MODERN MATHEMATICAL METHODS
3 (2025), No. 1, pp. 1-13
https://modernmathmeth.com/
ISSN 3023 - 5294

Research Article

On branched continued fraction expansions of hypergeometric functions F_M and their ratios

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ABSTRACT. The paper investigates the problem of constructing branched continued fraction expansions of hypergeometric functions $F_M(a_1, a_2, b_1, b_2; a_1, c_2; \mathbf{z})$ and their ratios. Recurrence relations of the hypergeometric function F_M are established, which provide the construction of formal branched continued fractions with simple structures, the elements of which are polynomials in the variables z_1, z_2, z_3. To construct the expansions, a method of based on the so-called complete group of ratios of hypergeometric functions was used, which is a generalization of the classical Gauss method.

Keywords: Hypergeometric function, recurrence relation, branched continued fraction, approximation by rational functions.

2020 Mathematics Subject Classification: 33C65, 30B99, 41A20.

1. INTRODUCTION

Recently, there has been increased interest in studying of the Lauricella-Saran family of hypergeometric functions [6, 9, 15, 21]. This is certainly due to the development of existing and new progressive methods of studying them. On the other hand, the diversity and importance of the practical applications of these functions in various fields of science and engineering have great importance [13, 19, 23, 25, 26].

The paper considers the hypergeometric function F_M, defined as follows [24]:

$$F_M(a_1, a_2, b_1, b_2; c_1, c_2; \mathbf{z}) = \sum_{p,q,r=0}^{+\infty} \frac{(a_1)_p (a_2)_{q+r} (b_1)_{p+r} (b_2)_q}{(c_1)_p (c_2)_{q+r}} \frac{z_1^p z_2^q z_3^r}{p!q!r!},$$

where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{C}$ herewith $c_1, c_2 \notin \{0, -1, -2, \ldots\}$, $\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{C}^3$, $(\cdot)_k$ is the Pochhammer symbol.

It is shown that representations of special functions in the form of branched continued fractions are their efficient rational approximations [7, 8, 10, 11, 12, 16, 17, 18].

The main goal of the paper is to construct formal branched continued fractional expansions of the hypergeometric functions F_M and their relations in the case when $a_1 = c_1$. Here we develop a method for constructing expansions using the so-called complete group of ratios of hypergeometric functions [1, 2, 22], which is a generalization of the classical Gauss method [14].

Received: 30.01.2025; Accepted: 18.03.2025; Published Online: 24.03.2025
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2. Recurrence Relations of Hypergeometric Function F_M

We begin by establishing recurrence relations of hypergeometric function

$$F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z}) = \sum_{p,q,r=0}^{\infty} \frac{(a_1)_p(a_2)_{q+r}(b_1)_{p+r}(b_2)_q}{(a_1)_p(c_2)_{q+r}} \frac{z_1^p z_2^q z_3^r}{p!q!r!} = \sum_{p,q,r=0}^{\infty} \frac{(a_2)_{q+r}(b_1)_{p+r}(b_2)_q}{(c_2)_{q+r}} \frac{z_1^p z_2^q z_3^r}{p!q!r!},$$

where $a_1,a_2,b_1,b_2,c_2\in\mathbb{C}$ herewith $a_1,c_2\notin\{0,-1,-2,\ldots\}$, that would provide the construction of formal branched continued fractions with the simplest structures and whose elements would be polynomials in the variables z_1, z_2, z_3. The following lemma is true.

Lemma 2.1. *The following recurrence relations are true:*

$$\begin{aligned} F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z}) &= F_M(a_1,a_2+1,b_1,b_2;a_1,c_2;\mathbf{z}) \\ &- \frac{b_2}{c_2} z_2 F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z}) \\ &- \frac{b_1}{c_2} z_3 F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}), \end{aligned} \tag{2.1}$$

$$\begin{aligned} F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z}) &= (1-z_1)F_M(a_1,a_2,b_1+1,b_2;a_1,c_2;\mathbf{z}) \\ &- \frac{a_2}{c_2} z_3 F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}), \end{aligned} \tag{2.2}$$

$$\begin{aligned} F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z}) &= F_M(a_1,a_2,b_1,b_2+1;a_1,c_2;\mathbf{z}) \\ &- \frac{a_2}{c_2} z_2 F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z}), \end{aligned} \tag{2.3}$$

$$\begin{aligned} F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z}) &= F_M(a_1,a_2,b_1,b_2;a_1,c_2+1;\mathbf{z}) \\ &+ \frac{a_2 b_2}{c_2(c_2+1)} z_2 F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+2;\mathbf{z}) \\ &+ \frac{a_2 b_1}{c_2(c_2+1)} z_3 F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+2;\mathbf{z}), \end{aligned} \tag{2.4}$$

where $a_1,a_2,b_1,b_2,c_2\in\mathbb{C}$ *herewith* $a_1,c_2\notin\{0,-1,-2,\ldots\}$.

Proof. Let us show the validity of the relation (2.1). We have

$$\begin{aligned} &F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z}) - F_M(a_1,a_2+1,b_1,b_2;a_1,c_2;\mathbf{z}) \\ &= \sum_{p,q,r=0}^{\infty} \frac{(b_1)_{p+r}(b_2)_q}{(c_2)_{q+r}} \left((a_2)_{q+r} - (a_2+1)_{q+r}\right) \frac{z_1^p z_2^q z_3^r}{p!q!r!} \\ &= \sum_{p\ge 0,q\ge 1,r=0} \frac{(b_1)_{p+r}(a_2+1)_{q+r-1}(b_2)_q}{(c_2)_{q+r}} (a_2-a_2-q) \frac{z_1^p z_2^q z_3^r}{p!q!r!} \\ &+ \sum_{p\ge 0,q=0,r\ge 1} \frac{(b_1)_{p+r}(a_2+1)_{q+r-1}(b_2)_q}{(c_2)_{q+r}} (a_2-a_2-r) \frac{z_1^p z_2^q z_3^r}{p!q!r!} \\ &+ \sum_{p\ge 0,q\ge 1,r\ge 1} \frac{(b_1)_{p+r}(a_2+1)_{q+r-1}(b_2)_q}{(c_2)_{q+r}} (a_2-a_2-q-r) \frac{z_1^p z_2^q z_3^r}{p!q!r!}. \end{aligned}$$

Then,

$$\begin{aligned}
&F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z})-F_M(a_1,a_2+1,b_1,b_2;a_1,c_2;\mathbf{z})\\
&=-\sum_{p\ge0,q\ge1,r\ge0}\frac{(b_1)_{p+r}(a_2+1)_{q+r-1}(b_2)_q}{(c_2)_{q+r}}\frac{z_1^pz_2^qz_3^r}{p!(q-1)!r!}\\
&-\sum_{p\ge0,q\ge0,r\ge1}\frac{(b_1)_{p+r}(a_2+1)_{q+r-1}(b_2+1)_{q-1}}{(c_2)_{q+r-1}}\frac{z_1^pz_2^qz_3^r}{p!q!(r-1)!}\\
&=-\frac{b_2}{c_2}z_2\sum_{p\ge0,q\ge1,r\ge0}\frac{(b_1)_{p+r}(a_2+1)_{q+r-1}(b_2+1)_{q-1}}{(c_2+1)_{q+r-1}}\frac{z_1^pz_2^{q-1}z_3^r}{p!(q-1)!r!}\\
&-\frac{b_1}{c_2}z_3\sum_{p\ge0,q\ge0,r\ge1}\frac{(b_1+1)_{p+r-1}(a_2+1)_{q+r-1}(b_2)_q}{(c_2+1)_{q+r-1}}\frac{z_1^pz_2^qz_3^{r-1}}{p!q!(r-1)!}\\
&=-\frac{b_2}{c_2}z_2F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z})\\
&-\frac{b_1}{c_2}z_3F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}).
\end{aligned}$$

The recurrence relation (2.1) has been proved. Similarly, we prove the relation (2.2). Indeed, we receive

$$\begin{aligned}
&F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z})-F_M(a_1,a_2,b_1+1,b_2;a_1,c_2;\mathbf{z})\\
&=\sum_{p,q,r=0}^{\infty}\frac{(a_2)_{q+r}(b_2)_q}{(c_2)_{q+r}}\left((b_1)_{p+r}-(b_1+1)_{p+r}\right)\frac{z_1^pz_2^qz_3^r}{p!q!r!}\\
&=\sum_{q\ge0,p\ge1,r=0}\frac{(a_2)_{q+r}(b_1+1)_{p+r-1}(b_2)_q}{(c_2)_{q+r}}\left(b_1-b_1-p\right)\frac{z_1^pz_2^qz_3^r}{p!q!r!}\\
&+\sum_{q\ge0,p=0,r\ge1}\frac{(a_2)_{q+r}(b_1+1)_{p+r-1}(b_2)_q}{(c_2)_{q+r}}\left(b_1-b_1-r\right)\frac{z_1^pz_2^qz_3^r}{p!q!r!}\\
&+\sum_{q\ge0,p\ge1,r\ge1}\frac{(a_2)_{q+r}(b_1+1)_{p+r-1}(b_2)_q}{(c_2)_{q+r}}\left(b_1-b_1-p-r\right)\frac{z_1^pz_2^qz_3^r}{p!q!r!}\\
&=-\sum_{q\ge0,p\ge1,r\ge0}\frac{(a_2)_{q+r}(b_1)_{p+r-1}(b_2)_q}{(c_2)_{q+r}}\frac{z_1^pz_2^qz_3^r}{(p-1)!q!r!}\\
&-\sum_{q\ge0,p\ge0,r\ge1}\frac{(a_2)_{q+r}(b_1)_{p+r-1}(b_2)_q}{(c_2)_{q+r-1}}\frac{z_1^pz_2^qz_3^r}{p!q!(r-1)!}\\
&=-z_1\sum_{q\ge0,p\ge1,r\ge0}\frac{(b_1+1)_{p+r-1}(a_2)_{q+r}(b_2)_q}{(c_2)_{q+r}}\frac{z_1^{p-1}z_2^qz_3^r}{(p-1)!q!r!}\\
&-\frac{a_2}{c_2}z_3\sum_{p\ge0,q\ge0,r\ge1}\frac{(b_1)_{p+r-1}(a_2+1)_{q+r-1}(b_2)_q}{(c_2+1)_{q+r-1}}\frac{z_1^pz_2^qz_3^{r-1}}{p!q!(r-1)!}\\
&=-z_1F_M(a_1,a_2,b_1+1,b_2;a_1,c_2;\mathbf{z})\\
&-\frac{a_2}{c_2}z_3F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}).
\end{aligned}$$

Next, we obtain

$$
\begin{aligned}
&F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z}) - F_M(a_1,a_2,b_1,b_2+1;a_1,c_2;\mathbf{z})\\
&= \sum_{p,q,r=0}^{\infty} \frac{(a_2)_{q+r}(b_1)_{p+r}}{(c_2)_{q+r}} \left((b_2)_q - (b_2+1)_q\right) \frac{z_1^p z_2^q z_3^r}{p!q!r!}\\
&= \sum_{p\geq 0,q\geq 1,r\geq 0} \frac{(a_2)_{q+r}(b_1)_{p+r}(b_2+1)_{q-1}}{(c_2)_{q+r}} (b_2 - b_2 - q) \frac{z_1^p z_2^q z_3^r}{p!q!r!}\\
&F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z}) - F_M(a_1,a_2,b_1,b_2+1;a_1,c_2;\mathbf{z})\\
&= -\frac{a_2}{c_2} z_2 \sum_{p\geq 0,q\geq 1,r\geq 0} \frac{(a_2+1)_{q+r-1}(b_1)_{p+r}(b_2+1)_{q-1}}{(c_2+1)_{q+r-1}} \frac{z_1^p z_2^{q-1} z_3^r}{p!(q-1)!r!}\\
&= -\frac{a_2}{c_2} z_2 F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z}).
\end{aligned}
$$

The relation (2.3) has been proved. Finally, we have

$$
\begin{aligned}
&F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z}) - F_M(a_1,a_2,b_1,b_2;a_1,c_2+1;\mathbf{z})\\
&= \sum_{p\geq 0,q\geq 1,r=0} \frac{(a_2)_{q+r}(b_1)_{p+r}(b_2)_q}{(c+1)_{q+r-1}} \left(\frac{1}{c_2} - \frac{1}{c_2+q}\right) \frac{z_1^p z_2^q z_3^r}{p!q!r!}\\
&+ \sum_{p\geq 0,q=0,r\geq 1} \frac{(a_2)_{q+r}(b_1)_{p+r}(b_2)_q}{(c+1)_{q+r-1}} \left(\frac{1}{c_2} - \frac{1}{c_2+r}\right) \frac{z_1^p z_2^q z_3^r}{p!q!r!}\\
&+ \sum_{p\geq 0,q\geq 1,r\geq 1} \frac{(a_2)_{q+r}(b_1)_{p+r}(b_2)_q}{(c+1)_{q+r-1}} \left(\frac{1}{c_2} - \frac{1}{c_2+q+r}\right) \frac{z_1^p z_2^q z_3^r}{p!q!r!}\\
&= \sum_{p\geq 0,q\geq 1,r=0} \frac{(a_2)_{q+r}(b_1)_{p+r}(b_2)_q}{c_2(c_2+1)_{q+r}} \frac{z_1^p z_2^q z_3^r}{p!(q-1)!r!}\\
&+ \sum_{p\geq 0,q=0,r\geq 1} \frac{(a_2)_{q+r}(b_1)_{p+r}(b_2)_q}{c_2(c_2+1)_{q+r}} \frac{z_1^p z_2^q z_3^r}{p!q!(r-1)!}\\
&+ \sum_{p\geq 0,q\geq 1,r\geq 1} \frac{(a_2)_{q+r}(b_1)_{p+r}(b_2)_q}{c_2(c_2+1)_{q+r}} \frac{z_1^p z_2^q z_3^r}{p!(q-1)!r!}\\
&+ \sum_{p\geq 0,q\geq 1,r\geq 1} \frac{(a_2)_{q+r}(b_1)_{p+r}(b_2)_q}{c_2(c_2+1)_{q+r}} \frac{z_1^p z_2^q z_3^r}{p!q!(r-1)!}\\
&= \frac{a_2 b_2}{c_2(c_2+1)} z_2 \sum_{p\geq 0,q\geq 1,r\geq 0} \frac{(a_2+1)_{q+r-1}(b_1)_{p+r}(b_2+1)_{q-1}}{(c_2+2)_{q+r-1}} \frac{z_1^p z_2^{q-1} z_3^r}{p!(q-1)!r!}\\
&+ \frac{a_2 b_1}{c_2(c_2+1)} z_3 \sum_{p\geq 0,q\geq 0,r\geq 1} \frac{(a_2+1)_{q+r-1}(b_1+1)_{p+r-1}(b_2)_q}{(c_2+2)_{q+r-1}} \frac{z_1^p z_2^q z_3^{r-1}}{p!q!(r-1)!}\\
&= \frac{a_2 b_2}{c_2(c_2+1)} z_2 F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+2;\mathbf{z})\\
&+ \frac{a_2 b_1}{c_2(c_2+1)} z_3 F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+2;\mathbf{z})
\end{aligned}
$$

that had to be proved. □

Lemma 2.2. *The following four-term recurrence relations are true:*

$$\begin{aligned} &F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z}) \\ &=\left((1-z_1)\left(1-\frac{b_2}{c_2}z_2\right)-\frac{a_2+b_1+1}{c_2}z_3\right)F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}) \\ &+\frac{(a_2+1)(b_1+1)}{c_2(c_2+1)}(1-z_1-z_3)z_3F_M(a_1,a_2+2,b_1+2,b_2;a_1,c_2+2;\mathbf{z}) \\ &+\frac{(a_2+1)b_2}{c_2(c_2+1)}(1-z_1)z_2(1-z_2)F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z}) \end{aligned} \tag{2.5}$$

and for $z_1 \neq 1$*,*

$$\begin{aligned} &(1-z_1)F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z}) \\ &=\left((1-z_1)\left(1-\frac{a_2+b_2+1}{c_2}z_2\right)-\frac{b_1}{c_2}z_3\right)F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z}) \\ &+\frac{(a_2+1)(b_2+1)}{c_2(c_2+1)}(1-z_1)z_2(1-z_2)F_M(a_1,a_2+2,b_1,b_2+2;a_1,c_2+2;\mathbf{z}) \\ &+\frac{(a_2+1)b_1}{c_2(c_2+1)}(1-z_1-z_3)z_3F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z}), \end{aligned} \tag{2.6}$$

where $a_1,a_2,b_1,b_2,c_2\in\mathbb{C}$ *herewith* $a_1,c_2\notin\{0,-1,-2,\ldots\}$.

Proof. Using the relation (2.1), (2.2), and (2.4), we have

$$\begin{aligned} &F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z}) = F_M(a_1,a_2+1,b_1,b_2;a_1,c_2;\mathbf{z}) \\ &-\frac{b_2}{c_2}z_2F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z})-\frac{b_1}{c_2}z_3F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}) \\ &=(1-z_1)F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2;\mathbf{z}) \\ &-\frac{a_2+1}{c_2}z_3F_M(a_1,a_2+2,b_1+1,b_2;a_1,c_2+1;\mathbf{z}) \\ &-\frac{b_2}{c_2}z_2F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z})-\frac{b_1}{c_2}z_3F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}) \\ &=(1-z_1)F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}) \\ &+\frac{(a_2+1)(b_1+1)}{c_2(c_2+1)}(1-z_1)z_3F_M(a_1,a_2+2,b_1+2,b_2;a_1,c_2+2;\mathbf{z}) \\ &+\frac{(a_2+1)b_2}{c_2(c_2+1)}(1-z_1)z_2F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z}) \\ &-\frac{a_2+1}{c_2}z_3F_M(a_1,a_2+2,b_1+1,b_2;a_1,c_2+1;\mathbf{z}) \\ &-\frac{b_2}{c_2}z_2F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z})-\frac{b_1}{c_2}z_3F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}). \end{aligned}$$

Next, applying (2.1)–(2.3), we obtain

$$
\begin{aligned}
&F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z}) = \left(1 - z_1 - \frac{b_1}{c_2} z_3\right) F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}) \\
&+\frac{(a_2+1)(b_1+1)}{c_2(c_2+1)}(1-z_1)z_3F_M(a_1,a_2+2,b_1+2,b_2;a_1,c_2+2;\mathbf{z}) \\
&+\frac{(a_2+1)b_2}{c_2(c_2+1)}(1-z_1)z_2F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z}) \\
&-\frac{a_2+1}{c_2}z_3\bigg(F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}) \\
&+\frac{b_1+1}{c_2+1}z_3F_M(a_1,a_2+2,b_1+2,b_2;a_1,c_2+2;\mathbf{z}) \\
&+\frac{b_2}{c_2+1}z_2F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z})\bigg) \\
&-\frac{b_2}{c_2}z_2\bigg((1-z_1)F_M(a_1,a_2+1,b_1+1,b_2+1;a_1,c_2+1;\mathbf{z}) \\
&-\frac{a_2+1}{c_2+1}z_3F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z})\bigg) \\
&=\left(1-z_1-\frac{a_2+b_1+1}{c_2}z_3\right)F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}) \\
&+\frac{(a_2+1)(b_1+1)}{c_2(c_2+1)}(1-z_1)z_3F_M(a_1,a_2+2,b_1+2,b_2;a_1,c_2+2;\mathbf{z}) \\
&+\frac{(a_2+1)b_2}{c_2(c_2+1)}(1-z_1)z_2F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z}) \\
&-\frac{(a_2+1)(b_1+1)}{c_2(c_2+1)}z_3^2F_M(a_1,a_2+2,b_1+2,b_2;a_1,c_2+2;\mathbf{z}) \\
&-\frac{(a_2+1)b_2}{c_2(c_2+1)}z_2z_3F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z}) \\
&-\frac{b_2}{c_2}z_2(1-z_1)F_M(a_1,a_2+1,b_1+1,b_2+1;a_1,c_2+1;\mathbf{z}) \\
&+\frac{(a_2+1)b_2}{c_2(c_2+1)}z_2z_3F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z}) \\
&=\left(1-z_1-\frac{a_2+b_1+1}{c_2}z_3\right)F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}) \\
&+\frac{(a_2+1)(b_1+1)}{c_2(c_2+1)}(1-z_1-z_3)z_3F_M(a_1,a_2+2,b_1+2,b_2;a_1,c_2+2;\mathbf{z}) \\
&+\frac{(a_2+1)b_2}{c_2(c_1+1)}(1-z_1)z_2F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z}) \\
&-\frac{b_2}{c_2}z_2(1-z_1)\bigg(F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}) \\
&+\frac{a_2+1}{c_2+1}z_2F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z})\bigg).
\end{aligned}
$$

Continuing the transformation, we have

$$
\begin{aligned}
&F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z})\\
=&\left(1-z_1-\frac{b_2}{c_2}z_2(1-z_1)-\frac{a_2+b_1+1}{c_2}z_3\right)F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z})\\
+&\frac{(a_2+1)(b_1+1)}{c_2(c_2+1)}(1-z_1-z_3)z_3F_M(a_1,a_2+2,b_1+2,b_2;a_1,c_2+2;\mathbf{z})\\
+&\frac{(a_2+1)b_2}{c_2(c_1+1)}(1-z_1)z_2F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z})\\
-&\frac{(a_2+1)b_2}{c_2(c_2+1)}(1-z_1)z_2^2F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z})\\
=&\left((1-z_1)\left(1-\frac{b_2}{c_2}z_2\right)-\frac{a_2+b_1+1}{c_2}z_3\right)F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+2;\mathbf{z})\\
+&\frac{(a_2+1)(b_1+1)}{c_2(c_2+1)}(1-z_1-z_3)z_3F_M(a_1,a_2+2,b_1+2,b_2;a_1,c_2+2;\mathbf{z})\\
+&\frac{(a_2+1)b_2}{c_2(c_2+1)}(1-z_1)z_2(1-z_2)F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z}).
\end{aligned}
$$

Next, we prove the relation (2.6). From (2.1), we have

$$
\begin{aligned}
&F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z})=F_M(a_1,a_2+1,b_1,b_2;a_1,c_2;\mathbf{z})\\
&-\frac{b_2}{c_2}z_2F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z})\\
&-\frac{b_1}{c_2}z_3F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z})\\
=&F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2;\mathbf{z})-\frac{a_2+1}{c_2}z_2F_M(a_1,a_2+2,b_1,b_2+1;a_1,c_2+1;\mathbf{z})\\
&-\frac{b_2}{c_2}z_2F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z})-\frac{b_1}{c_2}z_3F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z})\\
=&F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z})\\
&+\frac{(a_2+1)(b_2+1)}{c_2(c_2+1)}z_2F_M(a_1,a_2+2,b_1,b_2+2;a_1,c_2+2;\mathbf{z})\\
&+\frac{(a_2+1)b_1}{c_2(c_2+1)}z_3F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z}\\
&-\frac{a_2+1}{c_2}z_2\Bigg(F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z})\\
&+\frac{b_2+1}{c_2+1}z_2F_M(a_1,a_2+2,b_1,b_2+2;a_1,c_2+2;\mathbf{z})\\
&+\frac{b_1}{c_2+1}z_3F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z})\Bigg)\\
&-\frac{b_2}{c_2}z_2F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z})\\
&-\frac{b_1}{c_2}z_3F_M(a_1,a_2+1,b_1+1,b_2;a_1,c_2+1;\mathbf{z}).
\end{aligned}
$$

Then,

$$\begin{aligned}
&F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z})\\
=&\left(1-\frac{a_2+b_2+1}{c_2}z_2\right)F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z})\\
&+\frac{(a_2+1)(b_2+1)}{c_2(c_2+1)}z_2F_M(a_1,a_2+2,b_1,b_2+2;a_1,c_2+2;\mathbf{z})\\
&+\frac{(a_2+1)b_1}{c_2(c_2+1)}z_3F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z})\\
&-\frac{(a_2+1)(b_2+1)}{c_2(c_2+1)}z_2^2F_M(a_1,a_2+2,b_1,b_2+2;a_1,c_2+2;\mathbf{z})\\
&-\frac{(a_2+1)b_1}{c_2(c_2+1)}z_2z_3F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z})\\
&-\frac{b_1}{c_2}z_3\Big(F_M(a_1,a_2+1,b_1+1,b_2+1;a_1,c_2+1;\mathbf{z})\\
&-\frac{a_2+1}{c_2+1}z_2F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z})\Big)\\
=&\left(1-\frac{a_2+b_2+1}{c_2}z_2\right)F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z})\\
&+\frac{(a_2+1)(b_2+1)}{c_2(c_2+1)}z_2(1-z_2)F_M(a_1,a_2+2,b_1,b_2+2;a_1,c_2+2;\mathbf{z})\\
&+\frac{(a_2+1)b_1}{c_2(c_2+1)}z_3F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z})\\
&-\frac{b_1}{c_2}z_3F_M(a_1,a_2+1,b_1+1,b_2+1;a_1,c_2+1;\mathbf{z}).
\end{aligned}$$

From (2.2), we obtain

$$\begin{aligned}
&(1-z_1)F_M(a_1,a_2+1,b_1+1,b_2+1;a_1,c_2+1;\mathbf{z})\\
=&F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z})\\
&+\frac{a_2+1}{c_2+1}z_3F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z}).
\end{aligned}$$

Then, finally, we have

$$\begin{aligned}
&(1-z_1)F_M(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z})\\
=&\left((1-z_1)\left(1-\frac{a_2+b_2+1}{c_2}z_2\right)-\frac{b_1}{c_2}z_3\right)F_M(a_1,a_2+1,b_1,b_2+1;a_1,c_2+1;\mathbf{z})\\
&+\frac{(a_2+1)(b_2+1)}{c_2(c_2+1)}(1-z_1)z_2(1-z_2)F_M(a_1,a_2+2,b_1,b_2+2;a_1,c_2+2;\mathbf{z})\\
&+\frac{(a_2+1)b_1}{c_2(c_2+1)}(1-z_1-z_3)z_3F_M(a_1,a_2+2,b_1+1,b_2+1;a_1,c_2+2;\mathbf{z})
\end{aligned}$$

that had to be proved. □

3. Construction Branched Continued Fractions

Let $\mathfrak{I} = \{1, 2\}$. For each $i_0 \in \mathfrak{I}$, we set

$$R_M^{(i_0)}(a_1, a_2, b_1, b_2; a_1, c_2; \mathbf{z}) = \frac{(1 - \delta_{i_0}^2 z_1) F_M(a_1, a_2, b_1, b_2; a_1, c_2; \mathbf{z})}{F_M(a_1, a_2 + 1, b_1 + \delta_{i_0}^1, b_2 + \delta_{i_0}^2; a_1, c_2 + 1; \mathbf{z})}, \tag{3.7}$$

where δ_i^j is the Kronecker delta. In addition, let for $k \geq 1$

$$\mathfrak{I}_k = \{i(k) = (i_0, i_1, i_2, \ldots, i_k) : \ i_r \in \mathfrak{I}, \ 0 \leq r \leq k\}.$$

Then, we have the following result.

Theorem 3.1. *For each $i_0 \in \mathfrak{I}$ and $z_1 \neq 1$ the ratio (3.7), where $a_1, a_2, b_1, b_2, c_2 \in \mathbb{C}$ herewith $a_1, c_2 \notin \{0, -1, -2, \ldots\}$, has a formal branched continued fraction*

$$v_{i_0}(\mathbf{z}) + \sum_{i_1=1}^{2} \cfrac{u_{i(1)}(\mathbf{z})}{v_{i(1)}(\mathbf{z}) + \sum_{i_2=1}^{2} \cfrac{u_{i(2)}(\mathbf{z})}{v_{i(2)}(\mathbf{z}) + \ddots}}, \tag{3.8}$$

where

$$v_{i_0}(\mathbf{z}) = 1 - z_1 - \frac{a_2 + b_{i_0} + 1}{c_2}(1 - \delta_{i_0}^2 z_1) z_{4-i_0} - \frac{b_{3-i_0}}{c_2}(1 - \delta_{i_0}^1 z_1) z_{1+i_0} \tag{3.9}$$

and for $i(k) \in \mathfrak{I}_k$ and $k \geq 1$,

$$u_{i(k)}(\mathbf{z}) = \frac{(a_2 + k)\left(b_{i_k} + \sum_{r=0}^{k-1} \delta_{i_r}^{i_k}\right)}{(c_2 + k - 1)(c_2 + k)}(1 - \delta_{i_k}^2 z_1)^2 z_{4-i_k}(1 - \delta_{i_k}^1 z_1 - z_{4-i_k}), \tag{3.10}$$

$$v_{i(k)}(\mathbf{z}) = 1 - z_1 - \frac{a_2 + b_{i_k} + k + 1 + \sum_{r=0}^{k-1} \delta_{i_r}^{i_k}}{c_2 + k}(1 - \delta_{i_k}^2 z_1) z_{4-i_k}$$

$$- \frac{b_{3-i_k} + \sum_{r=0}^{k-1} \delta_{i_r}^{3-i_k}}{c_2 + k}(1 - \delta_{i_k}^1 z_1) z_{1+i_k}. \tag{3.11}$$

Proof. Let us divide (2.5) and (2.6) by

$$F_M(a_1, a_2 + 1, b_1 + 1, b_2; a_1, c_2 + 1; \mathbf{z}) \quad \text{and} \quad F_M(a_1, a_2 + 1, b_1, b_2 + 1; a_1, c_2 + 1; \mathbf{z}),$$

respectively. Then, in view of (3.7), we get

$$\begin{aligned} &R_M^{(1)}(a_1, a_2, b_1, b_2; a_1, c_2; \mathbf{z}) \\ =&(1 - z_1)\left(1 - \frac{b_2}{c_2} z_2\right) - \frac{a_2 + b_1 + 1}{c_2} z_3 \\ &+ \frac{\frac{(a_2 + 1)(b_1 + 1)}{c_2(c_2 + 1)}(1 - z_1 - z_3) z_3}{R_M^{(1)}(a_1, a_2 + 1, b_1 + 1, b_2; a_1, c_2 + 1; \mathbf{z})} + \frac{\frac{(a_2 + 1) b_2}{c_2(c_2 + 1)}(1 - z_1)^2 z_2 (1 - z_2)}{R_M^{(2)}(a_1, a_2 + 1, b_1 + 1, b_2; a_1, c_2 + 1; \mathbf{z})} \end{aligned}$$

and

$$\begin{aligned}
&R_M^{(2)}(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z})\\
=&(1-z_1)\left(1-\frac{a_2+b_2+1}{c_2}z_2\right)-\frac{b_1}{c_2}z_3\\
&+\frac{\frac{(a_2+1)b_1}{c_2(c_2+1)}(1-z_1-z_3)z_3}{R_M^{(1)}(a_1,a_2+1,b_1,b_2+1;a_1,c_2+2;\mathbf{z})}+\frac{\frac{(a_2+1)(b_2+1)}{c_2(c_2+1)}(1-z_1)^2z_2(1-z_2)}{R_M^{(2)}(a_1,a_2+1,b_1,b_2+1;a_1,c_2+2;\mathbf{z})}.
\end{aligned}$$

Let i_0 be an arbitrary index in $\mathfrak{I}$. We write these relations as follows

$$\begin{aligned}
&R_M^{(i_0)}(a_1,a_2,b_1,b_2;a_1,c_2;\mathbf{z})\\
=&1-z_1-\frac{a_2+b_{i_0}+1}{c_2}(1-\delta_{i_0}^2z_1)z_{4-i_0}-\frac{b_{3-i_0}}{c_2}(1-\delta_{i_0}^1z_1)z_{1+i_0}\\
&+\sum_{i_1=1}^{2}\frac{\frac{(a_2+1)(b_{i_1}+\delta_{i_1}^{i_0})}{c_2(c_2+1)}(1-\delta_{i_1}^2z_1)^2z_{4-i_1}(1-\delta_{i_1}^1z_1-z_{4-i_1})}{R_M^{(i_1)}(a_1,a_2+1,b_1+\delta_{i_0}^1,b_2+\delta_{i_0}^2;a_1,c_2+1;\mathbf{z})},
\end{aligned}\tag{3.12}$$

and this is the first step of constructing a formal branched continued fraction expansion. From (3.12) it is clear that for $i_1\in\mathfrak{I}$

$$\begin{aligned}
&R_M^{(i_1)}(a_1,a_2+1,b_1+\delta_{i_0}^1,b_2+\delta_{i_0}^2;a_1,c_2+1;\mathbf{z})\\
=&1-z_1-\frac{a_2+b_{i_1}+2+\delta_{i_0}^{i_1}}{c_2+1}(1-\delta_{i_1}^2z_1)z_{4-i_1}-\frac{b_{3-i_1}+\delta_{i_0}^{3-i_1}}{c_2+1}(1-\delta_{i_1}^1z_1)z_{1+i_1}\\
&+\sum_{i_2=1}^{2}\frac{\frac{(a_2+2)(b_{i_2}+\delta_{i_0}^{i_2}+\delta_{i_1}^{i_2})}{(c_2+1)(c_2+2)}(1-\delta_{i_2}^2z_1)^2z_{4-i_2}(1-\delta_{i_2}^1z_1-z_{4-i_2})}{R_M^{(i_2)}(a_1,a_2+2,b_1+\delta_{i_0}^1+\delta_{i_1}^1,b_2+\delta_{i_0}^2+\delta_{i_1}^2;a_1,c_2+2;\mathbf{z})},
\end{aligned}\tag{3.13}$$

and, thus, for $i_k\in\mathfrak{I}$ and $k\geq 2$

$$\begin{aligned}
&R_M^{(i_k)}\left(a_1,a_2+k,b_1+\sum_{r=0}^{k-1}\delta_{i_r}^1,b_2+\sum_{r=0}^{k-1}\delta_{i_r}^2;a_1,c_2+k;\mathbf{z}\right)\\
=&1-z_1-\frac{a_2+b_{i_k}+k+1+\sum_{r=0}^{k-1}\delta_{i_r}^{i_k}}{c_2+k}(1-\delta_{i_k}^2z_1)z_{4-i_k}-\frac{b_{3-i_k}+\sum_{r=0}^{k-1}\delta_{i_r}^{3-i_k}}{c_2+1}(1-\delta_{i_k}^1z_1)z_{1+i_k}\\
&+\sum_{i_{k+1}=1}^{2}\frac{\frac{(a_2+k+1)\left(b_{i_{k+1}}+\sum_{r=0}^{k}\delta_{i_r}^{i_{k+1}}\right)}{(c_2+k)(c_2+k+1)}(1-\delta_{i_{k+1}}^2z_1)^2z_{4-i_{k+1}}(1-\delta_{i_{k+1}}^1z_1-z_{4-i_{k+1}})}{R_M^{(i_{k+1})}\left(a_1,a_2+k+1,b_1+\sum_{r=0}^{k}\delta_{i_r}^1,b_2+\sum_{r=0}^{k}\delta_{i_r}^2;a_1,c_2+k+1;\mathbf{z}\right)}.
\end{aligned}\tag{3.14}$$

Substituting (3.13) into (3.11) and taking into account (3.9)–(3.11), in the second step of constructing the expansion we obtain

$$R_M^{(i_0)}(a_1, a_2, b_1, b_2; a_1, c_2; \mathbf{z})$$
$$= v_{i_0}(\mathbf{z}) + \sum_{i_1=1}^{2} \frac{u_{i(1)}(\mathbf{z})}{v_{i(1)}(\mathbf{z}) + \sum\limits_{i_2=1}^{2} \dfrac{u_{i(2)}(\mathbf{z})}{R_M^{(i_2)}(a_1, a_2+2, b_1+\delta_{i_0}^1+\delta_{i_1}^1, b_2+\delta_{i_0}^2+\delta_{i_1}^2; a_1, c_2+2; \mathbf{z})}}.$$

Further, thanks to (3.14), at the kth step we have

$$R_M^{(i_0)}(a_1, a_2, b_1, b_2; a_1, c_2; \mathbf{z})$$
$$= v_{i_0}(\mathbf{z}) + \sum_{i_1=1}^{2} \frac{u_{i(1)}(\mathbf{z})}{v_{i(1)}(\mathbf{z}) + \ddots + \sum\limits_{i_k=1}^{2} \dfrac{u_{i(k)}(\mathbf{z})}{R_M^{(i_k)}\left(a_1, a_2+k, b_1+\sum\limits_{r=0}^{k-1}\delta_{i_r}^1, b_2+\sum\limits_{r=0}^{k-1}\delta_{i_r}^2; a_1, c_2+k; \mathbf{z}\right)}},$$

where $v_{i_0}(\mathbf{z})$, $v_{i(r)}(\mathbf{z})$, $i(r) \in \mathfrak{I}_r$, $1 \le r \le k$, and $u_{i(r)}(\mathbf{z})$, $i(r) \in \mathfrak{I}_r$, $1 \le r \le k-1$, are defined by (3.9), (3.10), and (3.11), respectively. Finally, continuing the process of constructing the expansion, we obtain the formal branched continued fraction expansion (3.8) for the ratio (3.7) for each $i_0 \in \mathfrak{I}$. □

Setting $a_2 = b_1 = 0$ and replacing c_2 by $c_2 - 1$ in Theorem 3.1, we have the following:

Corollary 3.1. *For $i_0 = 1$ and $z_1 \neq 1$ the hypergeometric function*

$$F_M(a_1, 1, 1, b_2; a_1, c_2; \mathbf{z}),$$

where $a_1, b_2, c_2 \in \mathbb{C}$ herewith $a_1 \notin \{0, -1, -2, \ldots\}$ and $c_2 \notin \{1, 0, -1, -2, \ldots\}$, has a formal branched continued fraction

$$\frac{1}{v_{i_0}(\mathbf{z}) + \sum\limits_{i_1=1}^{2} \dfrac{u_{i(1)}(\mathbf{z})}{v_{i(1)}(\mathbf{z}) + \sum\limits_{i_2=1}^{2} \dfrac{u_{i(2)}(\mathbf{z})}{v_{i(2)}(\mathbf{z}) + \ddots}}},$$

where

$$v_{i_0}(\mathbf{z}) = 1 - z_1 - \frac{b_{i_0}+1}{c_2-1}(1-\delta_{i_0}^2 z_1) z_{4-i_0} - \frac{b_{3-i_0}}{c_2-1}(1-\delta_{i_0}^1 z_1) z_{1+i_0}$$

and for $i(k) \in \mathfrak{I}_k$ and $k \ge 1$

$$u_{i(k)}(\mathbf{z}) = \frac{k\left(b_{i_k} + \sum\limits_{r=0}^{k-1}\delta_{i_r}^{i_k}\right)}{(c_2+k-2)(c_2+k-1)}(1-\delta_{i_k}^2 z_1)^2 z_{4-i_k}(1-\delta_{i_k}^1 z_1 - z_{4-i_k}),$$

$$v_{i(k)}(\mathbf{z}) = 1 - z_1 - \frac{b_{i_k}+k+1+\sum\limits_{r=0}^{k-1}\delta_{i_r}^{i_k}}{c_2+k-1}(1-\delta_{i_k}^2 z_1) z_{4-i_k} - \frac{b_{3-i_k}+\sum\limits_{r=0}^{k-1}\delta_{i_r}^{3-i_k}}{c_2+k-1}(1-\delta_{i_k}^1 z_1) z_{1+i_k}.$$

Finally, let us give the following example:

Example 3.1. *From* [20, Formula (2.4.1)], [27, Formulas (5.7) and (5.18)], *and Corollary 3.1 it follows*

$$\begin{aligned}&\ln \frac{1-z_1-z_3}{(1-z_1)(1-z_2)}\\=&((1-z_1)z_2-z_3)F_M(1,1,1,1;1,2;\mathbf{z})\\=&\cfrac{(1-z_1)z_2-z_3}{(1-z_1)(1-z_2)-z_3+\cfrac{\frac{1}{2}z_3(1-z_1-z_3)}{(1-z_1)\left(1-\frac{1}{2}z_2\right)-\frac{3}{2}z_3+\ddots}+\cfrac{\frac{1}{2}(1-z_1)^2z_2(1-z_2)}{(1-z_1)\left(1-\frac{3}{2}z_2\right)-\frac{1}{2}z_3+\ddots}}.\end{aligned}$$

Note that in Example 3.1 this is a formal representation of a special function as a branched continued fraction. Some convergence problem of branched continued fractions can be found in [3, 4, 5].

4. Conclusions and Open Problems

The paper constructs branched continued fraction expansions for hypergeometric functions $F_M(a_1,a_2,b_1,b_2;c_1,c_2;\mathbf{z})$ and their relations in the case when $a_1=c_1$. In the general case, the problem of constructing such expansions remains open. Another problem is to prove that the branched continued fraction converges to the function whose expansion is. An equally important problem is to establish the domains of convergence of the constructed expansions, which, in turn, will be the domains of analytical continuation of the corresponding functions.

Acknowledgments

The authors were partially supported by the Ministry of Education and Science of Ukraine, project registration number 0123U101791.

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