} & df(x) \sin y \sin z + d \sin x \sin (y-z) - d \sin z \sin (y-x) - df(z) \sin y \sin x \\ & = f(x)[f(y-z) - f(y) f(z)] + [f(x-y+z) - f(x) f(y-z)] \\ & - [f(z-y+x) - f(z) f(y-x)] - f(z)[f(y-x) - f(y) f(x)] \\ & = 0 \end{aligned} \tag{2.3}$$

for all $x, y, z \in \mathbb{R}$. If we set $y = z = \frac{\pi}{2}$ in the above equality, then

$$df(x) - d \sin\left(\frac{\pi}{2} - x\right) - df\left(\frac{\pi}{2}\right) \sin x = 0$$
$$df(x) - d \cos(x) - df\left(\frac{\pi}{2}\right) \sin x = 0$$
$$f(x) - f\left(\frac{\pi}{2}\right) \sin x - \cos x = 0 \tag{2.4}$$

for all $x \in \mathbb{R}$. Substituting 0 for x in (2.4) yields $f(0) = 1$. If we put $x = y = \frac{\pi}{2}$ in (1.2), then we obtain

$$\left[f\left(\frac{\pi}{2}\right)\right]^2 = 1 - d \tag{2.5}$$

and hence

$$f\left(\frac{\pi}{2}\right) = c \quad \text{or} \quad f\left(\frac{\pi}{2}\right) = -c, \tag{2.6}$$

where $c := \sqrt{1-d}$. Consequently, by (2.4), we have

$$f(x) = c \sin x + \cos x \quad \text{or} \quad f(x) = -c \sin x + \cos x \tag{2.7}$$

for all $x \in \mathbb{R}$, where each expression satisfies (1.2). □

3. Stability and Superstability Results for (1.2)

First, we prove a theorem concerning the superstability of the functional equation

$$f(x-y) = f(x) f(y)$$

for all $x, y \in \mathbb{R}$, where $f : \mathbb{R} \to \mathbb{C}$ is an unknown function.

Theorem 3.3. *Let an arbitrary constant $\varepsilon > 0$ be fixed. If a function $f : \mathbb{R} \to \mathbb{C}$ satisfies the inequality*

$$|f(x-y) - f(x) f(y)| \le \varepsilon \tag{3.8}$$

for all $x, y \in \mathbb{R}$, then either for each $x \in \mathbb{R}$, $|f(x)| \le \frac{1+\sqrt{1+4\varepsilon}}{2}$ or $|f(-x)| \le \frac{1+\sqrt{1+4\varepsilon}}{2}$ or $f(x-y) = f(x) f(y)$ for all $x, y \in \mathbb{R}$.

Proof. If we set $\delta := \frac{1+\sqrt{1+4\varepsilon}}{2}$, then $\delta^2 - \delta = \varepsilon$ and $\delta > 1$. Suppose that there exists an $a \in \mathbb{R}$ such that $|f(a)| > \delta$ and $|f(-a)| > \delta$, say $|f(a)| = \delta + p$ and $|f(-a)| = \delta + q$ for some $p, q > 0$. It follows from (3.8) that

$$\begin{aligned} |f(2a)| & = |f(a) f(-a) - f(a) f(-a) + f(2a)| \\ & \ge |f(a) f(-a)| - |f(a) f(-a) - f(2a)| \\ & \ge (\delta + p)(\delta + q) - \varepsilon \\ & = \delta + (p+q)\delta + pq \\ & > \delta + (p+q). \end{aligned}$$

By using the same process, we get $|f(-2a)| > \delta + (p+q)$.

Now, we will show that

$$|f(2^n a)| > \delta + 2^{n-1}(p+q) \quad \text{and} \quad |f(-2^n a)| > \delta + 2^{n-1}(p+q) \tag{3.9}$$

for all $n \in \mathbb{N}$. To claim the above fact by using mathematical induction, we suppose that (3.9) holds for some $n \in \mathbb{N}$. From (3.8), we get

$$\begin{aligned} \left|f\left(2^{n+1}a\right)\right| &= \left|f(2^n a) f(-2^n a) - f(2^n a) f(-2^n a) + f\left(2^{n+1}a\right)\right| \\ &\geq |f(2^n a) f(-2^n a)| - \varepsilon \\ &> \left(\delta + 2^{n-1}(p+q)\right)^2 - \left(\delta^2 - \delta\right) \\ &> \delta + 2^n (p+q). \end{aligned}$$

Similarly, we get

$$\left|f\left(-2^{n+1}a\right)\right| > \delta + 2^n (p+q).$$

Hence, (3.9) is established for all $n \in \mathbb{N}$ via mathematical induction. For every $x, y, z \in \mathbb{R}$, it follows from (3.8) that

$$|f(z - x + y) - f(z) f(x - y)| \leq \varepsilon$$

and

$$|f(y - x + z) - f(y) f(x - z)| \leq \varepsilon.$$

This yields that

$$|f(z) f(x - y) - f(y) f(x - z)| \leq 2\varepsilon$$

for all $x, y, z \in \mathbb{R}$. Hence,

$$\begin{aligned} |f(z) f(x - y) - f(x) f(y) f(z)| &\leq |f(z) f(x - y) - f(y) f(x - z)| \\ &\quad + |f(y) f(x - z) - f(x) f(y) f(z)| \\ &\leq 2\varepsilon + |f(y)| \varepsilon, \end{aligned}$$

that is,

$$|f(x - y) - f(x) f(y)| \, |f(z)| \leq 2\varepsilon + |f(y)| \varepsilon$$

for all $x, y, z \in \mathbb{R}$. In particular,

$$|f(x - y) - f(x) f(y)| \leq \frac{2\varepsilon + |f(y)| \varepsilon}{|f(2^n a)|} \tag{3.10}$$

for all $x, y \in \mathbb{R}$ and any $n \in \mathbb{N}$. Letting $n \to \infty$ in (3.10) and considering (3.9), we conclude that $f(x - y) = f(x) f(y)$ for all $x, y \in \mathbb{R}$. □

From the above result, we obtain the following theorem.

Theorem 3.4. *Let d be a nonzero constant with $d < 1$ and ε be a real constant with $0 < \varepsilon < |d|$. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following inequality*

$$|f(x - y) - f(x) f(y) - d \sin x \sin y| \leq \varepsilon \tag{3.11}$$

for all $x, y \in \mathbb{R}$, then for each $x \in \mathbb{R}$, we have

$$|f(x)| \leq \frac{1 + \sqrt{1 + 8|d|}}{2} \quad \text{or} \quad |f(-x)| \leq \frac{1 + \sqrt{1 + 8|d|}}{2}. \tag{3.12}$$

Proof. As $0<\varepsilon<|d|$, it follows from (3.11) that

$$|f(x-y)-f(x)f(y)|\le 2|d|$$

for all $x,y\in\mathbb{R}$. According to Theorem 3.3, $f(x-y)=f(x)f(y)$ for all $x,y\in\mathbb{R}$ or (3.12) holds. If (3.12) does not hold, then it would follow from (3.11) that $|d\sin x\sin y|\le\varepsilon$ for all $x,y\in\mathbb{R}$, which is contrary to our hypothesis, $\varepsilon<|d|$. Therefore, for each $x\in\mathbb{R}$, we get

$$|f(x)|\le\frac{1+\sqrt{1+8|d|}}{2}\quad\text{or}\quad |f(-x)|\le\frac{1+\sqrt{1+8|d|}}{2}.$$

□

Next, we shall prove the stability of Equation (1.2). We start with the following lemma.

Lemma 3.1. *Let d be a nonzero constant with $d<1$ and ε be a real constant with $0<\varepsilon<|d|$. If a function $f:\mathbb{R}\to\mathbb{R}$ satisfies the inequality (3.11) for all $x,y\in\mathbb{R}$, then f is bounded.*

Proof. If we replace x,y by z and $y-x$ in (3.11) respectively, then we have

$$|f(z-y+x)-f(z)f(y-x)-d\sin z\sin(y-x)|\le\varepsilon \tag{3.13}$$

for all $x,y,z\in\mathbb{R}$. If we replace y by $y-z$ in (3.11), then we get

$$|f(x-y+z)-f(x)f(y-z)-d\sin x\sin(y-z)|\le\varepsilon \tag{3.14}$$

for all $x,y,z\in\mathbb{R}$. Using (3.13) and (3.14), we obtain

$$\begin{aligned}
&|f(x)f(y-z)-f(z)f(y-x)+d\sin x\sin(y-z)-d\sin z\sin(y-x)|\\
&=|[f(z-y+x)-f(z)f(y-x)-d\sin z\sin(y-x)]\\
&-[f(x-y+z)-f(x)f(y-z)-d\sin x\sin(y-z)]|\\
&=|f(z-y+x)-f(z)f(y-x)-d\sin z\sin(y-x)|\\
&+|f(x-y+z)-f(x)f(y-z)-d\sin x\sin(y-z)|\\
&\le 2\varepsilon
\end{aligned}\tag{3.15}$$

for all $x,y,z\in\mathbb{R}$. It follows from (3.15) that

$$\begin{aligned}
&|f(x)[f(y-z)-f(y)f(z)-d\sin y\sin z]+f(x)f(y)f(z)+df(x)\sin y\sin z\\
&-f(z)[f(y-x)-f(x)f(y)-d\sin x\sin y]-f(x)f(y)f(z)-df(z)\sin x\sin y\\
&+d\sin x\sin(y-z)-d\sin z\sin(y-x)|\\
&=|f(x)f(y-z)-f(z)f(y-x)+d\sin x\sin(y-z)-d\sin z\sin(y-x)|\\
&\le 2\varepsilon
\end{aligned}\tag{3.16}$$

for all $x, y, z \in \mathbb{R}$. Hence, in view of (3.11) and (3.16), we can now get

$$\begin{aligned}
&|df(x)\sin y \sin z + d\sin x \sin(y-z) - df(z)\sin x \sin y - d\sin z \sin(y-x)| \\
&= |f(x)[f(y-z) - f(y)f(z) - d\sin y \sin z] + f(x)f(y)f(z) + df(x)\sin y \sin z \\
&- [f(y-x) - f(x)f(y) - d\sin x \sin y] f(z) - f(x)f(y)f(z) - df(z)\sin x \sin y \\
&+ d\sin x \sin(y-z) - d\sin z \sin(y-x) - f(x)[f(y-z) - f(y)f(z) - d\sin y \sin z] \\
&+ [f(y-x) - f(x)f(y) - d\sin x \sin y] f(z)| \\
&\leq |f(x)[f(y-z) - f(y)f(z) - d\sin y \sin z] + f(x)f(y)f(z) + df(x)\sin y \sin z \\
&- [f(y-x) - f(x)f(y) - d\sin x \sin y] f(z) - f(x)f(y)f(z) - df(z)\sin x \sin y \\
&+ d\sin x \sin(y-z) - d\sin z \sin(y-x)| + |f(x)|\,|[f(y-z) - f(y)f(z) - d\sin y \sin z]| \\
&+ |[f(y-x) - f(x)f(y) - d\sin x \sin y]|\,|f(z)| \\
&\leq |f(x)f(y-z) - f(z)f(y-x) + d\sin x \sin(y-z) - d\sin z \sin(y-x)| + |f(x)|\varepsilon + |f(z)|\varepsilon \\
&= |[f(x-y+z) - f(z)f(y-x) - d\sin z \sin(y-x)] \\
&- [f(x-y+z) - f(x)f(y-z) - d\sin x \sin(y-z)]| \\
&+ |f(x)|\varepsilon + |f(z)|\varepsilon \\
&\leq 2\varepsilon + |f(x)|\varepsilon + |f(z)|\varepsilon \\
&\leq (2 + |f(x)| + |f(z)|)\varepsilon
\end{aligned}$$

for all $x, y, z \in \mathbb{R}$. If we set $y = z = \frac{\pi}{2}$ in the above inequality, then

$$\left| df(x) - df\left(\frac{\pi}{2}\right)\sin x - d\cos x \right| \leq \left(2 + |f(x)| + \left| f\left(\frac{\pi}{2}\right) \right| \right)\varepsilon \tag{3.17}$$

for all $x \in \mathbb{R}$.

Next, we assume that f is unbounded. Then there exists a sequence $\{x_n\} \subseteq \mathbb{R}$ such that $f(x_n) \neq 0$ for every $n \in \mathbb{N}$ and $|f(x_n)| \to \infty$ as $n \to \infty$. Replacing x by $\{x_n\}$ in (3.17), and dividing both sides of the resulting inequality by $|f(x_n)|$, and then let n close to infinity, we obtain $|d| \leq \varepsilon$ which is contrary to our hypothesis, say $\varepsilon < |d|$. Therefore, f must be bounded. □

Theorem 3.5. *Let d be a nonzero constant with $d < 1$ and ε be a real constant with $0 < \varepsilon < |d|$. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality (3.11) for all $x, y \in \mathbb{R}$, then*

$$\left| f(x) - f\left(\frac{\pi}{2}\right)\sin x - \cos x \right| \leq \frac{2(1 + M_f)}{|d|}\varepsilon$$

for all $x \in \mathbb{R}$, where $M_f := \sup_{x\in\mathbb{R}} |f(x)|$.

Proof. It follows from Lemma 3.1 that f is bounded, and hence, $M_f := \sup_{x\in\mathbb{R}} |f(x)|$ has to be finite. From the proof of Lemma 3.1, we obtain (3.17) holds. It follows from (3.17) that

$$\left| f(x) - f\left(\frac{\pi}{2}\right)\sin x - \cos x \right| \leq \frac{2(1 + M_f)}{|d|}\varepsilon \tag{3.18}$$

for all $x \in \mathbb{R}$. □

Remark 3.1. *Based on Theorems 2.2 and 3.4, it has been established that the newly proposed functional equation (1.2) is stable.*

4. Conclusion

To summarize, this paper has determined the general solution $f : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$f(x-y) - f(x) f(y) = d \sin x \sin y$$

for all $x, y \in \mathbb{R}$, where d is a nonzero constant with $d < 1$. The solution is given by

$$f(x) = c \sin x + \cos x \quad \text{or} \quad f(x) = -c \sin x + \cos x,$$

where $c = \sqrt{1-d}$. In addition, the stability of this functional equation has been examined through a theorem addressing its superstability. Specifically, it has been established that for a nonzero constant d with $d < 1$ and a real constant ε satisfying $0 < \varepsilon < |d|$, if a function $f : \mathbb{R} \to \mathbb{R}$ fulfills the functional inequality

$$|f(x-y) - f(x) f(y) - d \sin x \sin y| \leq \varepsilon$$

for all $x, y \in \mathbb{R}$, then for each $x \in \mathbb{R}$, we have

$$|f(x)| \leq \frac{1 + \sqrt{1 + 8|d|}}{2} \quad \text{or} \quad |f(-x)| \leq \frac{1 + \sqrt{1 + 8|d|}}{2},$$

and the following estimate holds

$$\left| f(x) - f\left(\frac{\pi}{2}\right) \sin x - \cos x \right| \leq \frac{2(1 + M_f)}{|d|} \varepsilon,$$

where $M_f := \sup_{x \in \mathbb{R}} |f(x)|$.

Acknowledgments

The authors are thankful to the referees and the editor for their comments and suggestions, which substantially improve the initial version of the paper.

Financial disclosure. This work was supported by Thammasat University Research Unit in Fixed Points and Optimization.

References

[1] S. M. Ulam: *Problems in Modern Mathematics*, Science Editions, Wiley, New York (1964), Chapter VI.
[2] D. H. Hyers: *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA, **27** (1941), 222–224.
[3] T. Aoki: *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Jpn., **2** (1950), 64–66.
[4] M. Th. Rassias: *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc., **72** (1978), 297–300.
[5] Z. Gajda: *On stability of additive mappings*, Int. J. Math. Math. Sci., **14** (3) (1991), 431–434.
[6] M. Th. Rassias: *On a modified Hyers–Ulam sequence*, J. Math. Anal. Appl., **158** (1) (1991), 106–113.
[7] S. Butler: *Problem no. 11030*, Am. Math. Mon., **110** (2003), 637–639.
[8] M. Th. Rassias: *Solution of a functional equation problem of Steven Butler*, Octogon Math. Mag., **12** (2004), 152–153.
[9] S. M. Jung, M. Th. Rassias and C. Mortici: *On a functional equation of trigonometric type*, Appl. Math. Comput., **252** (2015), 293–303.

Jiraporn Kunrattanaworawong
Vajiravudh College
Ratchawithi Road, 10300, Bangkok, Thailand
Email address: jiraporn.kunr@gmail.com

Wutiphol Sintunavarat
Thammasat University
Faculty of Science and Technology
Department of Mathematics and Statistics
Phaholyothin Road, 12120, Pathum Thani, Thailand
Email address: wutiphol@mathstat.sci.tu.ac.th