

A Browder-Petryshyn coincidence point theorem

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ABSTRACT. Let C be a subset of a Hilbert space, and let f and g be self-maps on C such that the range of f is a convex, closed, and bounded subset of the range of g . If f does not increase distances more than g , we demonstrate that f and g have coincidence points. This result generalizes a fixed point theorem of Browder-Petryshyn and provide a new result for certain firmly nonexpansive-type mappings. As applications, we establish the existence of solutions to both matrix and integral equations.

Keywords: Coincidence point, matrix equation, integral equation.

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1. INTRODUCTION AND PRELIMINARIES

The study of nonexpansive mappings, which are mappings that do not increase the distance between points, has been a central topic in fixed point theory since the mid-20th century. The Banach Contraction Principle [4] provides a foundational result for contractive mappings; however, nonexpansive mappings do not generally guarantee the existence of fixed points under the same conditions. Major improvements were achieved independently by Browder [8], Göhde [10], and Kirk [13], who established fixed point theorems for nonexpansive self-maps in uniformly convex Banach spaces, showing that such maps admit fixed points in closed, convex, and bounded subsets. These results marked a turning point and have since been extended to broader contexts, such as hyperconvex metric spaces (see Aronszajn and Panitchpakdi [3]) CAT(0) spaces (see Bridson and Haefliger [6]), and more general topological vector spaces. More recently, coincidence point theory, which initiated by Jungck [12], has provided a framework to investigate when two mappings share a coincidence point, enriching fixed point theory and leading to numerous extensions of classical results (see, for example, [1, 2, 5]).

Many important real-world problems can be framed as finding fixed or coincidence points of certain mappings. For instance, the existence of solutions to problems in semi-definite programming [14], digital signal processing [17] or fractional differential equations [18] frequently involves exploring the existence of coincidence points of matrix or integral equations. For more recent references on applications, see, for example, the study of convergence of the viscosity generalization of Halpern's iteration [16], the application to delay differential equations with finite constant delays [15], and applications in split feasibility problems [11]. When the mapping under consideration is nonexpansive, the associated problem can be resolved in Hilbert spaces via the Browder-Petryshyn fixed point theorem [7].

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In this work, we extend the Browder-Petryshyn theorem by exploring the existence of coincidence points for two self-mappings f and g defined on a suitable set C , under the condition that f does not increase distances more than g , or equivalently, that f is g -nonexpansive or g is f -expansive, meaning that

$$\|f(u) - f(v)\| \leq \|g(u) - g(v)\| \text{ for all } u, v \in C,$$

without requiring compactness or commutativity assumptions. Moreover, we provide a new result for firmly g -nonexpansive mappings, defined subsequently. As the first application, we establish the existence of a Hermitian solution to a matrix equation. As the second application, we investigate the existence of a solution to an integral equation. Before concluding the introduction, we recall the fixed point theorems of Browder and Petryshyn, and of Al-Thagafi and Shahzad.

Theorem 1.1 (Browder and Petryshyn [7]). *Let C be a closed, bounded, convex subset of a Hilbert space, and let $f: C \rightarrow C$ be a nonexpansive mapping, that is,*

$$\|f(u) - f(v)\| \leq \|u - v\| \text{ for all } u, v \in C.$$

Then f has a fixed point in C .

Theorem 1.2 (Al-Thagafi and Shahzad [2]). *Let (X, d) be a metric space and $f, g: X \rightarrow X$ be given mappings such that the closure of $f(X)$ is complete and subset of $g(X)$. If there exists $\lambda < 1$ such that*

$$d(f(x), f(y)) \leq \lambda d(g(x), g(y)) \text{ for all } x, y \in X,$$

then f and g have a coincidence point in X .

Remark 1.1. *Additional insight into the uniqueness of the coincidence point is provided by [2, Theorem 2.1].*

2. THE MAIN RESULT

The main result is the following theorem.

Theorem 2.3. *Let C be a subset of a Hilbert space and let $f, g: C \rightarrow C$ be given maps such that $f(C)$ is closed, convex and bounded subset of $g(C)$, and f is g -nonexpansive. Then f and g have a coincidence point in C .*

Proof. Since $f(C)$ is convex, then for a fixed element v_0 of C and for every $\lambda \in (0, 1)$, the map $h_\lambda: C \rightarrow C$ given by

$$h_\lambda(x) = \lambda f(x) + (1 - \lambda)f(v_0),$$

is well defined. We also have

$$\|h_\lambda(x) - h_\lambda(y)\| = \lambda \|f(x) - f(y)\| \leq \lambda \|g(x) - g(y)\|.$$

According to Theorem 1.2, h_λ and g have a coincidence point $u_\lambda \in C$. Since $f(C)$ is closed, convex and bounded in a Hilbert space, it is weakly compact. Hence, we may find a sequence $\lambda_j \rightarrow 1$ as $j \rightarrow \infty$ such that $g(u_{\lambda_j}) = g(u_j) = h_{\lambda_j}(u_j) = f(w_j)$ converges weakly to an element y_0 of the Hilbert space, where $\{w_j\}$ is a sequence of C . Now, since $f(C)$ is closed, $y_0 \in f(C) \subseteq g(C)$, so there exists $x_0 \in C$ such that $y_0 = g(x_0)$. We shall prove that x_0 is a coincidence point of f and g . Let u be any point in C , then

$$\begin{aligned} \|g(u_j) - g(u)\|^2 &= \|(g(u_j) - g(x_0)) + (g(x_0) - g(u))\|^2 \\ &= \|g(u_j) - g(x_0)\|^2 + \|g(x_0) - g(u)\|^2 \\ &\quad + 2(g(u_j) - g(x_0), g(x_0) - g(u)), \end{aligned}$$

where $(g(u_j) - g(x_0), g(x_0) - g(u)) \rightarrow 0$ as $j \rightarrow \infty$, since $g(u_j) - g(x_0)$ converges weakly to 0. Now, since $f(C) \subseteq g(C)$, we can take u such that $g(u) = f(x_0)$, we deduce

$$\lim_{j \rightarrow \infty} (\|g(u_j) - f(x_0)\|^2 - \|g(u_j) - g(x_0)\|^2) = \|g(x_0) - f(x_0)\|^2.$$

Now, since f is g -nonexpansive, we have

$$\|f(u_j) - f(x_0)\| \leq \|g(u_j) - g(x_0)\|.$$

Thus, we obtain

$$\begin{aligned} \|g(u_j) - f(x_0)\| &\leq \|g(u_j) - f(u_j)\| + \|f(u_j) - f(x_0)\| \\ &\leq \|g(u_j) - f(u_j)\| + \|g(u_j) - g(x_0)\|. \end{aligned}$$

Observe that since $s_j \rightarrow 1$, then

$$\begin{aligned} f(u_j) - g(u_j) &= \lambda_j f(u_j) + (1 - \lambda_j) f(v_0) - g(u_j) + (1 - \lambda_j) (f(u_j) - f(v_0)) \\ &= h_{\lambda_j} (u_j) - g(u_j) + (1 - \lambda_j) (f(u_j) - f(v_0)) \\ &= (1 - \lambda_j) (f(u_j) - f(v_0)). \end{aligned}$$

Thus, $\limsup_{j \rightarrow \infty} (\|g(u_j) - f(x_0)\| - \|g(u_j) - g(x_0)\|) \leq 0$, and therefore

$$\limsup_{j \rightarrow \infty} (\|g(u_j) - f(x_0)\|^2 - \|g(u_j) - g(x_0)\|^2) \leq 0.$$

We conclude that $\|g(x_0) - f(x_0)\|^2 = 0$, so $f(x_0) = g(x_0)$. $\square$

Remark 2.2. Note that since $f(C) \subseteq g(C)$, Theorem 2.3 holds true even if the convexity of $f(C)$ is replaced by this of $g(C)$.

The following corollaries follow immediately.

Corollary 2.1. Let C be a subset of a Hilbert space, and let $f, g: C \rightarrow C$ be given maps such that $f(C)$ is closed, convex and bounded, and g is surjective and f -expansive. Then f and g have a coincidence point in C .

Corollary 2.2. Let C be a convex, closed and bounded subset of a Hilbert space, and let $f, g: C \rightarrow C$ be given surjective maps such that f is g -nonexpansive. Then f and g have a coincidence point in C .

By selecting one of the mappings as the identity map, the following corollaries can be readily derived.

Corollary 2.3. Let C be a closed and bounded subset of a Hilbert space, and $f: C \rightarrow C$ be a map such that $f(C)$ is convex and $\|f(u) - f(v)\| \leq \|u - v\|$ for all $u, v \in C$. Then f has a fixed point in C .

Corollary 2.4. Let C be a closed convex and bounded subset of a Hilbert space and $g: C \rightarrow C$ be a surjective map such that $\|u - v\| \leq \|g(u) - g(v)\|$ for all $u, v \in C$. Then g has a fixed point in C .

Remark 2.3. The Corollary 2.3 extends Theorem 1.1. Since, it requires convexity only for $f(C)$, meaning that C itself need not be convex.

At the end of this section, we study the existence of coincidence point for certain firmly nonexpansive-type mappings. Let C be a convex set, let $f, g: C \rightarrow C$ be a given mapping, and for $x, y \in C$ consider the function $\varphi_{x,y}$ defined by

$$\varphi_{x,y}(t) = \|t(f(x) - f(y)) + (1 - t)(g(x) - g(y))\|, \text{ for all } t \in [0, 1].$$

Definition 2.1. Let C be a convex set, a mapping $f: C \rightarrow C$ is said to be firmly g -nonexpansive if for all $x, y \in C$ the function $\varphi_{x,y}$ is nonincreasing on $[0, 1]$.

Notice that $\varphi_{x,y}$ is a convex function of t . To compute the derivative of $\varphi_{x,y}$ at $t = 1$, set

$$u := g(x) - g(y), \quad v := f(x) - f(y), \quad w := v - u.$$

Then $\varphi_{x,y}(t) = \|u + tw\|$. Whenever $u + tw \neq 0$, the function $\varphi_{x,y}$ is differentiable and

$$\varphi'_{x,y}(t) = \frac{(u + tw, w)}{\|u + tw\|}.$$

In particular, if $v = f(x) - f(y) \neq 0$, then $\varphi_{x,y}$ is differentiable at $t = 1$ and

$$\varphi'_{x,y}(1) = \frac{(v, w)}{\|v\|} = \frac{(f(x) - f(y), (f(x) - f(y)) - (g(x) - g(y)))}{\|f(x) - f(y)\|}.$$

Thus $\varphi'_{x,y}(1) \leq 0$ is equivalent to the numerator being nonpositive

$$(f(x) - f(y), (f(x) - f(y)) - (g(x) - g(y))) \leq 0,$$

or equivalently,

$$(f(x) - f(y), g(x) - g(y)) \geq \|f(x) - f(y)\|^2.$$

Thus, we obtain the following equivalence

$$\varphi'_{x,y}(1) \leq 0 \iff (f(x) - f(y), g(x) - g(y)) \geq \|f(x) - f(y)\|^2.$$

The equivalence remains valid in case $f(x) = f(y)$.

Let us present some examples of firmly g -nonexpansive mapping based on the following definition of projection.

Definition 2.2 (g -projection). Let H be a real Hilbert space and $C \subseteq H$ a nonempty closed convex set. Let $g: H \rightarrow H$ be a mapping satisfying:

- (i) The restriction $g|_C: C \rightarrow g(C)$ is injective (so that the inverse is well-defined on $g(C)$),
- (ii) $g(C)$ is a nonempty closed and convex subset of H .

Then, for every $x \in H$, the g -projection of x onto C is the mapping $P_C^g: H \rightarrow C$ defined by

$$P_C^g(x) := (g|_C)^{-1}(P_{g(C)}(g(x))),$$

where $P_{g(C)}$ denotes the standard metric projection onto the closed convex set $g(C)$.

Remark 2.4. Under (i)–(ii), $P_{g(C)}(g(x))$ exists and is unique for all $x \in H$, and injectivity of $g|_C$ ensures that $P_C^g(x) \in C$ is uniquely defined. Moreover, a point $y = P_C^g(x)$ if and only if it satisfies the variational inequality

$$(2.1) \quad y \in C \quad \text{and} \quad (g(x) - g(y), g(z) - g(y)) \leq 0 \quad \forall z \in C,$$

or equivalently, the minimization problem

$$(2.2) \quad P_C^g(x) = \arg \min_{z \in C} \|g(x) - g(z)\|.$$

Thus, $P_C^g(x)$ is the point in C whose image under g is closest to $g(x)$ in the Hilbert norm.

Definition of g -projection unifies several important notions of projection used in optimization, variational inequalities, and numerical analysis. Below are some examples of firmly g -nonexpansive mappings:

1. **Metric projection.** Take $g = \text{id}_H$, the identity mapping on H . Then g is injective, and $g(C) = C$ is closed and convex. Hence the conditions of the definition are satisfied, and

$$P_C^{\text{id}}(x) = \text{id}^{-1}(P_C(\text{id}(x))) = P_C(x),$$

the classical nearest point projection. Moreover, (2.2) reduces to $\min_{z \in C} \|x - z\|$.

2. **Weighted projection.** Let $M: H \rightarrow H$ be a bounded linear isomorphism (bijective with bounded inverse), self-adjoint, and positive definite, i.e., $(Mx, x) \geq \alpha \|x\|^2$ for all $x \in H$ and some $\alpha > 0$. Define $g(x) = Mx$. Then g is injective and $g(C) = M(C)$ is closed and convex. The g -projection is

$$P_C^g(x) = M^{-1}(P_{M(C)}(Mx)),$$

equivalently solving

$$\min_{z \in C} \|Mx - Mz\| = \min_{z \in C} \|x - z\|_M,$$

where $\|u\|_M := \sqrt{(Mu, u)} = \|M^{1/2}u\|$. This construction is widely used in variable-metric and preconditioned optimization algorithms; see, e.g., [9].

3. **Bregman-type projection.** Let $\phi: H \rightarrow \mathbb{R}$ be strictly convex and Fréchet differentiable, and assume that $\nabla\phi(C)$ is closed and convex. Define $g = \nabla\phi$. Then the g -projection is

$$P_C^g(x) = (\nabla\phi|_C)^{-1}(P_{\nabla\phi(C)}(\nabla\phi(x))) = \arg \min_{z \in C} \|\nabla\phi(x) - \nabla\phi(z)\|.$$

The g -projection satisfies

$$(P_C^g(x) - P_C^g(y), \nabla\phi(x) - \nabla\phi(y)) \geq \|P_C^g(x) - P_C^g(y)\|^2$$

if and only if ϕ is quadratic, that is, $\phi(x) = \frac{1}{2}(Mx, x) + (b, x) + c$ with M is positive definite. In this case, $\nabla\phi(z) - \nabla\phi(x) = M(z - x)$ and the g -projection is a firmly nonexpansive operator in the M -inner product. For general strictly convex ϕ , the g -projection need not be firmly g -nonexpansive.

Next, we have the following result:

Corollary 2.5. Let C be a subset of a Hilbert space and let $f, g: C \rightarrow C$ be given maps such that $h(C)$ is closed, convex and bounded subset of $g(C)$ where $h = 2f - g$, and f is firmly g -nonexpansive. Then f and g have a coincidence point in C .

Proof. For $x, y \in C$,

$$\begin{aligned} \|h(x) - h(y)\|^2 &= \|(2f(x) - g(x)) - (2f(y) - g(y))\|^2 \\ &= \|2(f(x) - f(y)) - (g(x) - g(y))\|^2 \\ &= 4\|f(x) - f(y)\|^2 - 4(f(x) - f(y), g(x) - g(y)) + \|g(x) - g(y)\|^2 \\ &\leq \|g(x) - g(y)\|^2. \end{aligned}$$

Hence, by Theorem 2.3, h and g have a coincidence point, and so f and g . □

Example 2.1. Let $C = [0, 1] \subset \mathbb{R}$. Choose the constants

$$r = \frac{1}{5}, \quad A = \frac{1}{20}, \quad B = \frac{1}{4}.$$

Define the mappings

$$g(x) = \frac{1}{2} + r \cos(2\pi x), \quad \phi(s) = \frac{1}{2}s + A \sin(2\pi s) + B, \quad f(x) = \phi(g(x)).$$

1. Well-definedness: Since $\cos(2\pi x) \in [-1, 1]$, we obtain

$$g(C) = \left[\frac{1}{2} - r, \frac{1}{2} + r\right] = \left[\frac{1}{2} - \frac{1}{5}, \frac{1}{2} + \frac{1}{5}\right] = \left[\frac{3}{10}, \frac{7}{10}\right] \subset C.$$

The derivative of ϕ satisfies

$$\phi'(s) = \frac{1}{2} + 2\pi A \cos(2\pi s) \in \left[\frac{1}{2} - 2\pi A, \frac{1}{2} + 2\pi A\right].$$

Since $2\pi A = \frac{\pi}{10} < \frac{1}{2}$, it follows that

$$0 < \phi'(s) < 1 \quad \forall s \in g(C),$$

so ϕ is strictly increasing. Hence

$$f(C) = \phi(g(C)) \subset [0, 1] = C.$$

Thus $f, g : C \rightarrow C$ are well defined.

2. Firm g -nonexpansiveness: By the mean-value theorem, for any $s, t \in g(C)$ there exists ξ such that

$$\phi(s) - \phi(t) = \phi'(\xi)(s - t).$$

Multiplying by $(s - t)$

$$(\phi(s) - \phi(t))(s - t) = \phi'(\xi)(s - t)^2.$$

Since $0 < \phi'(\xi) < 1$, we have

$$\phi'(\xi)(s - t)^2 \geq (\phi'(\xi))^2(s - t)^2 = (\phi(s) - \phi(t))^2.$$

Thus,

$$(\phi(s) - \phi(t))(s - t) \geq (\phi(s) - \phi(t))^2.$$

Setting $s = g(x)$ and $t = g(y)$, we obtain

$$(f(x) - f(y))(g(x) - g(y)) \geq (f(x) - f(y))^2,$$

so f is firmly g -nonexpansive.

3. The mapping $h = 2f - g$: Using the definition of f ,

$$h(x) = 2\phi(g(x)) - g(x) = 2A \sin(2\pi g(x)) + 2B.$$

Observe that h is continuous and C is an interval, $h(C)$ is a connected interval in $\mathbb{R}$, hence convex. Precisely, since $\sin(2\pi g(x)) \in [-1, 1]$, thus

$$h(C) = [2B - 2A, 2B + 2A] = \left[\frac{1}{2} - \frac{1}{10}, \frac{1}{2} + \frac{1}{10}\right] = \left[\frac{2}{5}, \frac{3}{5}\right].$$

Finally, since

$$h(C) = \left[\frac{2}{5}, \frac{3}{5}\right] \subset \left[\frac{3}{10}, \frac{7}{10}\right] = g(C)$$

the inclusion $h(C) \subset g(C)$ holds. We conclude that the mappings f, g and h satisfy all the hypotheses of Corollary 2.5, and therefore f and g have a coincidence point in C .

3. EXISTENCE OF HERMITIAN SOLUTIONS TO A CLASS OF MATRIX EQUATIONS

Let $\mathcal{M}(n), \mathcal{H}(n), \mathcal{P}(n)$ and $\overline{\mathcal{P}}(n)$ be respectively the sets of all $n \times n$ arbitrary, Hermitian, positive-definite and positive semi-definite matrices. The spectral norm of a matrix A is the largest singular value of A and it is denoted by $\|A\|_2$. The notation $X \leq Y$ means that $Y - X \in \overline{\mathcal{P}}(n)$. If $X, Y \in \mathcal{H}(n)$ such that $X \leq Y$, then the order interval is defined by

$$[X, Y] := (X + \overline{\mathcal{P}}(n)) \cap (Y - \overline{\mathcal{P}}(n)).$$

Then $\mathcal{H}(n)$, endowed with Hilbert-Schmidt inner product $\langle X, Y \rangle = \text{tr}(Y^* X)$ is a real Hilbert space, whereas $\mathcal{H}(n)$ with the spectral norm $\|\cdot\|_2$ is not a Hilbert space, since this norm is not induced by any inner product. The associated norm is the Hilbert-Schmidt norm, which coincides with the Frobenius norm

$$\|X\|_F^2 = \text{tr}(X^* X).$$

It is worthy to note that $\|XY\|_F \leq \|X\|_2 \|Y\|_F$ for all $X, Y \in \mathcal{H}(n)$.

We next provide sufficient conditions for the existence of a Hermitian non trivial solution to the following matrix equation

$$(3.3) \quad (AB)^* XAB = A^* XA,$$

where A and B are $n \times n$ commuting matrices.

Let $M, N \in \overline{\mathcal{P}}(n)$ such that $M \leq N$ ($M \neq N$) and $C = [M, N]$ an order interval. Consider the following assumptions:

$$(A1) \quad M \leq (AB)^* M A B \text{ and } (AB)^* N A B \leq N,$$

$$(A2) \quad M \leq B^* M B \text{ and } B^* N B \leq N,$$

$$(A3) \quad \|B\|_2 = 1.$$

Proposition 3.1. *Under the assumptions (A1)-(A3), the matrix equation (3.3) has a solution in C .*

Proof. Firstly, observe that C is a compact convex set. Take

$$f(X) = (AB)^* X A B \text{ and } g(X) = A^* X A.$$

From (A1) and (A2), we deduce easily that $f, g: C \rightarrow C$ are well defined.

We shall show that $f(C)$ is convex, closed, bounded, and subset of $g(C)$. Clearly, from (A1), we deduce that $g(C)$ is bounded. Let $Y_1, Y_2 \in f(C)$ such that $(Y_1, Y_2) = (f(X_1), f(X_2))$ for some $X_1, X_2 \in C$, we deduce by definition of f and the convexity of C that

$$\begin{aligned} \lambda Y_1 + (1 - \lambda) Y_2 &= \lambda f(X_1) + (1 - \lambda) f(X_2) \\ &= f(\lambda X_1 + (1 - \lambda) X_2) \in f(C), \end{aligned}$$

for all $\lambda \in (0, 1)$ which proves that $f(C)$ is convex. Now, since f is linear between finite-dimensional vector spaces, it is continuous and thus maps the compact C to a compact set, which implies that $f(C)$ is closed and bounded.

We have that $f(C) \subseteq g(C)$, since from the commutativity of A and B , we have for every $X \in C$, $f(X) = g(h(X))$, where $h(X) = B^* X B$ with $h(C) \subset C$ comes from (A2).

Finally, from (A3), we have

$$\begin{aligned} \|f(X) - f(Y)\|_F &= \|B^*(g(X) - g(Y))B\|_F \\ &\leq \|B\|_2^2 \|g(X) - g(Y)\|_F \\ &= \|g(X) - g(Y)\|_F, \end{aligned}$$

for all $X, Y \in C$, which proves that f is g -nonexpansive.

We conclude by Theorem 2.3, that f and g have a coincidence point in C , that is, the matrix equation (3.3) has a solution in C . $\square$

Example 3.2. *Consider the following matrices:*

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \quad B = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}, \\ M &= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \quad N = 2 \begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}. \end{aligned}$$

Since (A1)-(A3) hold, we conclude by Proposition 3.1 that (3.3) has a solution in $C = [M, N]$. Note that $X = M$ is a solution of (3.3) in C .

Consider now the following matrices:

$$A = B = M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Again, since (A1)-(A3) hold, we conclude by Proposition 3.1 that (3.3) has a solution in $C = [M, N]$. Observe that $X = \frac{1}{2}(M + N)$ is a solution of (3.3) in C .

4. EXISTENCE OF SOLUTIONS TO A CLASS OF INTEGRAL EQUATIONS

Let a, b be two real constants. The set of square integrable functions of $L^2([a, b])$ is a Hilbert space endowed with the inner product $\langle f, g \rangle := \int_a^b f(s)g(s)ds$, where f and g are real functions. The L^2 -norm is given by $\|f\|_{L^2}^2 := \langle f, f \rangle$. We study the existence of a nontrivial solution x of the following integral equation:

$$(4.4) \quad \int_a^b K(t, s)x(s) ds = \int_a^b \int_a^b K(t, s)K(s, r)x(r) dr ds,$$

for all $t \in [a, b]$, where K is a continuous non-negative function defined on $[a, b]^2$.

Let

$$\begin{aligned} (fx)(t) &= \int_a^b \int_a^b K(t, s)K(s, r)x(r)dr ds, \\ (gx)(t) &= \int_a^b K(t, s)x(s)ds. \end{aligned}$$

A mapping $h: C \rightarrow C$ is said to be nondecreasing, if $x \leq y$ implies $hx \leq hy$ where the notation $x \leq y$ means $x(t) \leq y(t)$ for all $t \in [a, b]$.

Assume now that there exist two functions $u, v \in C[a, b]$ ($u \neq v$) such that $u \leq v$. Define a set C to be the closure of $C_{u,v}$, where

$$C_{u,v} := \{x \in C[a, b]: u \leq x \leq v\},$$

and consider the following assumptions:

- (B1) $u \leq fu$ and $fv \leq v$,
- (B2) $u \leq gu$ and $gv \leq v$,
- (B3) $\int_a^b \int_a^b K(t, s)^2 ds dt = 1$.

Proposition 4.2. *Under the assumptions (B1)-(B3), the integral equation (4.4) has a solution in C .*

Proof. Firstly, observe that f and g are nondecreasing, then from (B1) and (B2) the maps $f, g: C \rightarrow C$ are well defined.

We now claim that $f(C)$ is closed, convex and bounded. To see this, observe that the set $C_{u,v}$ is convex, then so is its closer which is obviously closed.

Let now $y_1, y_2 \in f(C)$, so there exist $x_1, x_2 \in C$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Let $\lambda \in (0, 1)$, then by definition of g it follows that

$$\begin{aligned} \lambda y_1 + (1 - \lambda)y_2 &= \lambda f(x_1) + (1 - \lambda)f(x_2) \\ &= f(\lambda x_1 + (1 - \lambda)x_2), \end{aligned}$$

which implies $\lambda y_1 + (1 - \lambda)y_2 \in f(C)$, so $f(C)$ is convex. Moreover, from the fact that $u, v \in C[a, b]$, it follows by the extreme value theorem that $C_{u,v}$ is bounded, and so it is its closer. We deduce that our claim holds.

Next, by using the Cauchy-Schwarz inequality and (B3), we get for all $x, y \in C$,

$$\begin{aligned} \|fx - fy\|_{L^2}^2 &= \int_a^b \left| \int_a^b K(t, s) \left(\int_a^b K(s, r)(x(r) - y(r))dr \right) ds \right|^2 dt \\ &\leq \int_a^b \int_a^b K(t, s)^2 ds dt \int_a^b \left| \int_a^b K(s, r)(x(r) - y(r))dr \right|^2 ds \\ &\leq \|gx - gy\|_{L^2}^2, \end{aligned}$$

which implies that f is g -nonexpansive.

Finally, note that $f = g \circ g$, which implies by (B2) that $f(C) \subseteq g(C)$. We conclude the result by Theorem 2.3. $\square$

Example 4.3. Consider the following integral equation

$$(4.5) \quad 8 \int_a^b s x(s) ds = 3 \int_a^b \int_a^b s^2 r x(r) dr ds,$$

where $a = 0$ and $b = 2$ with $K(t, s) = \frac{3}{8}ts$ for all $t, s \in [a, b]$. Clearly, K is non-negative, continuous and satisfies (B3). It is easy to see that the conditions (B1) and (B2) hold for the functions u and v defined by $u(t) = 0$ and $v(t) = \frac{3}{2}t$ for all $t \in [a, b]$. Hence, according to Proposition 4.2 the integral equation (4.5) has a solution in C . It is not difficult to see that the function x defined by $x(t) = \frac{3}{4}t$ for all $t \in [a, b]$ is in C and it is a solution of the integral equation (4.5).

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