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Research Article

Viscosity approximation involving generalized cocoercive mapping in Hadamard space

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ABSTRACT. In this paper, we introduce a new type of mapping which we term generalized cocoercive mapping in a CAT(0) space, we prove some properties of the new mapping, we also construct implicit viscosity type algorithm for approximating common solution of fixed point of (f, g)-generalized κ-strictly pseudononspreading mapping, quasi-nonexpansive mapping, family of θ_i-generalized demimetric mapping, mixed equilibrium problem and variational inequality problem involving the new mapping. Strong convergence is obtained under some mild conditions and without considering cases as in many results in the literature. Our results improved and generalized many results in the literature and our technique of proof is new and of independent interest.

Keywords: Viscosity approximation, generalized demimetric, generalized cocoercive.

2020 Mathematics Subject Classification: 47H09, 47J25.

1. INTRODUCTION

Let X be a nonempty set and $T : X \to X$ be a nonlinear map. Many physical problems can be formulated as the problem of finding

$$x \in X \text{ such that } Tx = x. \tag{1.1}$$

Fixed point of T is a point satisfying (1.1), the set of all fixed points of T is denoted by $Fix(T)$. The fixed point of the map T plays the role of an equilibrium of a system (usually of differential equations) defined in terms of the map T. The concept of equilibrium system is essential in biology, economics, noncooperative game theory, ergodic theory, physics, chemistry and so on. Therefore, fixed point theorems are related to these fields. Fixed point theory is an essential tool for establishing existence of solutions of differential equations, integro-differential equations, minimization, variational inequalities, mixed equilibrium, split feasibility problem and so on (see for example [6, 8, 12, 14, 17, 18, 19, 21, 22] and [25] for more details on fixed point theory).

Let (X, ρ) be a metric space and $x, y \in X$, a geodesic from x to y is a function $\gamma : [a, b] \subseteq \mathbb{R} \to X$ satisfying the following axioms:

$(G_1) : \ \gamma(a) = x, \gamma(b) = y$,
$(G_2) : \ \rho(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for each $t_1, t_2 \in [a, b]$.

Geodesic segment from x to y is the image of the function γ, ($i.e\ \gamma[a, b]$). A metric space (X, d) is called geodesic if every for every $x, y \in X$, there exists $\gamma : [a, b] \to X$ satisfying (G_1) and (G_2). We say that a metric space X is uniquely geodesic space if for every $x, y \in X$, there exists a unique function $\gamma : [a, b] \to X$ satisfying (G_1) and (G_2), in this case $\gamma[a, b]$ is denoted by $[x, y]$.

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Let X be a geodesic space and $x_1, x_2, x_3 \in X$, a geodesic triangle in X denoted by $\Delta(x_1, x_2, x_3)$ is a set consisting of x_1, x_2, x_3 as its vertices and three geodesic segments joining each pair of the vertices as its sides. For every geodesic triangle in X, there exists a plane triangle called a comparison triangle denoted by $\bar{\Delta}(x_1, x_2, x_3)$ and define as $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ satisfying $\rho(x_i, x_j) = \rho_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j})$ for every $i, j \in \{1, 2, 3\}$.

A geodesic space (X, ρ) is called a CAT(0) space if each geodesic triangle Δ in X satisfies the following inequality

$$\rho(x, y) \leq \rho_{\mathbb{R}^2}(\bar{x}, \bar{y}), \quad \forall (x, y) \in \Delta^2, (\bar{x}, \bar{y}) \in \bar{\Delta}^2,$$

where $\bar{\Delta}$ is a comparison triangle for Δ, it is known that CAT(0) space is uniquely geodesic space. Examples of CAT(0) spaces are Hilbert spaces and R-Trees (see [10, 11, 26]).

Let X be a CAT(0) space, $(1-t)x \oplus ty$ denote the unique point z in the geodesic segment joining x to y for every $x, y \in X$ such that $\rho(z, x) = t\rho(x, y)$ and $\rho(z, y) = (1-t)\rho(x, y)$, where $t \in [0, 1]$, let $[x, y] = \{(1-t)x \oplus ty : t \in [0, 1]\}$, then $K \subseteq X$ is convex if $[x, y] \subseteq K \; \forall x, y \in X$. A complete CAT(0) space is called a Hadamard space.

Berg and Nikolaev [4] introduced the notion of quasilinearization mapping in CAT(0) spaces, which they defined as a mapping $\langle ., . \rangle : (X \times X) \times (X \times X) \longrightarrow \mathbb{R}$ by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\Big(\rho^2(a, d) + \rho^2(b, c) - \rho^2(a, c) - \rho^2(b, d)\Big), \quad \forall (a, b, c, d) \in X^4, \tag{1.2}$$

called the quasilinearization mapping (see [4] for more information). It follows from the definition above that for $(a, b, c, d, e) \in X^5$:

(1) $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{dc} \rangle$,
(2) $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = \rho^2(a, d)$,
(3) $\langle \overrightarrow{ab}, \overrightarrow{dc} \rangle = \langle \overrightarrow{ab}, \overrightarrow{de} \rangle + \langle \overrightarrow{ab}, \overrightarrow{ec} \rangle$.

A metric space X is said to satisfy the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq \rho(a, b)\rho(c, d), \quad \forall (a, b, c, d) \in X^4. \tag{1.3}$$

It is known that a geodesic space is CAT(0) space if and only if (1.3) holds, (see [5]). Kakavandi and Amini [15] introduced the concept of dual space of a CAT(0) space X (see [15] for more details). Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in a Hadamard space X and $x \in X$, let

$$r(x, \{x_n\}_{n=1}^{\infty}) := \limsup_{n \to \infty} \rho(x, x_n)$$

the asymptotic radius of $\{x_n\}_{n=1}^{\infty}$ is given by

$$r(\{x_n\}_{n=1}^{\infty}) := \inf_{x \in X} r(x, \{x_n\}_{n=1}^{\infty})$$

and the asymptotic center of $\{x_n\}_{n=1}^{\infty}$ is the set

$$A(\{x_n\}_{n=1}^{\infty}) := \{x \in X : r(x, \{x_n\}_{n=1}^{\infty}) = r(\{x_n\}_{n=1}^{\infty})\}.$$

If X is a Hadamard space then the asymptotic center of $\{x_n\}$ is a singleton set (see [9]).

Definition 1.1. *Let X be a Hadamard space, a sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to Δ-converges to a point $x^* \in X$, if $\limsup_{n \to \infty} \rho(x, x_{n_k}) = x^*$, for every subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$.*

Many authors studied the notion of Δ-convergence (see [2, 6, 9, 19] for more details). Ugwunnadi et al. study a new class of nonspreading-type mappings in the setting of Hadamard space which they called generalized κ-strictly pseudononspreading mapping defined as follows: Let X be a metric space and $K \subseteq X$, a mapping $G : K \to X$ is generalized κ-strictly pseudononspreading, if there exists two functions $f, g : K \to [0, \tau]$, $\tau < 1$ and $0 \leq \kappa < 1$ satisfying two conditions:

$$(1-\kappa)\rho^2(Gx, Gy) \leq \kappa\rho^2(x, y) + (f(x) - \kappa)\rho^2(Gx, y) + (g(x) - \kappa)\rho^2(x, Gy) + \kappa(\rho^2(x, Gx) + \rho^2(y, Gy)), \ \ \forall x, y \in K$$

and $(f(x) + g(x)) \in (0, 1], \ \ \forall x \in K$.

They showed that generalized κ-strictly nonspreading mapping is a generalization of the class of strictly pseudononspreading and the class of generalized nonspreading mappings and studied some properties of the mapping with some examples and they also constructed a Halpern-type scheme which they showed that it converges strongly to fixed point of the map G. For more details see [34]. Takahashi [30] introduced a new class of nonlinear mappings in a real Hilbert space which is define as follows:

Let H be a real Hilbert space and K be a nonempty, closed and convex subset of H, a mapping $U : K \longrightarrow H$ is called α-demimetric, if $Fix(U) \neq \emptyset$ and there exists $\alpha \in (-\infty, 1)$ such that

$$\langle x - y, x - Ux\rangle \geq \frac{1-\alpha}{2}\|x - Ux\|^2.$$

The class of demimetric mappings is important in optimization theory since it contains many common types of operators. For example, the class of α-demicontractive mapping with $\alpha \in [0, 1)$, the metric projections, generalized hybrid mappings, the resolvents of maximal monotone operators (which are well-known useful tools for solving optimization problems) in Hilbert spaces are subclasses of the class of α-demimetric mappings (see [3, 30]). Many authors studied this class of mappings in Banach spaces (see [20, 30, 31, 32]). Aremu et al. [3] extended the above mapping in the setting of Hadamard spaces as follows: Let H be a Hadamard space and K be a nonempty, closed and convex subset of X, a mapping $U : K \longrightarrow X$ is α-demimetric, if $Fix(U) \neq \emptyset$ and there exists $\alpha \in (-\infty, 1)$ such that

$$\langle \overrightarrow{xy}, \overrightarrow{xUx}\rangle \geq \frac{1-k}{2}\rho^2(x, Ux), \ \ \ \forall x \in K, \ y \in Fix(U).$$

They gave an example of a demimetric mapping and established some fixed point theorems for this class of mappings and proved a strong convergence theorem for approximating a common solution of finite family of minimization problems and fixed point problems for this class of mappings in Hadamard spaces. Kawasaki and Takahashi [16] generalized the class of demimetric mappings as follows:

Let E be a smooth real Banach space and K be a nonempty, closed and convex subset of a E, let $\alpha \neq 0$, a map $U : K \longrightarrow E$ is α-generalized demimetric if $Fix(U) \neq \emptyset$ and

$$\alpha\langle x - y, J(x - Ux)\rangle \geq \|x - Ux\|^2, \ \ \ \forall x \in K, \ y \in Fix(U),$$

where J is a duality mapping on E. Takahashi [33] studied this class of mappings in Banach spaces. Recently, Ogwo et al. [23] introduced generalized demimetric in the setting of Hadamard space as follows:

Let X be a Hadamard space, let $\alpha \neq 0$, a mapping $U : X \longrightarrow X$ is α-generalized demimetric, if $Fix(T) \neq \emptyset$ and

$$\alpha\langle \overrightarrow{xy}, \overrightarrow{xUx}\rangle \geq \rho^2(x, Ux), \ \ \ \forall x \in X, \ y \in Fix(U).$$

They gave some examples and properties of generalized demimetric mappings in Hadamard spaces, they also proved a strong convergence theorem involving the map. Let K be a nonempty, closed and convex subset of a Hadamard space X, a mapping $U : K \longrightarrow K$ is Δ-demiclosed, if for any bounded sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\Delta - \lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} \rho(x_n, Ux_n) = 0$, then $x = Ux$. A mapping $U : K \longrightarrow X$ is

(i) nonexpansive [29], if
$$\rho(Ux, Uy) \leq \rho(x, y), \ \ \forall x, y \in K,$$

(ii) quasi-nonexpansive, if $Fix(U) \neq \emptyset$ and
$$\rho(q, Ux) \leq \rho(q, x) \ \ \forall (x, q) \in K \times Fix(U),$$

(iii) α-inverse strongly monotone [2] if there exists $\alpha > 0$ such that
$$\alpha\phi_U(x, y) \leq \rho^2(x, y) - \langle \overrightarrow{UxUy}, \overrightarrow{xy} \rangle, \quad \forall x, y \in K,$$
where,

(1.4)
$$\phi_U(x, y) = \rho^2(x, y) - 2\langle \overrightarrow{UxUy}, \overrightarrow{xy} \rangle + \rho^2(Ux, Uy).$$

It is known that (see [2]), $\phi_U(x, y) \geq 0, \quad \forall x, y \in K$.

(iv) firmly nonexpansive if
$$\langle \overrightarrow{UxUy}, \overrightarrow{xy} \rangle \geq \rho^2(Ux, Uy), \quad \forall x, y \in K.$$

The metric projection $P_K : X \longrightarrow K$ assigns to every $x \in X$, a unique point $P_K(x)$ in K such that
$$\rho(x, y) \geq \rho(x, P_K x), \ \ \forall y \in K,$$
the map P_K is firmly nonexpansive [7]. The Variational Inequality Problem $(V.I.P.)$ in Hilbert space is formulated as follows

(1.5)
$$\text{find } x \in K \text{ such that } \langle Bx, x - y \rangle \leq 0, \quad \forall y \in K,$$

where K is a nonempty closed and convex subset of H and B is a nonlinear mapping defined on K. Stampacchia [28] introduced $V.I.P.$ for modeling problems arising in mechanics and the regularity problem for partial differential equations, Stampacchia [28] studied a generalization of the Lax-Milgram theorem and called all problems of this kind $V.I.P.$s. The theory of $V.I.P.$ has numerous applications in diverse fields such as physics, engineering, economics, mathematical programming and others (see [27, 28] and references therein). Alizadeh et al. [2] extended (1.5) to the setting of Hadamard space H as follows:

(1.6)
$$\text{find } x \in K \text{ such that } \langle \overrightarrow{Bxx}, \overrightarrow{yx} \rangle \leq 0, \ \ \forall y \in K,$$

when B is inverse strongly monotone, they established the existence of $V.I.P$ (1.6) in a Hadamard space. Furthermore, they constructed the following algorithm:

(1.7)
$$\begin{cases} x_1 \in K, \\ y_n = P_K(\beta_n x_n \oplus (1 - \beta_n)Bx_n), \\ x_{n+1} = P_K(\alpha_n x_n \oplus (1 - \alpha_n)Ty_n), \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subseteq (0, 1)$, T and B are nonexpansive and inverse strongly monotone mappings, respectively. They also obtained Δ-convergence of algorithm (1.7) to a common solution of the $V.I.P$ (1.6) and fixed point of T. Recently G. C. Ugwunnadi et al. [35] constructed a viscosity type algorithm in a setting of Hadamard space which comprises of a demimetric mapping, a finite family of inverse strongly monotone mappings and an equilibrium problem for a bifunction, they succeded in obtaining a strong convergence of their proposed algorithm to a common solution of a variational inequality problem, fixed point problem and equilibrium problem in Hadamard space. Furthermore, they gave applications and numerical examples.

Let K be a nonempty subset of a Hadamard space X and $\varphi : K \longrightarrow \mathbb{R}$ and $F : K \times K \longrightarrow \mathbb{R}$ be a function and bifunction respectively, a minimization problem $(M.P)$ is a problem of searching for $x^* \in K$ such that

$$\varphi(x^*) \leq \varphi(x), \quad \forall x \in K, \tag{1.8}$$

the point x^* satisfying inequality (1.8) is called a minimizer of $\varphi(.)$ on K, we denote the solution set of (1.8) by $M.P(\varphi, K)$. An Equilibrium Problem $(E.P)$ is to find $x^* \in K$ satisfying

$$F(x^*, y) \geq 0, \quad \forall y \in K, \tag{1.9}$$

the point x^* satisfying inequality (1.9) is called an equilibrium point of $F(.,.)$ on K, we denote the solution set of (1.9) by $E.P(F, K)$. C. Izuchukwu et al. [13] introduced mixed equilibrium problem $(M.E.P)$ in a Hadamard space X as the problem of finding $x^* \in K$ such that

$$F(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \ \ \forall y \in K \tag{1.10}$$

the solution set of (1.10) is denoted by $MEP(F, \varphi, K)$. They obtained the following result for existence of solution:

Theorem 1.1 ([13]). *Let K be a nonempty closed and convex subset of an Hadamard space X. Let $\psi : K \longrightarrow \mathbb{R}$ be a real-valued function and $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction such that the following assumptions hold:*

- A_1: $F(x, x) \geq 0, \forall x \in K$.
- A_2: *For every $x \in K$, the set $\{y \in K : F(x, y) + \varphi(y) - \varphi(x) < 0\}$ is convex.*
- A_3: *There exists a compact subset $D \subset K$ containing a point $y_0 \in D$ such that $x \in K/D$ implies $F(x, y_0) + \varphi(y_0) - \varphi(x) < 0$,*

then, the $M.E.P(F, \varphi, K)$ in (1.10) has a solution.

For uniqueness of solution, they obtained the following result.

Theorem 1.2 ([13]). *Let K be a nonempty, closed and convex subset of a Hadamard space X, let $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction, let $\varphi : K \longrightarrow \mathbb{R}$ be a convex function such that the following conditions hold:*

- A_1: $F(x, x) = 0, \quad \forall x \in K$,
- A_2: *F is monotone, i.e , $F(x, y) + F(y, x) \leq 0, \ \ \forall x, y \in K$,*
- A_3: *$F(x, .) : K \longrightarrow \mathbb{R}$ is convex for each $x \in K$.*
- A_4: *Given any $q \in X$ and $\lambda > 0$, there exists a compact subset $D_q \subseteq K$ containing a point $y_q \in D_q$ such that $x \in K/D_q$ implies*

$$F(x, y_q) + \varphi(y_q) - \varphi(x) + \frac{1}{\lambda}\langle \overrightarrow{xy_q}, \overrightarrow{qx} \rangle < 0,$$

then (1.10) has a unique solution.

Based on the information provided above, our main goal in this paper is to introduce and study some properties of a new mapping called generalized cocoercive in Hadamard space, construct viscosity type algorithm for approximating common solution of fixed point of (f, g)-generalized κ-strictly pseudononspreading mapping, quasi-nonexpansive mapping, a finite family of θ_i-generalized demimetric mapping and mixed equilibrium problem for a bifunction and convex lower semi continuous function and variational inequality problem involving the generalized cocoercive mapping in Hadamard space. Our result generalizes and compliments some results in the literature. Furthermore our technique of proof is new and is of independent interest, as it does not involve the use of cases as done in [23], [34] and [35].

2. Preliminaries

In this section, we state some known and useful results which we are going to use in the proof of our main result.

Lemma 2.1. *Let X be a CAT(0) space and $x, y, z \in X$ and $t \in [0,1]$ then the following inequalities hold*

(i) $\rho(tx \oplus (1-t)y, z) \leq t\rho(x,z) + (1-t)\rho(y,z)$ ([9]),
(ii) $\rho^2(tx \oplus (1-t)y, z) \leq t\rho^2(x,z) + (1-t)\rho^2(y,z) - t(1-t)\rho(x,y)$ ([9]),
(iii) $\rho^2(tx \oplus (1-t)y, z) \leq t^2\rho^2(x,z) + (1-t)^2\rho^2(y,z) + 2t(1-t)\langle\overrightarrow{xz}, \overrightarrow{yz}\rangle$ ([7]).

Lemma 2.2 ([19]). *Every bounded sequence in a Hadamard space has a Δ-convergent subsequence.*

Lemma 2.3 ([15]). *Let H be a Hadamard space and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X then $\limsup\limits_{n\to\infty}\langle\overrightarrow{x_n x}, \overrightarrow{yx}\rangle \geq 0$, $\forall y \in X$ if and only if $\{x_n\}_{n=1}^{\infty}$ Δ-converges to x.*

Lemma 2.4 ([2]). *Let K be a nonempty closed and convex subset of a Hadamard space H and $A : K \longrightarrow H$ be α-inverse strongly monotone . Suppose $\mu \in [0,1]$ and a map $A_\mu : K \longrightarrow X$ define by $A_\mu(x) = (1-\mu)x + \mu Ax$ for each $x \in K$. If $\mu \in (0, 2\mu)$, then A_μ is nonexpansive and $Fix(A) = Fix(A_\mu)$.*

Lemma 2.5 ([2]). *Let K be a nonempty convex subset of a Hadamard space H and $A : K \longrightarrow H$ be a mapping and for $\mu \in (0,1]$, a map $A_\mu : K \longrightarrow X$ defined as $A_\mu(x) = (1-\mu)x + \mu A(x)$, for all $x \in K$ then,*

$$VI(K,A) = VI(K,A_\mu).$$

Remark 2.1 ([24]). *As a consequence of the Lemma 2.5 above, we have*

$$Fix(P_K A) = VI(K,A) = VI(K,A_\mu) = Fix(P_K A_\mu).$$

Lemma 2.6 ([29]). *Let X be a CAT(0) space, $\{x_i\}_{i=1}^{N} \subseteq X$ and $\alpha_i \in [0,1]$, $\forall i \in \{1,2,...,N\}$ satisfying $\sum\limits_{i=1}^{N}\alpha_i = 1$, then,*

$$\rho\Big(\bigoplus_{i=1}^{N}\alpha_i x_i, z\Big) \leq \sum_{i=1}^{N}\alpha_i\rho(x_i,z), \ \forall z \in X. \tag{2.11}$$

Lemma 2.7 ([6]). *Let X be a CAT(0) space and $q \in X$, let $\{x_i\}_{i=1}^{N} \subseteq X$ and $\{\alpha_i\}_{i=1}^{N} \subseteq [0,1]$ such that $\sum\limits_{i=1}^{N}\alpha_i = 1$, then*

$$\rho^2\left(q, \bigoplus_{i=1}^{N}\alpha_i x_i\right) \leq \sum_{i=1}^{N}\alpha_i\rho^2(q,x_i) - \sum_{i,j=1, i\neq j}\rho^2(x_i,x_j).$$

Lemma 2.8 ([8]). *Let H be a Hadamard space and $T : H \longrightarrow H$ be a nonexpansive mapping, then T is Δ-demiclosed.*

Theorem 2.3 ([34]). *Let K be a nonempty closed and convex subset of a Hadamard space H and $G : K \to K$ be (f,g)-generalized κ-strictly pseudononspreading mapping with $\kappa \in [0,1)$ and $f, g : K \to [0,\tau]$, $\tau < 1$ and $(f(x) + g(x)) \in (0,1]$ for every $x \in K$. Assume $Fix(G) \neq \emptyset$ and $f(q) \neq 0$ with $\beta \in \left[\frac{\kappa}{f(q)}, 1\right)$, $\forall q \in Fix(G)$, then $Fix(G)$ is closed and convex.*

Remark 2.2 ([34]). *Observe that if G is $(f,g)-$generalized κ-strictly pseudononspreading with $Fix(G)\neq\emptyset$ and $f(q)\neq 0$, for each $q\in Fix(G)$, then for every $q\in Fix(G)$ and $x\in dom(G)$, we obtain*

$$\rho^2(q,Gx)\leq\rho^2(q,x)+\frac{\kappa}{f(q)}\rho^2(x,Gx). \tag{2.12}$$

Lemma 2.9 ([23]). *Let X be a CAT(0) space and $U:X\longrightarrow X$ be β generalized demimetric mapping with $\beta\neq 0$. Suppose that $U_\alpha x=\alpha x\oplus(1-\alpha)Ux$ with $\beta\leq\frac{2}{1-\alpha}$ and $\alpha\in(0,1)$, then U_α is quasi-nonexpansive and $Fix(U_\alpha)=Fix(U)$.*

Lemma 2.10 ([36]). *Let $\{\gamma_n\}_{n=1}^{\infty}\subset(0,1)$ and $\{\delta_n\}_{n=1}^{\infty}\subset\mathbb{R}$ satisfying*

(i) $\sum_{n=1}^{\infty}\gamma_n=\infty$,

(ii) $\limsup_{n\longrightarrow\infty}\delta_n\leq 0$ *or* $\sum_{n=1}^{\infty}|\gamma_n\delta_n|<\infty$,

if $\{a_n\}_{n=1}^{\infty}\subset[0,\infty)$ such that

$$a_{n+1}\leq(1-\gamma_n)a_n+\gamma_n\delta_n,\ \forall n\geq 0$$

then

$$\lim_{n\longrightarrow\infty}a_n=0.$$

Theorem 2.4 ([13]). *Let K be a nonempty, closed and convex subset of a Hadamard space X, $F:K\times K\longrightarrow\mathbb{R}$ be a bifunction, let $\varphi:K\longrightarrow\mathbb{R}$ be a convex function such that conditions A_1-A_4 of Theorem 1.2 holds. For $\lambda>0$, we have that $T_\lambda^{\varphi,F}$ is single valued. Moreover, if $K\subset Dom(T_\lambda^{\varphi,F})$, then*

(i) *$T_\lambda^{\varphi,F}$ is firmly nonexpansive on K.*

(ii) *If $Fix(T_\lambda^{\varphi,F})\neq\emptyset$, then*

$$\rho^2(x,T_\lambda^{\varphi,F}x)+\rho^2(T_\lambda^{\varphi,F}x,q)\leq\rho^2(x,q),\ \forall x\in K,q\in Fix(T_\lambda^{\varphi,F}). \tag{2.13}$$

(iii) *$Fix(T_\lambda^{\varphi,F})=MEP(K,F,\varphi)$.*

Lemma 2.11 ([34]). *Let K be a nonempty closed and convex subset of a Hadamard space H and $G:K\to K$ be (f,g)-generalized κ-strictly pseudononspreading mapping with $0\leq\kappa<1$ and $f,g:K\to[0,\tau]$, $\tau<1$ and $(f(x)+g(x))\in(0,1]$, $\forall x\in K$. Suppose $(2\kappa+f(x))<1$, $\forall x\in K$ and $\{x_n\}$ is a bounded sequence in K such that $\Delta-\lim_{n\to\infty}x_n=x^*$ and $\lim_{n\to\infty}\rho(x_n,Gx_n)=0$ then $x^*\in Fix(G)$.*

3. Main Results

In this section, we present the main results of this paper.

Theorem 3.5. *Let H be a CAT(0) space, K be a nonempty subset of H, $A:K\to H$ be a mapping and $\phi_A:H\times H\to\mathbb{R}$ be a mapping defined as in (1.4) then the following holds:*

(i) $\phi_A(x,y)=\phi_A(y,x),\ \ \forall x,y\in K.$

(ii) $\Big(\rho(x,y)-\rho(Ax,Ay)\Big)^2\leq\phi_A(x,y)\leq\Big(\rho(x,y)+\rho(Ax,Ay)\Big)^2,\ \ \forall x,y\in K.$

Proof. (i) Let $x,y\in K$, then from the definition of $\phi_A(.,.)$ and property of $\rho(.,.)$, we have

$$\begin{aligned}\phi_A(x,y)&=\rho^2(x,y)-2\langle\overrightarrow{AxAy},\overrightarrow{xy}\rangle+\rho^2(Ax,Ay)\\&=\rho^2(y,x)-2\langle\overrightarrow{AyAx},\overrightarrow{yx}\rangle+\rho^2(Ay,Ax)\\&=\phi_A(y,x).\end{aligned}$$

(ii) Applying Cauchy-Schwarz inequality twice, we obtain

$$\begin{aligned}\Big(\rho(x,y)-\rho(Ax,Ay)\Big)^2 &= \rho^2(x,y)-2\rho(x,y)\rho(Ax,Ay)+\rho^2(Ax,Ay)\\ &\leq \phi_A(x,y)\\ &= \rho^2(x,y)-2\langle\overrightarrow{xy},\overrightarrow{AxAy}\rangle+\rho^2(Ax,Ay)\\ &= \rho^2(x,y)+2\langle\overrightarrow{yx},\overrightarrow{AxAy}\rangle+\rho^2(Ax,Ay)\\ &\leq \rho^2(x,y)+2\rho(x,y)\rho(Ax,Ay)+\rho^2(Ax,Ay)\\ &= \Big(\rho(x,y)+\rho(Ax,Ay)\Big)^2.\end{aligned}$$

□

Remark 3.3. *We observe that from Theorem 3.5 (ii), we can easily obtain that $\phi_A(x,y)\geq 0, \forall x,y\in K$.*

4. Generalized Cocoercive Mapping in CAT(0) Spaces

The definition of generalized cocoercive is motivated from the newly introduced map called generalized inverse strongly monotone map (see [1]). Let H be a real Hilbert space with an inner product $\langle .,.\rangle$ and a norm $\|.\|$ and K be a nonempty subset of H. For $\alpha>0$, a mapping $B:K\to H$ is called generalized α-inverse strongly monotone (see [1]), if

$$\langle Bx-By,x-y\rangle\geq\alpha\|Bx-By\|^2-\|Bx\|\|By\|,\;\;\forall x,y\in K. \tag{4.14}$$

By setting $A=I-B$, where I is identity map, we see that $(I-A)$ is generalized α-cocoercive.

Definition 4.2. *Let H be a CAT(0) space, K be a nonempty subset of H, for $\alpha>0$, a mapping $B:K\to H$ is called generalized α-cocoercive, there exists a map $A:K\to H$ such that for all $x,y\in K$,*

$$\rho^2(x,y)+\rho(x,Ax)\rho(y,Ay)-\langle\overrightarrow{AxAy},\overrightarrow{xy}\rangle\geq\alpha\phi_A(x,y), \tag{4.15}$$

where ϕ_A is defined as in (1.4).

Remark 4.4. *It is not difficult to see that every α-inverse strongly monotone is generalized α-cocoercive mapping.*

Proof. For $\alpha>0$, let $B:K\to H$ be α-inverse strongly monotone mapping then there exists $A:K\to H$ such that

$$\begin{aligned}\rho^2(x,y)-\langle\overrightarrow{AxAy},\overrightarrow{xy}\rangle &\geq \alpha\phi_A(x,y)\\ &\geq \alpha\phi_A(x,y)-\rho(x,Ax)\rho(y,Ay),\end{aligned}$$

which implies B is generalized α-cocoercive. □

The converse is not true in general:

Example 4.1. *Consider $[-1,1]$ and for $d>0$, let $A:[-1,1]\to\mathbb{R}$ be defined by*

$$Ax=\begin{cases}x+d, & x\leq 0\\ x-d, & x>0\end{cases} \tag{4.16}$$

then A satisfy the following inequality:

$$|x-y|^2-(Ax-Ay)(x-y)\geq\frac{1}{4}\phi_A(x,y)-|x-Ax||y-Ay|,\;\;\forall x,y\in[-1,1].$$

Solution 1. *Define a map $B : [-1, 1] \to \mathbb{R}$ by*

$$Bx = \begin{cases} -d, & x \leq 0 \\ d, & x > 0 \end{cases}$$

then for $x \in [-1, 1]$, we have

$$\begin{aligned} x - Bx &= x - \begin{cases} -d, & x \leq 0 \\ d, & x > 0 \end{cases} \\ &= \begin{cases} x + d, & x \leq 0 \\ x - d, & x > 0 \end{cases} \\ &= Ax. \end{aligned}$$

Thus, $A = I - B$ and also, B satisfy the following inequality:

$$(Bx - By)(x - y) \geq \frac{1}{4}|Bx - By|^2 - |Bx||By|, \ \ \forall x, y \in [-1, 1],$$

and for each $\alpha > 0$,

$$(Bx - By)(x - y) \ngeq \frac{1}{4}|Bx - By|^2 \ \ \forall x, y \in [-1, 1]$$

which implies B is generalized $\frac{1}{4}$-inverse strongly monotone mapping which is not α-inverse strongly monotone mapping for any $\alpha > 0$ (see [1] for more details).

Lemma 4.12. *Let K be a nonempty closed and convex subset of a CAT(0) space H, for $\alpha > \frac{1}{2}$, let $A : K \to H$ be a generalized cocoercive mapping. Let $\mu \in [0, 1]$ and define $A_\mu : K \to H$ by*

$$A_\mu x = \mu Ax \oplus (1 - \mu)x, \quad \forall x \in K, \tag{4.17}$$

if $\mu \in (0, 2\alpha - 1)$, then A_μ is nonexpansive mapping and $Fix(A_\mu) = Fix(T)$.

Proof. Let $x, y \in K$, then

$$\begin{aligned} \rho^2(A_\mu x, A_\mu y) &\leq (1-\mu)^2\rho^2(x, y) + \mu(1-\mu)\rho^2(x, Ay) - \mu(1-\mu)\rho^2(y, Ay) \\ &+ \mu(1-\mu)\rho^2(Ax, y) + \mu^2\rho^2(Ax, Ay) - \mu(1-\mu)\rho^2(x, Ax) \\ &\leq (1-\mu)^2\rho^2(x, y) + \mu^2\rho^2(Ax, Ay) \\ &+ 2\mu(1-\mu)\Big[\rho^2(x, y) - \alpha\phi_A(x, y) + \rho(x, Ax)\rho(y, Ay)\Big] \\ &= (1-\mu)^2\rho^2(x, y) + \mu^2\rho^2(Ax, Ay) + 2\mu(1-\mu)\rho(x, Ax)\rho(y, Ay) \\ &- 2\alpha\mu(1-\mu)\phi_A(x, y) \\ &\leq (1-\mu)^2\rho^2(x, y) - 2\alpha\mu(1-\mu)\phi_A(x, y) + 2\mu(1-\mu)\rho(x, Ax)\rho(y, Ay) \\ &+ \mu^2\phi_A(x, y) + 2\mu^2\langle\overrightarrow{AxAy}, \overrightarrow{xy}\rangle - \mu^2\rho^2(x, y) \\ &= \rho^2(x, y) + (-2\alpha\mu(1-\mu) + \mu^2)\phi_A(x, y) \\ &+ 2\mu^2(\langle\overrightarrow{AxAy}, \overrightarrow{xy}\rangle - \rho^2(x, y) - \rho(x, Ax)\rho(y, Ay)) + 2\mu\rho(x, Ax)\rho(y, Ay) \\ &\leq \rho^2(x, y) + \mu\Big((\mu - 2\alpha)\phi_A(x, y) + 2\rho(x, Ax)\rho(y, Ay)\Big) \\ &\leq \rho^2(x, y) + \mu\Big((\mu - 2\alpha + 1)\max\{\phi_A(x, y), 2\rho(x, Ax)\rho(y, Ay)\}\Big) \\ &\leq \rho^2(x, y). \end{aligned}$$

Consequently, we get A_μ is nonexpansive. □

Lemma 4.13. *Let K be a nonempty closed and convex subset of a Hadamard space H. For $\alpha_i > \frac{1}{2}$, let $A_i : K \longrightarrow H$, $i \in 1, 2, ..., N$ be a family of α_i-generalized cocoercive mappings, define $A_{\mu_i} = (1 - \mu_i)I \oplus \mu_i A_i$, for $\mu_i \in (0, 2\alpha_i - 1)$ $\gamma_i \in (0, 1)$, $i \in \{1, 2, ..., N\}$ such that $\sum_{i=1}^{N} \gamma_i = 1$ then the map $\bigoplus_{i=1}^{N} \gamma_i P_K A_{\mu_i}$ is a nonexpansive mapping. If in addition, $\bigcap_{i=1}^{N} Fix(P_K A_{\mu_i}) \neq \emptyset$, then $Fix\Big(\bigoplus_{i=1}^{N} \gamma_i P_K A_{\mu_i}\Big) = \bigcap_{i=1}^{N} Fix(P_K A_{\mu_i})$.*

Proof. It follows from Lemma 2.5. □

Theorem 4.6. *Let H be a Hadamard space, K be a nonempty closed and convex subset of H, $\alpha > \frac{1}{2}$ and A be generalized α-cocoercive mapping of K into H, then $VI(K, A) \neq \emptyset$.*

Proof. Let $\mu \in (0, 2\alpha - 1]$, we define a mapping $B : K \to K$ by

$$Bx = P_K(\mu Ax \oplus (1 - \mu)x), \ \forall x \in K. \tag{4.18}$$

Since P_K and A_μ are nonexpansive, then

$$\begin{aligned} \rho(Bx, By) &\leq \rho(A_\mu x, A_\mu y), \\ &\leq \rho(x, y) \end{aligned}$$

which implies, B is nonexpansive, there exist $\hat{x} \in K$ satisfying

$$\begin{aligned} \hat{x} = B\hat{x} &\Leftrightarrow \hat{x} = P_K(\mu A\hat{x} \oplus (1 - \mu)\hat{x}) \\ &\Leftrightarrow \hat{x} \in VI(K, A). \end{aligned}$$

Therefore, $VI(K, A) \neq \emptyset$. □

Theorem 4.7. *Let H be a Hadamard space and H^* be its dual space, let K be a nonempty closed and convex subset of H. Let $A_i : K \longrightarrow H$ be family of generalized α_i-cocoercive mapping, where $\alpha_i > \frac{1}{2}$, let $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction satisfying condition A_1 to A_4 of Theorem 1.2, $\varphi : K \longrightarrow \mathbb{R}$ be a convex and lower semi-continuous function, let $G : K \to K$ be (f, g)-generalized κ-strictly pseudononspreading mapping with $\kappa \in [0, 1)$, where $f, g : K \to [0, \tau]$, $\tau < 1$, such that f, g, κ, τ satisfies assumptions in Theorem 2.3 [34], Remark 2.2 [34] and Lemma 2.11 [34]. Let $S : K \to K$ be quasi-nonexpansive mapping, let $T_i : K \longrightarrow K$ be family of θ_i-generalized demimetric mapping with $\theta_i \neq 0$. Assume that $\Omega = Fix(G) \cap Fix(S) \cap M.E.P(F, \varphi, K) \bigcap_{i=1}^{N} \Big(VI(K, A_i) \cap Fix(T_i)\Big) \neq \emptyset$ and $\{z_n\}_{n=1}^{\infty}$ is a sequence generated by*

$$\begin{cases} z_0, z_1 \in X, \\ v_n = \alpha_n z_n \oplus (1 - \alpha_n) z_{n+1}, \\ w_n = T_{r_n}^{F,\varphi} v_n, \\ x_n = \delta_n w_n \oplus \beta_{n,0} G w_n \oplus \bigoplus_{i=1}^{N} \beta_{n,i} P_K A_{\mu_i} w_n, \\ y_n = a_n w_n \oplus b_{n,0} S w_n \oplus \bigoplus_{i=1}^{N} b_{n,i} T_i^{\alpha} x_n, \\ z_{n+1} = \lambda_n h(v_n) \oplus (1 - \lambda_n) y_n, \ n \in \mathbb{N}, \end{cases} \tag{4.19}$$

where $h : H \to H$ is γ-contraction for some $\gamma \in [0,1)$ and for each $i \in \{1,2,...,m\}$

$$A_{\mu_i}x = (1-\mu_i)x \oplus \mu_i A_i x, \text{ with } \mu_i \in (0, 2\alpha_i - 1),$$
$$T_i^{\alpha}x = \alpha x \oplus (1-\alpha)T_i x$$

with assumption that T_i^{α} and S are Δ-demiclosed, with $\theta_i \leq \frac{2}{1-\alpha}, \alpha \in (0,1)$, $\{r_n\} \subseteq (0,1)$ and $\{\alpha_n\}, \{\beta_{n,0}\}, \{\beta_{n,i}\}, \{\delta_n\}, \{b_{n,i}\}, \{b_{n,0}\}, \{a_n\}, \{\lambda_n\}$ are sequences in $\mathbb{R}$ satisfying the following conditions:

C_1: $\{\alpha_n\}, \{\lambda_n\}$ are in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \lambda_n = 0$ and
$$\sum_{n=1}^{\infty} \alpha_n = +\infty = \sum_{n=1}^{\infty} \lambda_n,$$

C_2: $\{\delta_n\}, \{b_{n,0}\}, \{b_{n,i}\} \subseteq (f_1, f_2) \subseteq (0,1)$ and $\delta_n + \beta_{n,0} + \sum_{i=1}^{N} \beta_{n,i} = 1,$

C_3: $\{a_n\}, \{b_{n,0}\}, \{b_{n,i}\} \subseteq (f_1, f_2) \subseteq (0,1)$ with $a_n + b_{n,0} + \sum_{i=1}^{m} b_{n,i} = 1,$

C_4: $\delta_n f(q) < \kappa$, for every $q \in Fix(G)$

then $\{z_n\}$ converges strongly to $z^* \in \Omega$ satisfying $z^* = P_\Omega \circ h(z^*)$.

Proof. The proof is divided into steps:

Step 1: We show that $\{z_n\}$ is bounded. Let $q \in \Omega$, then utilizing Lemma 2.1 (i), we have

$$\rho(q, v_n) \leq \alpha_n \rho(q, z_n) + (1-\alpha_n)\rho(q, z_{n+1}). \tag{4.20}$$

Using (2.13), we get

$$\rho(q, w_n) \leq \rho(q, v_n). \tag{4.21}$$

Using Lemma 4.13 and Remark 2.2, we get

$$\begin{aligned} \rho^2(q, x_n) &\leq \delta_n \rho^2(q, w_n) + \beta_{n,0}\rho^2(q, Gw_n) + \sum_{i=1}^{N} \beta_{n,i}\rho^2(q, P_K A_{\mu_i} w_n) \\ &\quad - \delta_n \beta_{n,0}\rho^2(w_n, Gw_n) \\ &\leq \rho^2(q, w_n) - \Big(\delta_n - \frac{\kappa}{f(q)}\Big)\beta_{n,0}\rho^2(w_n, Gw_n). \end{aligned} \tag{4.22}$$

By applying Condition C_4, we obtain

$$\rho(q, x_n) \leq \rho(q, w_n). \tag{4.23}$$

Using Lemma 2.9, (4.23) and Condition C_3, we get

$$\begin{aligned} \rho(q, y_n) &\leq a_n \rho(q, w_n) + b_{n,0}\rho(q, Sw_n) + \sum_{i=1}^{N} b_{n,i}\rho(q, T_i^{\alpha} x_n) \\ &\leq \rho(q, w_n). \end{aligned} \tag{4.24}$$

From Condition C_1, we get

$$\lambda_n + k_0 < 1,$$

therefore, we let

$$\nu_n = \frac{1-(\lambda_n + k_0)}{1-k_0}, \ \forall n \in \mathbb{N}, \text{ for some } k_0 \in (0,1),$$

by utilizing (4.20), (4.21) and (4.24), we get

$$\begin{aligned}\rho(q, z_{n+1}) &\leq \lambda_n \rho(q, h(v_n)) + (1-\lambda_n)\rho(q, y_n)\\ &\leq \lambda_n \rho(q, h(q)) + (1-\lambda_n(1-\gamma))\rho(q, v_n)\\ &\leq \lambda_n \rho(q, h(q)) + (1-\lambda_n(1-\gamma))\Big(\alpha_n \rho(q, z_n) + (1-\alpha_n)\rho(q, z_{n+1})\Big).\end{aligned}$$

Since $(1-\lambda_n(1-\gamma))(1-\alpha) < 1$, then there exists $k_0 \in (0,1)$ such that

$$(1-\lambda_n(1-\gamma))(1-\alpha_n) \leq k_0 < 1,$$

this implies

$$1-(1-\lambda_n(1-\gamma))(1-\alpha_n) \geq (1-k_0).$$

By using Condition C_1, we can easily get

$$1 \geq \alpha_n - \lambda_n(1-\alpha_n(1-\gamma)) + k_0,$$

therefore we have,

$$\begin{aligned}\rho(q, z_{n+1}) &\leq \frac{\lambda_n}{1-[1-\lambda_n(1-\gamma)](1-\alpha_n)}\rho(q, h(q))\\ &\quad + \frac{(1-\lambda_n(1-\gamma))\alpha_n}{1-[1-\lambda_n(1-\gamma)](1-\alpha_n)}\rho(q, z_n)\\ &\leq \frac{\lambda_n}{1-k_0}\rho(q, h(q)) + \frac{(1-\lambda_n(1-\gamma))\alpha_n}{1-k_0}\rho(q, z_n)\\ &\leq (1-\nu_n)\rho(q, h(q)) + \nu_n \rho(q, z_n)\\ &\leq \max\{\rho(q, h(q)), \rho(q, z_n)\},\end{aligned}$$

which we can show by induction that,

$$\rho(q, z_{n+1}) \leq \max\{\rho(q, h(q)), \rho(q, z_0)\}, \quad \forall n \in \mathbb{N}.$$

Hence, $\{\rho(q, z_n)\}$ is bounded, which implies $\{z_n\}$ is bounded. Utilizing (4.20), (4.21), (4.23), (4.24), Lemma 4.13, definition of S and Lemma 2.9, we get $\{v_n\}, \{w_n\}, \{x_n\}, \{y_n\}, \{P_K A_{\mu_i} w_n\}$, $\{Sw_n\}$ and $\{T_i^{\alpha} x_n\}$ are bounded respectively, for each $i \in \{1, 2, ..., N\}$. Also, we observe that

$$\begin{aligned}\rho^2(q, h(v_n)) &\leq \Big(\rho(q, h(q)) + \rho(h(q), h(v_n))\Big)^2\\ &\leq \Big(\rho(q, h(q)) + \gamma\rho(q, v_n)\Big)^2,\end{aligned}$$

which implies $\{\rho^2(q, h(v_n))\}$ is bounded as $\{\rho(q, v_n)\}$ is bounded.

Step 2: Next, we show that

$$\lim_{n\to\infty} \rho(z_{n+1}, v_n) = \lim_{n\to\infty} \rho(z_{n+1}, w_n) = \lim_{n\to\infty} \rho(z_{n+1}, x_n) = 0.$$

Indeed, utilizing Lemma 2.9 and (4.22) in algorithm (4.19), we get

$$\begin{aligned}\rho^2(q,y_n) &\leq a_n\rho^2(q,w_n)+b_{n,0}\rho^2(q,Sw_n)+\sum_{i=1}^{N}b_{n,i}\rho^2(q,T_i^{\alpha}x_n)\\ &\leq a_n\rho^2(q,w_n)+b_{n,0}\rho^2(q,Sw_n)\\ &+\sum_{i=1}^{N}b_{n,i}\Big[\rho^2(q,w_n)-\Big(\delta_n-\frac{\kappa}{f(q)}\Big)\beta_{n,0}\rho^2(w_n,Gw_n)\Big]\\ &\leq \rho^2(q,w_n)-b_{n,i}\beta_{n,0}\Big(\delta_n-\frac{\kappa}{f(q)}\Big)\rho^2(w_n,Gw_n).\end{aligned} \tag{4.25}$$

Substituing (4.21), (4.25) and (4.20) in algorithm (4.19), we get

$$\begin{aligned}\rho^2(q,z_{n+1}) \leq \lambda_n\rho^2(q,h(v_n))+(1-\lambda_n)\Big(\rho^2(q,v_n)\\ -b_{n,i}\beta_{n,0}\Big(\delta_n-\frac{\kappa}{f(q)}\Big)\rho^2(w_n,Gw_n)\Big),\end{aligned}$$

which implies for each $i\in\{1,2,...,N\}$,

$$\begin{aligned}b_{n,i}\beta_{n,0}\Big(\delta_n-\frac{\kappa}{f(q)}\Big)\rho^2(w_n,Gw_n) \leq \frac{\lambda_n}{1-\lambda_n}\rho^2\Big(q,h(v_n)-\rho^2(q,z_{n+1})\Big)\\ +\Big(\rho^2(q,v_n)-\rho^2(q,z_{n+1})\Big).\end{aligned} \tag{4.26}$$

Using (4.20) in (4.26), we get

$$\begin{aligned}b_{n,i}\beta_{n,0}\Big(\delta_n-\frac{\kappa}{f(q)}\Big)\rho^2(w_n,Gw_n) \leq \frac{\lambda_n}{1-\lambda_n}\rho^2\Big(q,h(v_n)-\rho^2(q,z_{n+1})\Big)\\ +\alpha_n\Big(\rho^2(q,z_n)-\rho^2(q,z_{n+1})\Big).\end{aligned}$$

Using Conditions C_1, C_2 and C_3, we get

$$\lim_{n\to\infty}\rho(w_n,Gw_n)=0. \tag{4.27}$$

By using Lemma 2.7, Lemma 4.13 and Condition C_3 in algorithm (4.19), we get for each $i\in\{1,2,...,N\}$,

$$\rho^2(q,x_n)\leq\rho^2(q,w_n)-\delta_n\beta_{n,i}\rho^2(w_n,P_KA_{\mu_i}w_n)+\frac{\kappa}{f(q)}\rho^2(w_n,Gw_n), \tag{4.28}$$

which implies, by using Lemma 2.9, (4.21), Conditions C_3, C_4 and (4.28), we have

$$\begin{aligned}\rho^2(q,y_n) &\le a_n\rho^2(q,w_n)+b_{n,0}\rho^2(q,Sw_n)+\sum_{i=1}^N b_{n,i}\rho^2(q,T_i^\alpha x_n)\\ &\le a_n\rho^2(q,w_n)+b_{n,0}\rho^2(q,Sw_n)+\sum_{i=1}^N b_{n,i}\rho^2(q,x_n)\\ &\le a_n\rho^2(q,w_n)+b_{n,0}\rho^2(q,Sw_n)\\ &+\sum_{i=1}^N b_{n,i}\Big(\rho^2(q,w_n)-\delta_n\beta_{n,j}\rho^2(w_n,P_KA_{\mu_i}w_n)+\frac{\kappa}{f(q)}\rho^2(w_n,Gw_n)\Big)\\ &\le \rho^2(q,v_n)-b_{n,i}\delta_n\beta_{n,j}\rho^2(w_n,P_KA_{\mu_i}w_n)+\sum_{i=1}^N b_{n,i}\frac{\kappa}{f(q)}\rho^2(w_n,Gw_n). \end{aligned}\tag{4.29}$$

By applying (4.20) and (4.29) in algorithm (4.19), we get for each $i,j\in\{1,2,...,N\}$

$$\begin{aligned}b_{n,i}\delta_n\beta_{n,j}\rho^2(w_n,P_KA_{\mu_i}w_n) &\le \frac{\lambda_n}{1-\lambda_n}\Big(\rho^2(q,h(v_n))-\rho^2(q,z_{n+1})\Big)\\ &+\rho^2(q,v_n)-\rho^2(q,z_{n+1})\\ &+\sum_{i=1}^N b_{n,i}\frac{\kappa}{f(q)(1-\lambda_n)}\rho^2(w_n,Gw_n)\\ &\le \frac{\lambda_n}{1-\lambda_n}\Big(\rho^2(q,h(v_n))-\rho^2(q,z_{n+1})\Big)\\ &+\alpha_n\Big(\rho^2(q,z_n)-\rho^2(q,z_{n+1})\Big)\\ &+\sum_{i=1}^N b_{n,i}\frac{\kappa}{f(q)(1-\lambda_n)}\rho^2(w_n,Gw_n). \end{aligned}\tag{4.30}$$

By applying Conditions C_1, C_2, C_3 and (4.27) in (4.30), we obtain

$$\lim_{n\to\infty}\rho(w_n,P_KA_{\mu_i}w_n)=0,\quad \forall i\in\{1,2,...,N\}.\tag{4.31}$$

Similarly, by applying Lemma 2.9, Condition C_3 and (4.21) in algorithm (4.19), we get

$$\begin{aligned}\rho^2(q,y_n) &\le a_n\rho^2(q,w_n)+b_{n,0}\rho^2(q,Sw_n)\\ &+\sum_{i=1}^N b_{n,i}\rho^2(q,T_i^\alpha x_n)-a_nb_{n,0}\rho^2(w_n,Sw_n)\\ &\le \rho^2(q,v_n)-a_nb_{n,0}\rho^2(w_n,Sw_n). \end{aligned}\tag{4.32}$$

By applying (4.32) and algorithm (4.20) in (4.19), we get

$$\begin{aligned}\rho^2(q,z_{n+1}) &\le \lambda_n\rho^2(q,h(v_n))+(1-\lambda_n)\Big(\rho^2(q,v_n)-a_nb_{n,0}\rho^2(w_n,Sw_n)\Big)\\ &\le \lambda_n\rho^2(q,h(v_n))+(1-\lambda_n)\Big(\Big[\alpha_n\rho^2(q,z_n)\\ &+(1-\alpha_n)\rho^2(q,z_{n+1})\Big]-a_nb_{n,0}\rho^2(w_n,Sw_n)\Big), \end{aligned}$$

which implies

$$a_n b_{n,0}\rho^2(w_n, Sw_n) \le \frac{\lambda_n}{1-\lambda_n}\Big(\rho^2(q,h(v_n)) - \rho^2(q,z_{n+1})\Big) + \alpha_n\Big(\rho^2(q,z_n) - \rho^2(q,z_{n+1})\Big). \tag{4.33}$$

Utilizing Conditions C_1, C_2 and C_4 in (4.33), we obtain

$$\lim_{n\to\infty} \rho(w_n, Sw_n) = 0. \tag{4.34}$$

Similarly, by applying Lemma 2.9, (4.20), (4.21), (4.23) and (4.24) in algorithm (4.19), we get

$$\begin{aligned} a_n b_{n,i}\rho^2(w_n, T_i^\alpha x_n) &\le \frac{\lambda_n}{1-\lambda_n}\Big(\rho^2(q,h(v_n)) - \rho^2(q,z_{n+1})\Big) \\ &\quad + \rho^2(q,v_n) - \rho^2(q,z_{n+1}) \\ &\le \frac{\lambda_n}{1-\lambda_n}\Big(\rho^2(q,h(v_n)) - \rho^2(q,z_{n+1})\Big) \\ &\quad + \alpha_n\Big(\rho^2(q,z_n) - \rho^2(q,z_{n+1})\Big). \end{aligned} \tag{4.35}$$

Using Conditions C_1 and C_3 in (4.35), we have

$$\lim_{n\to\infty} \rho(w_n, T_i^\alpha x_n) = 0,\ \forall i \in \{1,2,...,N\}. \tag{4.36}$$

Using Lemma 2.6, we have

$$\rho(w_n, x_n) \le \beta_{n,0}\rho(w_n, Gw_n) + \sum_{i=1}^{N} \beta_{n,i}\rho(w_n, P_K A_{\mu_i} w_n).$$

Using (4.27) and (4.31), we get

$$\lim_{n\to\infty} \rho(w_n, x_n) = 0. \tag{4.37}$$

Utilizing Lemma 2.1 (ii), (4.20) and (4.24), we obtain

$$\begin{aligned} \rho^2(v_n, w_n) &\le \frac{\lambda_n}{1-\lambda_n}\Big(\rho^2(q,h(v_n)) - \rho^2(q,z_{n+1})\Big) \\ &\quad + \rho^2(q,v_n) - \rho^2(q,z_{n+1}) \\ &\le \frac{\lambda_n}{1-\lambda_n}\Big(\rho^2(q,h(v_n)) - \rho^2(q,z_{n+1})\Big) \\ &\quad + \alpha_n\Big(\rho^2(q,z_n) - \rho^2(q,z_{n+1})\Big). \end{aligned} \tag{4.38}$$

Applying Condition C_1 in (4.38), we get

$$\lim_{n\to\infty} \rho(v_n, w_n) = 0. \tag{4.39}$$

Using Lemma 2.6 in algorithm (4.19),

$$\rho(w_n, y_n) \le b_{n,0}\rho(w_n, Sw_n) + \sum_{i=1}^{N} b_{n,i}\rho(w_n, T_i^\alpha x_n).$$

By applying (4.34), (4.36) and Conditions C_3 and C_4, we get

$$\lim_{n\to\infty} \rho(w_n, y_n) = 0. \tag{4.40}$$

Combining (4.40) and (4.37), we have

$$\lim_{n\to\infty} \rho(y_n, x_n) = 0. \tag{4.41}$$

Using Lemma 2.1 (i), we get

$$\begin{aligned}\rho(v_n, z_{n+1}) &\leq \lambda_n \rho(v_n, h(v_n)) + (1-\lambda_n)\rho(v_n, y_n)\\ &\leq \lambda_n \rho(v_n, h(v_n)) + (1-\lambda_n)\Big(\rho(v_n, w_n) + \rho(w_n, y_n)\Big).\end{aligned}$$

Utilizing (4.39) and (4.40), we get

$$\lim_{n\to\infty} \rho(v_n, z_{n+1}) = 0. \tag{4.42}$$

Using (4.39) and (4.42), we have

$$\lim_{n\to\infty} \rho(w_n, z_{n+1}) = 0. \tag{4.43}$$

Utilizing (4.37) and (4.43), we get

$$\lim_{n\to\infty} \rho(x_n, z_{n+1}) = 0. \tag{4.44}$$

Applying (4.40) and (4.43), we obtain

$$\lim_{n\to\infty} \rho(y_n, z_{n+1}) = 0. \tag{4.45}$$

By utilizing (4.36) and (4.37), we get

$$\lim_{n\to\infty} \rho(x_n, T_i^\alpha x_n) = 0. \tag{4.46}$$

Let $\{z_{n_j}\}$ be a subsequence of $\{z_n\}$ such that $\Delta - \lim_{j\to\infty} z_{n_j} = \bar{z}$, for some $\bar{z} \in H$ and

$$\limsup_{n\to\infty} \langle \overrightarrow{h(z^*)z^*}, \overrightarrow{z_{n+1}z^*} \rangle = \lim_{j\to\infty} \langle \overrightarrow{h(z^*)z^*}, \overrightarrow{z_{n_j+1}z^*} \rangle$$

where $z^* = P_\Omega h(z^*)$.

Step 3: Next, we show that $\bar{z} \in \Omega$. Since $\Delta - \lim_{j\to\infty} z_{n_j} = \bar{z}$, then using (4.43) and (4.44) respectively, we also have $\Delta - \lim_{j\to\infty} w_{n_j} = \bar{z}$ and $\Delta - \lim_{j\to\infty} x_{n_j} = \bar{z}$. Utilizing Lemma 2.11 and (4.27), we get

$$\bar{z} \in Fix(G).$$

Using Lemma 4.13, Lemma 2.8, (4.31) and Remark 2.1, we get

$$\bar{z} \in VI(K, A).$$

By applying (4.39) and Lemma 2.8, we have

$$\bar{z} \in M.E.P(F, \varphi).$$

Utilizing Δ-demiclosed assumption on S and (4.34), we get

$$\bar{z} \in Fix(S).$$

Using (4.46), Lemma 2.9 and Δ-demiclosed assumption on T_i^α, for each $i \in \{1, 2, ..., N\}$, we obtain

$$\bar{z} \in Fix(T).$$

Hence, $\bar{z} \in \Omega$.

Step 4: Next, we show that $\limsup_{n\to\infty}\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_nz^*}\rangle \leq 0$.

Using (4.45), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_nz^*}\rangle &= \limsup_{n\to\infty}\left(\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_nz_{n+1}}\rangle + \langle\overrightarrow{h(z^*)z^*},\overrightarrow{z_{n+1}z^*}\rangle\right)\\ &\leq \rho(z^*,h(z^*))\lim_{n\to\infty}\rho(y_n,z_{n+1}) + \limsup_{n\to\infty}\langle\overrightarrow{h(z^*)z^*},\overrightarrow{z_{n+1}z^*}\rangle\\ &= \langle\overrightarrow{h(z^*)z^*},\overrightarrow{\bar{z}z^*}\rangle \leq 0. \end{aligned} \tag{4.47}$$

Step 5: Finally, we show that the sequence $\{z_n\}$ converges strongly to z^*. By utilizing Lemma 2.1 (iii), (4.20), (4.21) and (4.24), we have

$$\begin{aligned}\rho^2(z^*,z_{n+1}) &\leq \lambda_n^2\rho^2(z^*,h(v_n)) + (1-\lambda_n)^2\rho^2(z^*,y_n)\\ &\quad + 2\lambda_n(1-\lambda_n)\langle\overrightarrow{h(v_n)z^*},\overrightarrow{y_nz^*}\rangle\\ &\leq \lambda_n^2M + (1-\lambda_n)^2\rho^2(z^*,v_n) + 2\lambda_n(1-\lambda_n)\left(\langle\overrightarrow{h(v_n)h(z^*)},\overrightarrow{y_nz^*}\rangle + \langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_nz^*}\rangle\right)\\ &\leq \lambda_n^2M + (1-\lambda_n)^2\rho^2(z^*,v_n) + 2\lambda_n(1-\lambda_n)\left(\gamma\rho(z^*,v_n)\rho(z^*,y_n) + \langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_nz^*}\rangle\right)\\ &\leq \lambda_n^2M + (1-\lambda_n)^2\rho^2(z^*,v_n) + 2\lambda_n(1-\lambda_n)\left(\gamma\rho^2(z^*,v_n) + \langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_nz^*}\rangle\right)\\ &= \lambda_n^2M + \left[1-\lambda_n\Big(2(1-\gamma)-\lambda_n(1-2\gamma)\Big)\right]\rho^2(z^*,v_n) + 2\lambda_n(1-\lambda_n)\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_nz^*}\rangle\\ &\leq \lambda_n^2M + \left[1-\lambda_n\Big(2(1-\gamma)-\lambda_n(1-2\gamma)\Big)\right]\Big(\alpha_n\rho^2(z^*,z_n) + (1-\alpha_n)\rho^2(z^*,z_{n+1})\Big)\\ &\quad + 2\lambda_n(1-\lambda_n)\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_nz^*}\rangle.\end{aligned}$$

By utilizing Condition C_1, there exists $M_0 < 1$ and $\bar{N}\in\mathbb{N}$ such that

$$\left[1-\lambda_n\Big(2(1-\gamma)-\lambda_n(1-2\gamma)\Big)\right] \leq M_0, \quad \forall n\geq\bar{N}.$$

Now,

$$\begin{aligned}\rho^2(z^*,z_{n+1}) &\leq \lambda_n^2M + M_0\Big(\alpha_n\rho^2(z^*,z_n) + (1-\alpha_n)\rho^2(z^*,z_{n+1})\Big)\\ &\quad + 2\lambda_n(1-\lambda_n)\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_nz^*}\rangle\\ &\leq \lambda_n^2M + M_0\Big(\alpha_n\rho^2(z^*,z_n) + \rho^2(z^*,z_{n+1})\Big)\\ &\quad + 2\lambda_n(1-\lambda_n)\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_nz^*}\rangle,\end{aligned}$$

which implies

$$(1-M_0)\rho^2(z^*,z_{n+1}) \leq \lambda_n^2M + M_0\alpha_n\rho^2(z^*,z_n) + 2\lambda_n(1-\lambda_n)\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_nz^*}\rangle,$$

which implies

$$\begin{aligned}\rho^2(z^*,z_{n+1}) &\leq \frac{\lambda_n^2M}{1-M_0} + \frac{M_0\alpha_n}{1-M_0}\rho^2(z^*,z_n) + \frac{2\lambda_n(1-\lambda_n)}{1-M_0}\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_nz^*}\rangle\\ &\leq \frac{\lambda_n^2M}{1-M_0} + (1-\lambda_n)\rho^2(z^*,z_n) + \frac{2\lambda_n(1-\lambda_n)}{1-M_0}\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_nz^*}\rangle\\ &= (1-\lambda_n)\rho^2(z^*,z_n) + \lambda_n\left[\frac{\lambda_nM}{1-M_0} + \frac{2(1-\lambda_n)}{1-M_0}\langle\overrightarrow{h(z^*)z^*},\overrightarrow{y_nz^*}\rangle\right].\end{aligned}$$

Using Condition C_1, Lemma 2.10 and (4.47), we obtain $\lim_{n\to\infty} z_n = z^*$. □

Corollary 4.1. *Let H be a Hadamard space and H^* be its dual space, let K be a nonempty closed and convex subset of H. Let $A_i : K \to H$ be family of α_i-inverse strongly monotone mapping, let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying condition A_1 to A_4 of Theorem 1.2, $\varphi : K \to \mathbb{R}$ be a convex and lower semi-continuous function, let $G : K \to K$ be (f, g)-generalized κ-strictly pseudononspreading mapping with $\kappa \in [0, 1)$, where $f, g : K \to [0, \tau]$, $\tau < 1$ such that f, g, κ, τ satisfies assumptions in Theorem 2.3 [34], Remark 2.2 [34] and Lemma 2.11 [34]. Let $S : K \to K$ be nonexpansive mapping, let $T_i : H \to H$ be family of θ_i-generalized demimetric mapping with $\theta_i \neq 0$. Assume that* $\Omega = Fix(G) \cap Fix(S) \cap M.E.P(F, \varphi, K) \bigcap_{i=1}^{N} \Big(VI(K, A_i) \cap Fix(T_i)\Big) \neq \emptyset$ *and* $\{z_n\}_{n=1}^{\infty}$ *is a sequence generated by*

$$\begin{cases} z_0, z_1 \in X, \\ v_n = \alpha_n z_n \oplus (1-\alpha_n) z_{n+1}, \\ w_n = T_{r_n}^{F,\varphi} v_n, \\ x_n = \delta_n w_n \oplus \beta_{n,0} G w_n \oplus \bigoplus_{i=1}^{N} \beta_{n,i} P_K A_{\mu_i} w_n, \\ y_n = a_n w_n \oplus b_{n,0} S w_n \oplus \bigoplus_{i=1}^{N} b_{n,i} T_i^{\alpha} x_n, \\ z_{n+1} = \lambda_n h(v_n) \oplus (1-\lambda_n) y_n, \quad n \in \mathbb{N}, \end{cases} \tag{4.48}$$

where $h : H \to H$ is γ-contraction, for some $\gamma \in [0, 1)$ and for each $i \in \{1, 2, ..., m\}$,

$$A_{\mu_i} x = (1-\mu_i) x \oplus \mu_i A_i x, \text{ with } \mu_i \in (0, 2\alpha_i - 1),$$
$$T_i^{\alpha} x = \alpha x \oplus (1-\alpha) T_i x$$

with assumption that T_i^{α} and S are Δ-demiclosed, with $\theta_i \le \frac{2}{1-\alpha}, \alpha \in (0, 1)$, $\{r_n\} \subseteq (0, 1)$ and $\{\alpha_n\}, \{\beta_{n,0}\}, \{\beta_{n,i}\}, \{\delta_n\}, \{b_{n,i}\}, \{b_{n,0}\}, \{a_n\}, \{\lambda_n\}$ are sequences in $\mathbb{R}$ satisfying the following conditions:

C_1: *$\{\alpha_n\}, \{\lambda_n\}$ are in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \lambda_n = 0$ and* $\sum_{n=1}^{\infty} \alpha_n = +\infty = \sum_{n=1}^{\infty} \lambda_n$,

C_2: $\{a_{n,0}\}, \{\beta_{n,0}\}, \{\beta_{n,i}\}, \{a_{n,i}\} \subseteq (f_1, f_2) \subseteq (0, 1)$ *and* $\beta_{n,0} + \sum_{i=1}^{N} \beta_{n,i} = a_{n,0} + \sum_{i=1}^{N} a_{n,i} = 1$,

C_3: $\{\delta_n\}, \{b_{n,0}\}, \{b_{n,i}\} \subseteq (f_1, f_2) \subseteq (0, 1)$ *and* $\delta_n + \beta_{n,0} + \sum_{i=1}^{N} \beta_{n,i} = 1$,

C_4: $\{a_n\}, \{b_{n,0}\}, \{b_{n,i}\} \subseteq (f_1, f_2) \subseteq (0, 1)$ *with* $a_n + b_{n,0} + \sum_{i=1}^{m} b_{n,i} = 1$,

then $\{z_n\}$ converges strongly to $z^* \in \Omega$ satisfying $z^* = P_{\Omega} \circ h(z^*)$.

Corollary 4.2. *Let H be a real Hilbert space, let K be a nonempty closed and convex subset of H. Let $A_i : K \to H$ be family of generalized α_i-cocoercive mapping, where $\alpha_i > \frac{1}{2}$, let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying condition A_1 to A_4 of Theorem 1.2, $\varphi : K \to \mathbb{R}$ be a convex and lower semi-continuous function, let $G : K \to K$ be (f, g)-generalized κ-strictly pseudononspreading mapping with $\kappa \in [0, 1)$, where $f, g : K \to [0, \tau]$, $\tau < 1$ such that f, g, κ, τ satisfies assumptions in Theorem*

2.3 [34], *Remark* 2.2 [34] *and Lemma* 2.11 [34]. *Let* $S : K \to K$ *be quasi-nonexpansive mapping, let* $T_i : K \to K$ *be family of* θ_i*-generalized demimetric mapping with* $\theta_i \neq 0$. *Assume that*

$\Omega = Fix(G) \cap Fix(S) \cap M.E.P(F, \varphi, K) \bigcap_{i=1}^{N} \Big(VI(K, A_i) \cap Fix(T_i) \Big) \neq \emptyset$ *and* $\{z_n\}_{n=1}^{\infty}$ *is a sequence generated by*

$$\begin{cases} z_0, z_1 \in X, \\ v_n = \alpha_n z_n + (1 - \alpha_n) z_{n+1}, \\ w_n = T_{r_n}^{F,\varphi} v_n, \\ x_n = \delta_n w_n + \beta_{n,0} G w_n + \sum_{i=1}^{N} \beta_{n,i} P_K A_{\mu_i} w_n, \\ y_n = a_n w_n + b_{n,0} S w_n + \sum_{i=1}^{N} b_{n,i} T_i^{\alpha} x_n, \\ z_{n+1} = \lambda_n h(v_n) + (1 - \lambda_n) y_n, \ \ n \in \mathbb{N}, \end{cases} \tag{4.49}$$

where $h : H \to H$ *is* γ*-contraction for some* $\gamma \in [0, 1)$ *and for each* $i \in \{1, 2, ..., m\}$,

$$A_{\mu_i} x = (1 - \mu_i) x + \mu_i A_i x, \ \text{with} \ \mu_i \in (0, 2\alpha_i - 1),$$
$$T_i^{\alpha} x = \alpha x + (1 - \alpha) T_i x$$

with assumption that T_i^{α} *and* S *are* Δ*-demiclosed, with* $\theta_i \leq \frac{2}{1-\alpha}, \alpha \in (0, 1)$, $\{r_n\} \subseteq (0, 1)$ *and* $\{\alpha_n\}, \{\beta_{n,0}\}, \{\beta_{n,i}\}, \{\delta_n\}, \{b_{n,i}\}, \{b_{n,0}\}, \{a_n\}, \{\lambda_n\}$ *are sequences in* $\mathbb{R}$ *satisfying the following conditions:*

C_1: $\{\alpha_n\}, \{\lambda_n\}$ *are in* $(0, 1)$ *such that* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \lambda_n = 0$ *and*
$$\sum_{n=1}^{\infty} \alpha_n = +\infty = \sum_{n=1}^{\infty} \lambda_n,$$
C_2: $\{a_{n,0}\}, \{\beta_{n,0}\}, \{\beta_{n,i}\}, \{a_{n,i}\} \subseteq (f_1, f_2) \subseteq (0, 1)$ *and*
$$\beta_{n,0} + \sum_{i=1}^{N} \beta_{n,i} = a_{n,0} + \sum_{i=1}^{N} a_{n,i} = 1,$$
C_3: $\{\delta_n\}, \{b_{n,0}\}, \{b_{n,i}\} \subseteq (f_1, f_2) \subseteq (0, 1)$ *and* $\delta_n + \beta_{n,0} + \sum_{i=1}^{N} \beta_{n,i} = 1$,

C_4: $\{a_n\}, \{b_{n,0}\}, \{b_{n,i}\} \subseteq (f_1, f_2) \subseteq (0, 1)$ *with* $a_n + b_{n,0} + \sum_{i=1}^{m} b_{n,i} = 1$,

then $\{z_n\}$ *converges strongly to* $z^* \in \Omega$ *satisfying* $z^* = P_\Omega \circ h(z^*)$.

5. Conclusion and Future Work

In this paper, we introduce and study some properties of a new mapping called generalized cocoercive in Hadamard space. We propose implicit viscosity-type algorithm for obtaining an element in the set of solutions of some mixed equilibrium problem, set of fixed point of (f, g)-generalized κ-strictly pseudononspreading mapping, set of fixed point of quasi-nonexpansive mapping, set of common solutions of family of variational inequality problems involving generalized cocoercive mappings and the set of common fixed point of family generalized demimetric mappings. The method of proof adapted in this paper is new, of independent interest and simpler than the standard technique used in many recent results using cases. For possible future work, one may be interested

- to get the result in more general metrically convex spaces, e.g hyperbolic spaces.
- to study an explicit version of the scheme studied in this paper.

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