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Research Article

Normalized solutions to the fractional Schrödinger equations with potential and saturable nonlinearity

WEN LIAO AND QIONGFEN ZHANG*

ABSTRACT. This paper is concerned with the existence of normalized solutions to a kind of fractional Schrödinger equations driven by a fractional operator with a parametric potential term and a saturable nonlinear term. We achieve the minimization of the energy functional and prove the existence of normalized solutions for the equation under specific conditions that we assume for the potential and saturable nonlinearity.

Keywords: Normalized solutions, Schrödinger equations, saturable nonlinearity, potential function.

2020 Mathematics Subject Classification: 34C37, 35A15, 37J45, 47J30.

1. INTRODUCTION

This paper studies the existence of normalized solutions for the following fractional Schrödinger equations

$$\imath\frac{\partial\varphi}{\partial t} = (-\Delta)^s\varphi + \mu V(x)\varphi - \mu\frac{\psi(x)+\varphi^2}{1+\psi(x)+\varphi^2}\varphi, \ \text{ in } \mathbb{R}^{\mathrm{N}}, \tag{1.1}$$

where $(-\Delta)^s$ is the fractional Laplacian, $\varphi = \varphi(t,x)$, $s \in (0.25, 1)$, $2s < N < 4s$, $\mu > 0$, $\lambda \in \mathbb{R}$ is a parameter, the potential function $V : \mathbb{R}^N \to [0, +\infty)$ is continuous and bounded. The function ψ is bounded in $\mathbb{R}^N$. A solution of the problem (1.1) is called a standing wave solution if it has the form $\varphi(t,x) = \ell^{i\lambda t}$. Indeed, u is a time-independent and real-valued function that satisfies the following fractional Schrödinger equation:

$$(-\Delta)^s u + \mu V(x)u + \lambda u = \mu\frac{\psi(x)+u^2}{1+\psi(x)+u^2}u, \ \text{in } \mathbb{R}^{\mathrm{N}}, \tag{1.2}$$

is defined by

$$(-\Delta)^s u(x) = C(N,s)P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy,$$

here, the symbol $P.V.$ is the Cauchy principal value and $C(N,s)$ is a constant that depends on N and s. For more information, please see [19] and the reference therein. $(-\Delta)^s$ originates from applied scientific fields such as obstacle problems, phase transition phenomena, fractional quantum mechanics, and Markov processes (see [10, 19, 20, 21]). We are examining the feasibility of solving the constraint minimization problem described as below:

$$I_a = \inf_{u\in S(a)} E(u),$$

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*Corresponding author: Qiongfen Zhang; qfzhangcsu@163.com
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where a is a positive constant and

$$S(a) = \{u \in W : \|u\|_2^2 = \int_{\mathbb{R}^N} |u|^2 dx = a^2\}. \tag{1.3}$$

Moreover, the weighted fractional Sobolev space W is defined as

$$W = \{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} \mu V(x)|u|^2 dx < +\infty\}, \tag{1.4}$$

with norm

$$\|u\|_W = \left(\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^N} \mu V(x)|u|^2 dx\right)^{\frac{1}{2}}. \tag{1.5}$$

We will study the energy functional $E : W \to \mathbb{R}$, which is defined by

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \mu V(x)|u|^2 dx - \frac{\mu}{2}\int_{\mathbb{R}^N} \left[u^2 - \ln\left(1 + \frac{u^2}{1+\psi(x)}\right)\right] dx. \tag{1.6}$$

In addition, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is given by:

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^2 dx = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy < \infty\}, \tag{1.7}$$

with norm

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)^{\frac{1}{2}}. \tag{1.8}$$

In recent years, the following fractional Schrödinger equation has been studied by many scholars:

$$(-\Delta)^s u + \mu u + \lambda V(x)u = h(u), \ \text{in } \mathbb{R}^{\mathrm{N}}. \tag{1.9}$$

When $h(u) = |u|^{p-2}u$, Laskin [11] obtained a couple of normalized solution by using a fiber map and concentration-compactness principle. Zuo [25] obtained minimization of the energy functional associated with the problem (1.9). When $h(u) = u \log u^2$, Alves [1, 2] employed minimization techniques and used the Lusternik-Schnirelmann category to prove existence of multiple normalized solutions. The solution to the problem (1.9) can be studied from two aspects. On the one hand, one can choose the fixed frequency $\mu \in \mathbb{R}$ and studied the existence of nontrivial solution of problem (1.9). On the other hand, taking μ as unknown, then we can look to prescribed L^2-norm solutions. A large number of people have conducted comprehensive studies utilizing variational and topological methods, see [5, 6, 12] and the references therein. As we all know, some previous works have explored this aspect, although they did not incorporate a potential term, see [3, 13, 17, 23].

We want to point out that the nonlinearity $f(x,u) = \mu\frac{\psi(x)+u^2}{1+\psi(x)+u^2}u$ for $\mu > 0$ is usually called saturable nonlinear term, which is used to describe photo refractive media [8]. Lin [14] firstly studied normalized solution for the Schrödinger equation with saturable nonlinearity. Later, Lin [15] used a convexity argument to expand the result when $g(x)$ becomes nonzero. Very recently, Sun [22] used variational methods to prove the existence of normalized solutions for a kind of quasilinear Schrödinger equations.

Inspired by the above paper, our objective is to expand the current findings to the fractional Schrödinger equations and investigate the existence of normalized solutions of fractional Schrödinger equations with a saturable nonlinear term $\mu\frac{\psi(x)+u^2}{1+\psi(x)+u^2}u$ and an external potential V. However, due to the presence of the saturable nonlinear term and the potential function, we have to face several difficulties, such as the scaling result of the energy functional $E(\cdot)$ and the strong convergence of the minimizing sequence in the weighted fractional Sobolev space W. In view of this, it is necessary to introduce some inequality and new ideas.

In order to present our main result, we require the following assumptions on $V(x)$ and $\psi(x)$.

(V_1) There exists a positive constant $C_0 > 0$ such that the following measure of the set Ω is finite, where

$$\Omega = \{x \in \mathbb{R}^N : V(x) < C_0\}. \tag{1.10}$$

(V_2) $V \in L^\infty(\mathbb{R})$.

(Ψ_1) The function $\psi(x)$ is radially symmetric with $-1 < \psi_1 \le \psi(|x|) \le \psi_2$ for $x \in \mathbb{R}^N$, where ψ_1, ψ_2 are two constants.

In conclusion, we will now present our primary result.

Theorem 1.1. *Assume that the condition $(V_1), (V_2)$ and (Ψ_1) holds, and let $2s < N < 4s$, $p \in (2, \min\{\frac{N}{N-2s}, 2 + \frac{4s}{N}\})$. Then, for each $a > 0$, there exist a sufficiently large positive $\mu^* > 0$ and $\Upsilon = \Upsilon(\mu_*, a) > 0$ such that $I_a < 0$ for all $\mu > \mu^*$ satisfying $\mu\|V\|_\infty < \Upsilon$. Moreover, the infimum I_a is attained by a function $u \in S(a)$, which is a normalized solution of (1.1) with $\lambda = \lambda_a$ as a Lagrange multiplier.*

2. Auxiliary Lemmas and Proof of Theorem 1.1

In this section, we give out some auxiliary lemmas, and then we give the proof of Theorem 1.1.

Lemma 2.1 ([16], Lemma 2.2)**.** *For each $2 < p \le \min\{4, 2^*\}$ ($2^* = \infty$ if $N = 1, 2$; $2^* = \frac{2N}{N-2}$ if $N \ge 3$), there exists a positive constant*

$$A_p = \begin{cases} \frac{1}{2}, & \text{if } p = 4; \\ \frac{p^{\frac{p-2}{2}}(4-p)^{\frac{4-p}{2}}}{2p}, & \text{if } 2 < p \le \min\{4, 2^*\} \text{ and } p \ne 4, \end{cases}$$

such that

$$t^2 - \ln(1 + \frac{t^2}{1+\psi(x)}) \le \frac{\psi(x)}{1+\psi(x)} t^2 + \frac{A_p}{(1+\psi(x))^{\frac{p}{2}}} t^p \text{ for all } \ge 0. \tag{2.11}$$

Lemma 2.2 ([9], Fractional Gagliardo-Nirenberg Inequality)**.** *For each $p \in (2, \frac{2N}{N-2s})$, there exists a constant $C(s, N, p)$, such that*

$$\int_{\mathbb{R}^N} |u|^p dx \le C(s, N, p)(\int_{\mathbb{R}^N} |u|^2 dx)^{\frac{p}{2} - \frac{N(p-2)}{4s}} (\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^2 dx)^{\frac{N(p-2)}{4s}}. \tag{2.12}$$

Lemma 2.3. *Assuming that condition (Ψ_1) is satisfied, it can be concluded that the energy functional $E(u)$ is both coercive and bounded from below on $S(a)$ for all $a > 0$.*

Proof. For any $u \in S(c)$, observe that condition (Ψ_1) guarantees $1 + \psi(x) \ge 1 + \psi_1(x) > 0$ for all $x \in \mathbb{R}^N$, this make the logarithmic term well defined. And controlling the range of the $\psi(x)$ make the function $\frac{1}{1+\psi(x)}$ bounded. It follow from Lemmas 2.1 and 2.2 that

$$\begin{aligned} E(u) &= \frac{1}{2}\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \mu V(x)|u|^2 dx - \frac{\mu}{2}\int_{\mathbb{R}^N} \left[u^2 - \ln\left(1 + \frac{u^2}{1+\psi(x)}\right)\right] dx \\ &\ge \frac{1}{2}\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^2 dx - C(s, N, p, A_p)\frac{\mu}{2}\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^{\frac{N(p-2)}{4s}} dx - \frac{\mu a^2}{2}. \end{aligned}$$

Since $p \in (2, 2+\frac{4s}{N})$, we infer that $0 < \frac{N(p-2)}{4s} < 1$. We can therefore conclude that the energy functional $E(u)$ is coercive and bounded from below on $S(a)$. We complete the proof. □

Even if it is not explicitly stated, in the following series of lemmas we always make the assumption that $(u_n)_{n\in\mathbb{N}} \subset S(a)$ is a minimizing sequence for I_a.

Lemma 2.4. *The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in W.*

Proof. It can be deduced from the definition of a minimizing sequence that

$$I_a = \lim_{n\to\infty} E(u_n). \tag{2.13}$$

According to Lemma 2.3, it is established that $(\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u_n|^2)_{n\in\mathbb{N}}$ is a bounded sequence for I_a. As a result, the sequence $(\int_{\mathbb{R}^N} \mu V(x)|u_n|^2 dx)_{n\in\mathbb{N}}$ is also bounded. In a word, $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in W. This concludes the proof. □

Lemma 2.5. *Assuming that conditions (V_1) and (V_2) are satisfied, then for any $\mu_2 > \mu > 0$ there exists $\Upsilon = \Upsilon(\mu_2, a) > 0$ such that $I_a < 0$ if $\lambda\|V\|_\infty < \Upsilon$.*

Proof. Let $(u_n) \subset S(a)$ be a minimizing sequence for $I_a = \inf_{u\in S(a)} E(u)$. Then (u_n) is bounded on $S(a)$ by Lemma 2.3. It is evident that the embedding $W(\mathbb{R}^N) \hookrightarrow L^h(\mathbb{R}^N)$ is compact for any $h \in [2, 2_s^*]$ (see Wang et al. [4]). Since there exists $\overline{u} \in H$ such that $u_n \rightharpoonup \overline{u}$ weakly in $W(\mathbb{R}^N)$, $u_n \rightharpoonup \overline{u}$ strongly in $L^h(\mathbb{R}^N)$. According to the Lagrange multipliers rule, there exists $\lambda_n \in \mathbb{R}^N$ such that

$$(-\Delta)^s u_n + \mu V(x) u_n + \lambda_n u_n = \mu \frac{\psi(x) + u_n^2}{1+\psi(x)+u_n^2} u_n, \text{ in } \mathbb{R}^{\mathrm{N}}. \tag{2.14}$$

In particular, we have

$$\begin{aligned}\lambda_n a^2 &= -\frac{1}{2}\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} \mu V(x)|u_n|^2 dx + \int_{\mathbb{R}^N} \mu \frac{\psi(x)+u_n^2}{1+\psi(x)+u_n^2} u_n^2 dx + o(1)\\ &\le \int_{\mathbb{R}^N} \mu \frac{\psi(x)+u_n^2}{1+\psi(x)+u_n^2} u_n^2 dx + o(1) < \mu a + o(1),\end{aligned}$$

which implies that $\{\lambda_n\}$ is bounded, thus allows us to use Bolzano-Weierstrass to extract a convergent subsequence of λ_n, then restrict all further analysis to this subsequence. And then we make the assumption that $\lambda_{n_r} \to \overline{\lambda}$ as $n \to \infty$.

For a fixed $u \in S(a)$, we choose a constant $\mu_1 > 0$ such that for $\mu > \mu_1$,

$$\frac{1}{2}\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \mu V(x)|u|^2 dx - \frac{\mu}{2}\int_{\mathbb{R}^N} [u^2 - \ln(1 + \frac{u^2}{1+\psi(x)})] dx < 0, \tag{2.15}$$

which implies that $I_a < 0$. Furthermore, there exists $\mu_2 \ge \mu_1 > 0$ such that

$$\frac{1}{2}\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{\mu_2}{2}\int_{\mathbb{R}^N} [u^2 - \ln(1 + \frac{u^2}{1+\psi(x)})] dx = B_{\mu_2} < 0, \tag{2.16}$$

hence, fixed the number $\Upsilon = \frac{-B_{\mu_2}}{a^2}$ and in light of the assumption $\lambda\|V\|_\infty < \Upsilon$, we infer that $E(u_n) < 0$, which implies that $I_a < 0$. Consequently, we complete the proof. □

Lemma 2.6. *Assuming that condition (V_2) is satisfied, then for any $a > 0$, there exist $\Upsilon(a) > 0$ and $\eta > 0$ such that*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p dx \ge \eta \text{ if } \mu\|\mathrm{V}\|_\infty < \Upsilon. \tag{2.17}$$

Proof. From Lemma 2.5, it is easy to find $\Upsilon(a) > 0$ and $\eta > 0$ such that $I_a < -\eta$ if $\mu\|V\|_\infty < \Upsilon$. Now, since$(u_n) \subset S(a)$ is a minimizing sequence for I_a, then we have

$$\begin{aligned} I_a + o(1) =& E(u_n) \\ =& \frac{1}{2}\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \mu V(x)|u_n|^2 dx \\ & - \int_{\mathbb{R}^N} \frac{\mu\psi(x)}{2(1+\psi(x))}|u_n|^2 dx - \int_{\mathbb{R}^N} \frac{\mu A_p}{2(1+\psi(x))^{\frac{p}{2}}}|u_n|^p dx. \end{aligned}$$

Furthermore, we obtain that

$$-\eta + o(1) \geq -\frac{\mu a^2}{2} - \int_{\mathbb{R}^N} \frac{\mu A_p}{2(1+\psi(x))^{\frac{p}{2}}}|u_n|^p dx. \tag{2.18}$$

Hence (2.17) holds. We complete the proof. □

In the following lemma, we establish a quantitative relationship between the two ordered values of a and the corresponding values of I_a.

Lemma 2.7. *Assuming that condition (V_2) is satisfied and if $0 < a_1 < a_2$ is achieved, then $\frac{I_{a_2}}{a_2^2} < \frac{I_{a_1}}{a_1^2}$.*

Proof. To prove this lemma, we borrow the ideas form Lv and Li [18]. Let $\xi > 1$ such that $a_2 = \xi a_1$, and let $(u_n) \subset S(a_1)$ be a minimizing sequence with respect to I_{a_1}, that is

$$E(u_n) \to I_{a_1}, \text{ as n} \to \infty. \tag{2.19}$$

Setting $v_n = \xi u_n$, we have that $(v_n) \subset S(a_2)$, and so

$$I_{a_2} \leq E(v_n) = \xi^2 E(u_n) - \frac{\mu\xi^2}{2}\int_{\mathbb{R}^N} \ln(1 + \frac{u_n^2}{1+\psi(x)})dx + \frac{\mu}{2}\int_{\mathbb{R}^N} \ln(1+\frac{(\xi u_n)^2}{1+\psi(x)})dx. \tag{2.20}$$

Letting $n \to \infty$, it follows from $\xi > 1$ that

$$\frac{\mu}{2}\int_{\mathbb{R}^N} \ln(1+\frac{(\xi u_n)^2}{1+\psi(x)})dx - \frac{\mu\xi^2}{2}\int_{\mathbb{R}^N} \ln(1+\frac{u_n^2}{1+\psi(x)})dx < 0, \tag{2.21}$$

which implies that

$$I_{a_2} \leq \xi^2 E(u_n) - \frac{\mu\xi^2}{2}\int_{\mathbb{R}^N} \ln(1+\frac{u_n^2}{1+\psi(x)})dx + \frac{\mu}{2}\int_{\mathbb{R}^N} \ln(1+\frac{(\xi u_n)^2}{1+\psi(x)})dx < \xi^2 E(u_n), \tag{2.22}$$

that is

$$I_{a_2} a_1^2 < I_{a_1} a_2^2. \tag{2.23}$$

The proof is completed. □

Lemma 2.8. *Assuming that condition (V_2) is satisfied and if $u_n \rightharpoonup u$ in W, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ and $u \neq 0$, then $u \in S(a)$, $E(u) = I_a$ and $u_n \to u$ in W.*

Proof. We note that if $\|u\|_2 = m \neq a$, in light of Fatou's lemma and the assumption $u \neq 0$, then we get that $m \in (0, a)$. From the continuity of embedding $W \hookrightarrow L^h(\mathbb{R}^N)$ for any $h \in [2, 2_s^*]$ and two kinds of Brézis-Lieb lemmas in [7, 24], we get that

$$\begin{aligned} \int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx &= \int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} (u_n - u)|^2 dx + \int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^2 dx + o(1), \\ \int_{\mathbb{R}^N} |u_n|^2 dx &= \int_{\mathbb{R}^N} |u_n - u|^2 dx + \int_{\mathbb{R}^N} |u|^2 dx + o(1). \end{aligned} \tag{2.24}$$

Let $v_n = u_n - u$, $\|v_n\|_2 = r_n$ and assume that $\|v_n\|_2 \to r$, by (2.14), we infer that $a^2 = m^2 + r^2$ and $r_n \in (0, a)$ for sufficiently big n, which implies that

$$\begin{aligned} I_a + o(1) &= E(u_n) \\ &= E(v_n) + E(u) + o(1) \\ &\geq I_{r_n} + I_m + o(1) \\ &\geq \frac{r_n^2}{a^2} I_a + I_m + o(1). \end{aligned}$$

Letting $n \to \infty$, we can derive the inequality

$$I_a \geq \frac{r^2}{a^2} I_a + I_m. \tag{2.25}$$

Using the fact that $m \in (0, a)$ and Lemma 2.7 in the above inequality, we can infer that

$$I_a > \frac{r^2}{a^2} I_a + \frac{m^2}{a^2} I_a = I_a. \tag{2.26}$$

This inequality presents a contradiction. Therefore $\|u\|_2 = a$, namely, $u \in S(a)$. It can be deduced from the premises $\|u_n\|_2 = \|u\|_2 = a$ and $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^N)$ (since $W \hookrightarrow L^2(\mathbb{R}^N)$ is a continuous embedding) that $u_n \to u$ in $L^2(\mathbb{R}^N)$. Furthermore, the combination of this deduction with the interpolation inequality allows us to conclude that $u_n \to u$ in $L^p(\mathbb{R}^N)$. In addition, since $\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \int_{\mathbb{R}^N} \mu V(x)|u_n|^2 dx$ is convex and continuous in W, it can be inferred that it is a weak lower semicontinuous, namely

$$\liminf_{n\to+\infty} \Big(\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \int_{\mathbb{R}^N} \mu V(x)|u_n|^2 dx \Big) \geq \int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^N} \mu V(x)|u|^2 dx. \tag{2.27}$$

According to $I_a = \lim\limits_{n\to\infty} E(u_n)$, we infer that $I_a \geq E(u)$, reuse the definition of I_a and $u \in S(a)$, we get that $I_a = E(u)$, and hence

$$\lim_{n\to\infty} E(u_n) = E(u). \tag{2.28}$$

Finally, based on the convergence $u_n \to u$ in $L^2(\mathbb{R}^N)$ and $u_n \to u$ in $L^p(\mathbb{R}^N)$, we can get that $u_n \to u$ in W. Thus, we have successfully completed the proof. □

Lemma 2.9. *Let $p \in [1, \frac{N}{N-2}]$. Assuming that conditions (V_1) and (V_2) are satisfied, and let $R > 0$ and $\mu^* > 0$ are given, for any $\mu > \mu^*$, we could get*

$$\limsup_{n\to+\infty} \int_{B_R^c(0)} |u_n|^p \leq \frac{\eta}{2},$$

where $B_R^c(0) = \{x \in \mathbb{R}^N : |x| > R\}$, $a > 0$ is as given in Lemma 2.5.

Proof. We now prove this lemma by a contradiction. We borrow the ideas from Bartsch and Wang [4], for $R > 0$, we consider two sets

$$\Phi(R) := \{x \in \mathbb{R}^N : |x| > R, V(x) \geq C_0\} \text{ and } \Theta(R) := \{x \in \mathbb{R}^N : |x| > R, V(x) < C_0\}. \tag{2.29}$$

Let $C>0$ represent a constant with a variable value that may change from line to line. By Lemma 2.4, we know that $\|u_n\|_W^2 \le C$ for all $n\in\mathbb{N}$, hence we get that

$$\begin{aligned}\int_{\Phi(R)} u_n^2dx &\le \frac{1}{\mu C_0+1}\int_{\mathbb{R}^N}(\mu V(x)+1)u_n^2dx\\ &\le \frac{1}{\mu C_0+1}\Big(\int_{\mathbb{R}^N}(\mu V(x)+1)u_n^2+\int_{\mathbb{R}^{2N}}|(-\Delta)^{\frac{s}{2}}u_n|^2\Big)dx\\ &=\frac{1}{\mu C_0+1}\|u_n\|_W^2\\ &\le \frac{C}{\mu C_0+1}. \end{aligned}\tag{2.30}$$

Clearly, the term on the right-hand side of the above inequality can be arbitrarily small when $\mu>\mu^*$ (large enough). According to the Hölder inequality, the continuous embedding $H^s(\mathbb{R}^N)\hookrightarrow L^{2p}(\mathbb{R}^N)$ for any $p\in[1,\frac{N}{N-2}]$ (see ([19], Theorem 6.5)) and condition (V_1), we obtain

$$\begin{aligned}\int_{\Theta(R)} u_n^2dx &\le \left(\int_{\Theta(R)}|u_n|^{2p}dx\right)^{\frac1p}\left(\int_{\Theta(R)}dx\right)^{\frac1q}\\ &\le \left(\int_{\mathbb{R}^N}|u_n|^{2p}dx\right)^{\frac1p}\left(\int_{\Theta(R)}dx\right)^{\frac1q}\\ &= C\|u_n\|_{H^s}^2|\Theta(R)|^{\frac1q}\\ &\le C|\Theta(R)|^{\frac1q}. \end{aligned}\tag{2.31}$$

With the help of condition (V_1), the term on the right-hand side of the above inequality can also be arbitrarily small if R is sufficiently large since $|\Theta(R)|\to 0$ as $R\to+\infty$, where $\frac1p+\frac1q=1$. It follows from Lemma 2.2 that

$$\begin{aligned}\int_{B_R^c(0)}|u_n|^pdx &\le C(s,N,p)\left(\int_{B_R^c(0)}|u_n|^2dx\right)^{\frac p2-\frac{N(p-2)}{4s}}\left(\int_{B_R^c(0)}|(-\Delta)^{\frac s2}u_n|^2dx\right)^{\frac{N(p-2)}{4s}}\\ &\le C(s,N,p)C^{\frac{N(p-2)}{4s}}\|u_n\|_W^{\frac{N(p-2)}{4s}}\left(\int_{\Phi(R)}u_n^2dx+\int_{\Theta(R)}u_n^2dx\right)^{\frac p2-\frac{N(p-2)}{4s}}\\ &\le C\left(\int_{\Phi(R)}u_n^2dx+\int_{\Theta(R)}u_n^2dx\right)^{\frac p2-\frac{N(p-2)}{4s}}. \end{aligned}\tag{2.32}$$

To sum up, we prove the lemma. □

Lemma 2.10. *Assuming the conditions (V_1) and (V_2) are satisfied, and let $\mu^{**}>0$ is given, for any $\mu>\mu^{**}$ the sequence $(u_n)_{n\in\mathbb{N}}$ admits a nontrivial weak limit u in W.*

Proof. By Lemma 2.4, it is know that there exists $u\in W$ and a subsequence of $(u_n)_{n\in\mathbb{N}}$, which is still denoted as itself, such that

$$u_n\rightharpoonup u \text{ in } W,\ u_n(x)\to u(x) \text{ a.e. in } \mathbb{R}^N.\tag{2.33}$$

Now, employing the method of argument by contradiction, we make the assumption that $u=0$ for some $\mu>\mu^*$ as given in Lemma 2.9. Because the compactness of the embedding over the bounded domain, we have $u_n \to 0$ in $L^p(B_R(0))$ for any $R > 0$. In light of Lemmas 2.6 and 2.9, we have

$$\begin{aligned}\eta &\leq \liminf_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^p dx \\ &= \liminf_{n\to+\infty} \int_{B_R^c(0)} |u_n|^p dx \\ &\leq \limsup_{n\to+\infty} \int_{B_R^c(0)} |u_n|^p dx \\ &\leq \frac{\eta}{2}, \end{aligned} \tag{2.34}$$

this inequality presents a contradiction. Therefore, we can conclude that there exists $\mu^* \leq \mu^{**}$ such that u is nontrivial for any $\mu > \mu^{**}$. □

In the end, we present the proof of our main result.

Proof of Theorem 1.1. By Lemmas 2.4 and 2.10, we know that there exists a minimizing sequence $(u_n)_{n\in\mathbb{N}} \subset S(a)$ for I_a, which is bounded in W and its weak limit u is nontrivial. According to the Lemma 2.8, we have that $u \in S(a)$, $E(u) = I_a$ and $u_n \to u$ in W. Then, we use the Lagrange multiplier method, there exists $\lambda_a \in \mathbb{R}$ solving the equation

$$E'(u) + \lambda_a J'(u) = 0 \text{ in } \mathrm{W}^*, \tag{2.35}$$

where W^* is the dual space of W and $J : W \to \mathbb{R}$ is defined by

$$J(u) = \|u\|_2, u \in W. \tag{2.36}$$

By (2.13), we deduce that

$$(-\Delta)^s u + \lambda_a u + \mu V(x)u - \mu\frac{\psi(x) + u^2}{1 + \psi(x) + u^2}u = 0 \text{ in } \mathbb{R}^{\mathrm{N}}. \tag{2.37}$$

We complete the proof of Theorem 1.1. □

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References

[1] C. O. Alves, N. V. Thin: *On existence of multiple normalized solutions to a class of elliptic problems in whole $\mathbb{R}^N$ via Lusternik-Schnirelmann category*, SIAM J. Math. Anal., **55** (2023), 1264–1283.

[2] C. O. Alves, C. Ji: *Multiple normalized solutions to a logarithmic Schrödinger equation via Lusternik–Schnirelmann category*, J. Geom. Anal., **34** (2024), Article ID: 198.

[3] L. Appolloni, S. Secchi: *Normalized solutions for the fractional NLS with mass supercritical nonlinearity*, J. Differential Equations, **286** (2021), 248–283.

[4] T. Bartsch, Z. Q. Wang: *Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^n$*, Comm. Partial Differential Equations, **20** (1995), 1725–1741.

[5] H. Berestycki, P. L. Lions: *Nonlinear scalar field equations I: existence of a ground state*, Arch. Ration. Mech. Anal., **82** (1983), 313–345.

[6] H. Berestycki, P. L. Lions: *The concentration-compactness principle in the calculus of variations. The locally compact case, part 1*, Ann. Inst. Henri. Poincaré Anal. Non Linéaire, **11** (1984), 109–145.

[7] H. Brézis, E. Lieb: *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Am. Math. Soc., **88** (3) (1983), 486–490.
[8] N. K. Efremidis, J. Hudock, D. N. Christodoulides, J. W. Fleischer, O. Cohen and M. Segev: *Two-dimensional optical lattice solitons*, Phys Rev Lett., **91** (2003), 213–906.
[9] R. L. Frank, E. Lenzmann and L. Silvestre: *Uniqueness of radial solutions for the fractional Laplacian*, Comm. Pure Appl. Math., **69** (2016), 1671–1726.
[10] Q. Y. Guan, Z. M. Ma: *Boundary problems for fractional Laplacians*, Stoch. Dyn., **5** (2005), 385–424.
[11] N. Laskin: *Fractional Schrödinger equation*, Phys. Rev. E, **66** (2002), Article ID: 56108.
[12] S. Liang, P. Pucci and B. Zhang: *Multiple solutions for critical Choquard-Kirchhoff type equations*, Adv. Nonlinear Anal., **10** (2021), 400–419.
[13] Q. Li, W. Zou: *The existence and multiplicity of the normalized solutions for fractional Schrödinger equations involving Sobolev critical exponent in the L^2-subcritical and L^2-supercritical cases*, Adv. Nonlinear Anal., **11** (2022), 1531–1551.
[14] T. C. Lin, M. R. Belić, M. S. Petrović and G. Chen: *Ground states of nonlinear Schrödinger systems with saturable nonlinearity in $\mathbb{R}^2$ for two counterpropagating beams*, J. Math. Phys., **55** (2014), Article ID: 011505.
[15] T. C. Lin, X. Wang and Z. Q. Wang: *Orbital stability and energy estimate of ground states of saturable nonlinear Schrödinger equations with intensity functions in $\mathbb{R}^2$*, J. Diff. Equ., **263** (2017), 2750–2786.
[16] T. C. Lin, T. F. Wu: *Multiple positive solutions of saturable nonlinear Schrödinger equations with intensity functions*, Discrete Contin. Dyn. Syst., **70** (2020), 2165–2187.
[17] H. Luo, Z. Zhang: *Normalized solutions to the fractional Schrödinger equations with combined nonlinearities*, Calc. Var. Partial Differ. Equ., **59** (2020), Article ID: 143.
[18] Y-C Lv, G-D Li: *Multiplicity of normalized solutions to a fractional logarithmic Schrödinger equation*,Fractal fract., **8** (7) (2024), Article ID: 391.
[19] E. D. Nezza, G. Palatucci and E. Valdinoci: *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., **136** (2012), 521–573.
[20] L. Silvestre: *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math., **60** (2007), 67–112.
[21] Y. Sire, E. Valdinoci: *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result*, J. Funct. Anal., **256** (2009), 1842–1864.
[22] Y. Zhang, J. Sun: *Normalized solutions of quasilinear Schrödinger equations with saturable nonlinearity*, Appl. Math. Lett., **138** (2023), Article ID: 108531.
[23] M. Zhen, B. Zhang: *Normalized ground states for the critical fractional NLS equation with a perturbation*, Rev. Mat. Complut., **35** (2022), 89–132.
[24] J. Zuo, T. An and A. Fiscella: *A critical Kirchhoff-type problem driven by a $p(\cdot)$-fractional Laplace operator with variable $s(\cdot)$-order*, Math. Methods Appl. Sci., **44** (2021), 1071–1085.
[25] J. B. Zuo, C. G. Liu and C. Vetro: *Normalized solutions to the fractional Schrödinger equation with potential*, Mediterr. J. Math., **216** (2023), 1660–5446.

WEN LIAO
GUILIN UNIVERSITY OF TECHNOLOGY
SCHOOL OF MATHEMATICS AND STATISTICS
GUILIN, GUANGXI 541004, PR CHINA

GUANGXI COLLEGES AND UNIVERSITIES
KEY LABORATORY OF APPLIED STATISTICS
GUILIN, GUANGXI 541004, PR CHINA
Email address: 1936692552@qq.com

QIONGFEN ZHANG
GUILIN UNIVERSITY OF TECHNOLOGY
SCHOOL OF MATHEMATICS AND STATISTICS
GUILIN, GUANGXI 541004, PR CHINA

GUANGXI COLLEGES AND UNIVERSITIES
KEY LABORATORY OF APPLIED STATISTICS
GUILIN, GUANGXI 541004, PR CHINA
Email address: qfzhangcsu@163.com