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Research Article

New fixed point theorems for (ϕ, F)-Gregus contraction in b-rectangular metric spaces

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ABSTRACT. In this paper, we introduce the notion of (ϕ, F)-Gregus contraction and (ϕ, F)-Gregus type quadratic contraction in b-rectangular metric spaces. Further, we study the existence and uniqueness of fixed point for these mappings in this spaces. Our results are legitimately validated by illustrative examples.

Keywords: Fixed point, b-metric space, b-rectangular metric spaces, (ϕ, F)-Gregus contraction, (ϕ, F)-Gregus type quadratic contraction.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. INTRODUCTION

Banach contraction principle is one of the earlier and main results in fixed point theory. Banach contraction principle [1] was proved in complete metric spaces. Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, b-metric spaces were introduced by Bakhtin [2] and Czerwik [4], in such a way that triangle inequality is replaced by the b-triangle inequality. Various mathematician considered a lot of interesting extensions and generalizations [3, 6, 14]. Piri and Kumam [12] introduced new type of contractions called F-contraction and F-weak contraction and proved new fixed point theorems concerning F-contractions. Very recently, Kari et al. [7] introduced the notion of $(\theta - \phi)$-contraction in these metric spaces and proved a fixed point theorem.

Definition 1.1 ([5]). *Let X be a nonempty set $s \geq 1$ be a given real number and let $d : X \times X \to [0, +\infty[$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$ each distinct from x and y:*

(i) $d(x, y) = 0$ *if only if* $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ *(b-rectangular inequality).*

Then, (X, d) is called a b-rectangular metric space.

In 1971, S. Reich [14] presented the following lemma to establish some remarks concerning contraction mappings

Lemma 1.1 ([14]). *Let (X, d) be a b-rectangular metric space.*

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(i) Suppose that sequences $\{x_n\}$ and $\{y_n\} \in X$ are such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, with $x \neq y$, $x_n \neq x$ and $y_n \neq y$ for all $n \in \mathbb{N}$. Then, we have

$$\frac{1}{s}d(x,y) \leq \lim_{n\to\infty} \inf \; d(x_n, y_n) \leq \lim_{n\to\infty} \sup d(x_n, y_n) \leq sd(x,y).$$

(ii) If $y \in X$ and $\{x_n\}$ is a Cauchy sequence in X with $x_n \neq x_m$ for any $m, n \in \mathbb{N}, m \neq n$, converging to $x \neq y$, then

$$\frac{1}{s}d(x,y) \leq \lim_{n\to\infty} \inf d(x_n, y) \leq \lim_{n\to\infty} \sup d(x_n, y) \leq sd(x,y) \;\; \forall x \in X.$$

Lemma 1.2 ([9]). *Let (X,d) be a b-rectangular metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim d(x_n, x_{n+2}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that

(i) $\epsilon \leq \lim_{k\to\infty} \inf d(x_{m_{(k)}}, x_{n_{(k)}}) \leq \lim_{k\to\infty} \sup d(x_{m_{(k)}}, x_{n_{(k)}}) \leq s\epsilon$,
(ii) $\epsilon \leq \lim_{k\to\infty} \inf d(x_{n_{(k)}}, x_{m_{(k)+1}}) \leq \lim_{k\to\infty} \sup d(x_{n_{(k)}}, x_{m_{(k)+1}}) \leq s\epsilon$,
(iii) $\epsilon \leq \lim_{k\to\infty} \inf d(x_{m_{(k)}}, x_{n_{(k)+1}}) \leq \lim_{k\to\infty} \sup d(x_{m_{(k)}}, x_{n_{(k)+1}}) \leq s\epsilon$,
(iv) $\frac{\epsilon}{s} \leq \lim_{k\to\infty} \inf d(x_{m_{(k)+1}}, x_{n_{(k)+1}}) \leq \lim_{k\to\infty} \sup d(x_{m_{(k)+1}}, x_{n_{(k)+1}}) \leq s^2\epsilon$.

Definition 1.2 ([16]). *Let F be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that*

(i) F is strictly increasing,
(ii) For each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers

$$\lim_{n\to\infty} x_n = 0 \text{ if and only if } \lim_{n\to\infty} F(x_n) = -\infty,$$

(iii) There exists $k \in]0,1[$ such that $\lim_{x\to 0} x^k F(x) = 0$.

In 2018, the following result was appeared.

Theorem 1.1 ([15]). *Let (X,d,s) be a complete b-metric space and T be a self-map on X. Assume that there exist $\tau > 0$ and a function $F :]0,+\infty[\to \mathbb{R}$ satisfying a sequence $t_n \in]0,+\infty[$ such that $\tau + F(d(Tx,Ty)) \leq F(d(x,y))$ holds for all $x, y \in X$ with $Tx \neq Ty$. Then, T has a unique fixed point.*

Recently, Piri and Kuman [12] extended the result of Wardowski [17, Definition 1.6] as follow:

Definition 1.3 ([12]). *Let F be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that*

(i) F is strictly increasing,
(ii) For each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers

$$\lim_{n\to\infty} x_n = 0 \text{ if and only if } \lim_{n\to\infty} F(x_n) = -\infty,$$

(iii) F is continuous.

The following definition introduced by Wardowski [17] will be used to prove our result.

Definition 1.4 ([17]). *Let F be the family of functions $F : \mathbb{R}^+ \to \mathbb{R}$ and $\phi :]0,+\infty[\to]0,+\infty[$ satisfy the following:*

(i) F is strictly increasing,
(ii) For each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers

$$\lim_{n\to\infty} x_n = 0 \text{ if and only if } \lim_{n\to\infty} F(x_n) = -\infty,$$

(iii) $\liminf_{s\to\alpha^+} \phi(s) > 0, \ \forall s > 0$,
(iv) *There exists* $k \in]0,1[$ *such that*

$$\lim_{x\to 0^+} x^k F(x) = 0.$$

Theorem 1.2 ([17]). *Each F-contraction T on a complete metric space (X,d) has a unique fixed point. Moreover, for each $x_0 \in X$, the corresponding Picard sequence $\{T^n x_0\}$ converges to that fixed point.*

Recently, Kari and Rossafi [10] gave the following definition.

Definition 1.5 ([10]). *Let F be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ and $\phi :]0,+\infty[\to]0,+\infty[$ satisfy the following:*

(i) *F is strictly increasing,*
(ii) *For each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers*

$$\lim_{n\to\infty} x_n = 0 \text{ if and only if } \lim_{n\to\infty} F(x_n) = -\infty,$$

(iii) $\liminf_{s\to\alpha^+} \phi(s) > 0, \ \forall s > 0$,
(iv) *There exists* $k \in]0,1[$ *such that*

$$\lim_{x\to 0^+} x^k F(x) = 0,$$

(v) *For each sequence $\alpha_n \in \mathbb{R}^+$ of positive numbers such that $\phi(\alpha_n) + F(s\,\alpha_{n+1}) \le F(\alpha_n)$ for all $n \in \mathbb{N}$, then $\phi(\alpha_n) + F(s^n \alpha_{n+1}) \le F(s^{n-1}\alpha_n)$ for all $n \in \mathbb{N}$.*

Definition 1.6 ([9]). *Let $\mathfrak{F}$ be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ and $\phi :]0,+\infty[\ \to \ ]0,+\infty[$ satisfy the following:*

(i) *F is strictly increasing,*
(ii) *For each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers*

$$\lim_{n\to\infty} x_n = 0 \text{ if and only if } \lim_{n\to\infty} F(x_n) = -\infty,$$

(iii) $\liminf_{s\to\alpha^+} \phi(s) > 0, \ \forall s > 0$,
(iv) *F is continuous.*

Definition 1.7 ([17]). *Let (X,d) be a metric space. A mapping $T : X \to X$ is called an (ϕ,F)-contraction on (X,d), if there exists $F \in \mathbb{F}$ and ϕ such that*

$$F(d(Tx,Ty) + \phi(d(x,y)) \le F(d(x,y))$$

for all $x,y \in X$ for which $Tx \ne Ty$.

In this paper, using the idea introduced by Wardowski [17], we introduce the concept of (ϕ,F)-Gregus contraction and Gregus type quadratic contraction in b-rectangular metric spaces and prove some fixed point results for such spaces. Our results are validated by suitable examples.

2. Main Result

Now, we introduce the following:

Definition 2.8. *Let (X,d) be a b-rectangular metric space with parameter $s > 1$ space and $T : X \to X$ be a mapping. T is said to be a (ϕ,F)-Gregus contraction if there exist $F \in \mathfrak{F}$ and $\phi \in \Phi$ such that*

$$d(Tx,Ty) > 0 \implies F[s^2 d(Tx,Ty)] + \phi(d(x,y)) \le F[M(x,y)] \tag{2.1}$$

where

$$M(x,y) = ad(x,y) + (1-a)\max\{d(x,Tx), d(y,Ty), d(y,Tx)\}, 0 \le a \le 1.$$

Example 2.1. *Let $F(x) = 1 - \frac{1}{x}$. Then, it is easy to prove that $F(x)$ is strictly increasing for $x > 0$ as $x_n \to 0^+, F(x_n) = 1 - \frac{1}{x} \to -\infty$. Also, the function $F(x) = 1 - \frac{1}{x}$ is continuous for $x > 0$. Again, if we choose $\phi(s) = s$ which satisfies $\liminf_{s\to\alpha+} \phi(s) > 0$ for all $s > 0$. Therefore, $F(x) = 1 - \frac{1}{x}$ belongs to $\mathfrak{F}$. Let $T(x) = \frac{x}{4}$. We now compute*

$$M(x, y) = ad(x, y) + (1 - a) \max \{d(x, Tx), d(y, Ty), d(y, Tx)\}, 0 \le a \le 1.$$

Using the metric $d(x, y) = |x - y|$, we have $d(x, Tx) = \left|x - \frac{x}{4}\right| = \left|\frac{3x}{4}\right|$, $d(y, Ty) = \left|y - \frac{y}{4}\right| = \left|\frac{3y}{4}\right|$, $d(y, Tx) = \left|y - \frac{x}{4}\right| = \left|\frac{4y-x}{4}\right|$. Thus, we can express $M(x, y)$ as:

$$M(x, y) = a|x - y| + (1 - a) \max \left\{ \frac{3x}{4}, \frac{3y}{4}, \frac{|4y - x|}{4} \right\}.$$

Now, we have

$$\begin{aligned} F[s^2 d(Tx, Ty)] + \phi(d(x, y)) &= F[\frac{s^2}{16}|x - y|] + \phi(|x - y|) \\ &\le F[s^2 d(Tx, Ty)] + F[a|x - y| + (1 - a) \max \left\{ \frac{3x}{4}, \frac{3y}{4}, \frac{|4y - x|}{4} \right\} \\ &\le F[M(x, y)]. \end{aligned}$$

Thus, all the conditions of Definition 2.8 are satisfied.

Remark 2.1. *The above example does not satisfy corresponding Definition presented in* [10] *and* [17].

Now, we present our main result.

Theorem 2.3. *Suppose (X, d) be a complete b-rectangular metric space and $T : X \to X$ be an $(\phi - F)$-Gregus contraction $(\mathfrak{F})$ i.e, there exist $F \in \mathfrak{F}$ and ϕ such that for any $x, y \in X$, satisfying* (2.1) *then, T has a unique fixed point.*

Proof. Suppose $x_0 \in X$ be an arbitrary point in X and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then proof is finished. We can suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Substituting $x = x_{n-1}$ and $y = x_n$, from (2.1), for all $n \in \mathbb{N}$, we have

$$F[d(x_n, x_{n+1})] \le F[s^2 d(x_n, x_{n+1})] + \phi(d(x_{n-1}, x_n)) \le F(M(x_{n-1}, x_n)), \forall n \in \mathbb{N}, \tag{2.2}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= ad(x_{n-1}, x_n) \\ &\quad + (1 - a) \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+1})\} \\ &= ad(x_{n-1}, x_n) + (1 - a) \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= d(x_n, x_{n+1}). \end{aligned}$$

If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$, by (2.2), we have
$F[d(x_n, x_{n+1})] \le F[d(x_n, x_{n+1})] - \phi(d(x_{n-1}, x_n)) < F(d(x_n, x_{n+1}))$. Since F is increasing, we have

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \tag{2.3}$$

which is a contradiction. Hence, $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$. Thus,

$$F[d(x_n, x_{n+1})] \le F[d(x_{n-1}, x_n)] - \phi(d(x_{n-1}, x_n)). \tag{2.4}$$

Repeating this step, we conclude that

$$\begin{aligned}F(d(x_n,x_{n+1})) &\le F(d(x_{n-1},x_n)) - \phi(d(x_{n-1},x_n))\\ &\le F(d(x_{n-2},x_{n-1})) - \phi(d(x_{n-1},x_n)) - \phi(d(x_{n-2},x_{n-1}))\\ &\le \cdots \le F(d(x_0,x_1)) - \sum_{i=0}^{n}\phi(d(x_i,x_{i+1})).\end{aligned}$$

Since $\liminf_{\alpha\to s^+}\phi(\alpha) > 0$, we have $\liminf_{n\to\infty}\phi(d(x_{n-1},x_n)) > 0$, then from the definition of the limit, there exists $n_0\in\mathbb{N}$ and $A>0$ such that for all $n\ge n_0$, $\phi(q(x_{n-1},x_n)) > A$, hence

$$\begin{aligned}F(d(x_{n-1},x_{n+1})) &\le F(d(x_0,x_1)) - \sum_{i=0}^{n_0-1}\phi(d(x_i,x_{i+1})) - \sum_{i=n_0-1}^{n}\phi(d(x_i,x_{i+1}))\\ &\le F(d(x_0,x_1)) - \sum_{i=n_0-1}^{n} A\\ &= F(d(x_0,x_1)) - (n-n_0)A \qquad (2.5)\end{aligned}$$

for all $n\ge n_0$. Taking the limit as $n\to\infty$ in the above inequality, we get

$$\lim_{n\to\infty}F(d(x_n,x_{n+1})) \le \lim_{n\to\infty}[F(d(x_0,x_1)) - (n-n_0)A], \qquad (2.6)$$

that is, $\lim_{n\to\infty}F(d(x_n,x_{n+1})) = -\infty$. Then, from the condition (ii) of Definition 1.3, we conclude that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \qquad (2.7)$$

Next, we shall prove that

$$\lim_{n\to\infty} d(x_n,x_{n+2}) = 0.$$

We assume that $x_n\ne x_m$ for every $n,m\in\mathbb{N}, n\ne m$. Suppose that $x_n = x_m$ for some $n = m+k$ with $k>0$ and using (2.2)

$$d(x_m,x_{m+1}) = d(x_n,x_{n+1}) < d(x_{n-1},x_n). \qquad (2.8)$$

Continuing this process, so that $d(x_m,x_{n+1}) = d(x_n,x_{n+1}) < d(x_m,x_{m+1})$. It is a contradiction. Therefore, $d(x_n,x_m) > 0$ for every $n,m\in\mathbb{N}, n\ne m$. Now, applying (2.1) with $x = x_{n-1}$ and $y = x_{n+1}$, we have

$$\begin{aligned}F(d(x_n,x_{n+2})) &= F[d(Tx_{n-1},Tx_{n+1})]\\ &\le F[s^2 d(Tx_{n-1},Tx_{n+1})]\\ &\le F(M(x_{n-1},x_{n+1})) - \phi(d(x_{n-1},x_n)), \qquad (2.9)\end{aligned}$$

where

$$\begin{aligned}M(x_{n-1},x_{n+1}) &= a d(x_{n-1},x_{n+1})\\ &\quad + (1-a)\max\{d(x_{n-1},x_{n+1}), d(x_{n-1},x_n), d(x_{n+1},x_{n+2}), d(x_{n+1},x_n)\}\\ &= a d(x_{n-1},x_{n+1}) + (1-a)\max\{d(x_{n-1},x_{n+1}), d(x_{n-1},x_n)\}\\ &= d(x_{n-1},x_{n+1}).\end{aligned}$$

So, we get

$$F(d(x_n,x_{n+2}) \le F(\max\{d(x_{n-1},x_n), d(x_{n-1},x_{n+1})\}) - \phi(d(x_{n-1},x_{n+1})). \qquad (2.10)$$

Take $a_n = d(x_n,x_{n+2})$ and $b_n = d(x_n,x_{n+1})$. Thus by (2.10), one can write

$$F(a_n) \le F(\max(a_{n-1},b_{n-1})) - \phi(d(a_{n-1})). \qquad (2.11)$$

Since F is increasing, we get

$$a_n < \max\{a_{n-1}, b_{n-1}\}.$$

By (2.2), we have

$$b_n \leq b_{n-1} \leq \max(a_{n-1}, b_{n-1})$$

which implies that

$$\max\{a_n, b_n\} \leq \max\{a_{n-1}, b_{n-1}\}, \ \ \forall\, n \in \mathbb{N}.$$

Therefore, the sequence $\max\{a_{n-1}, b_{n-1}\}_{n\in\mathbb{N}}$ is non-negative decreasing sequence of real numbers. Thus, there exists $\lambda \geq 0,$ such that

$$\lim_{n\to\infty} \max\{a_n, b_n\} = \lambda.$$

By (2.6) assume that $\lambda > 0$, we have

$$\lambda = \lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \sup \max\{a_n, b_n\} = \lim_{n\to\infty} \max\{a_n, b_n\}.$$

Taking the $\lim\sup_{n\to\infty}$ in (2.10) and applying the continuity of F and the property of ϕ, we get

$$\begin{aligned} F(\lim_{n\to\infty} \sup a_n) &\leq F(\lim_{n\to\infty} \sup \max\{a_{n-1}, b_{n-1}\}) - \lim_{n\to\infty} \sup \phi(a_{n-1}) \\ &\leq F(\lim_{n\to\infty} \sup \max\{a_{n-1}, b_{n-1}\}) - \lim_{n\to\infty} \inf \phi(a_{n-1}) \\ &< F(\lim_{n\to\infty} \max\{a_{n-1}, b_{n-1}\}). \end{aligned}$$

Therefore,

$$F(\lambda) < F(\lambda)$$

which is a contradiction. Hence,

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.12}$$

Next, we shall prove that $\{x_n\}_n \in \mathbb{N}$ is a Cauchy sequence, i.e, $\lim_{n\to\infty} d(x_n, x_m) = 0$, for all $n, m \in \mathbb{N}$. Suppose to the contrary. By Lemma 1.2, then there is $\epsilon > 0$ such that for an integer k there exists two sequences $\{m_k\}$ and $\{n_k\}$ such that

(i) $\epsilon \leq \lim_{k\to\infty} \inf d(x_{m_{(k)}}, x_{n_{(k)}}) \leq \lim_{k\to\infty} \sup d(x_{m_{(k)}}, x_{n_{(k)}}) \leq s\epsilon$,
(ii) $\epsilon \leq \lim_{k\to\infty} \inf d(x_{n_{(k)}}, x_{m_{(k)+1}}) \leq \lim_{k\to\infty} \sup d(x_{n_{(k)}}, x_{m_{(k)+1}}) \leq s\epsilon$,
(iii) $\epsilon \leq \lim_{k\to\infty} \inf d(x_{m_{(k)}}, x_{n_{(k)+1}}) \leq \lim_{k\to\infty} \sup d(x_{m_{(k)}}, x_{n_{(k)+1}}) \leq s\epsilon$,
(iv) $\frac{\epsilon}{s} \leq \lim_{k\to\infty} \inf d(x_{m_{(k)+1}}, x_{n_{(k)+1}}) \leq \lim_{k\to\infty} \sup d(x_{m_{(k)+1}}, x_{n_{(k)+1}}) \leq s^2\epsilon$.

From (2.2) and by setting $x = x_{m_k}$ and $y = x_{n_k}$, we have

$$\begin{aligned} \lim_{k\to\infty} M(x_{m_k}, x_{n_k}) =& a d(x_{m_k}, x_{n_k}) \\ &+ (1-a)\max\left\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1}), d(x_{n_k}, x_{m_k+1})\right\} \\ \leq& s\epsilon. \end{aligned} \tag{2.13}$$

Now, using (2.1) with $x = x_{m_k}$ and $y = x_{n_k}$, we get

$$F[s^2 d(x_{m_k+1}, x_{n_k+1}] \leq F(M(x_{m_k}, x_{n_k})) - \phi(d(x_{m_k}, x_{n_k})). \tag{2.14}$$

Letting $k \to \infty$ the above inequality and applying (2.13) and (iv), we get

$$\begin{aligned} F(\frac{\epsilon}{s}s^2) &= F(\epsilon s) \\ &\leq F(s^2 \lim_{k\to\infty} \sup d(x_{m_k+1}, x_{n_k+1})) \\ &= \lim_{k\to\infty} \sup F(s^2 d(x_{m_k+1}, x_{n_k+1})) \\ &\leq \lim_{k\to\infty} \sup F(M(x_{m_k}, x_{n_k}) - \lim_{k\to\infty} \sup \phi(d(x_{m_k}, x_{n_k})) \\ &= F(M(x_{m_k}, x_{n_k})) - \lim_{k\to\infty} \sup \phi(d(x_{m_k}, x_{n_k})) \\ &\leq F(M(x_{m_k}, x_{n_k})) - \lim_{k\to\infty} \inf \phi(d(x_{m_k}, x_{n_k})) \\ &< F(\lim_{k\to\infty} \sup M(x_{m_k}, x_{n_k})) \\ &\leq F(s\epsilon). \end{aligned}$$

Therefore,

$$F(s\epsilon) < F(s\epsilon).$$

Since F is increasing, we get

$$s\epsilon < s\epsilon$$

which is a contradiction. So $\lim_{n,m\to\infty} d(x_m, x_n) = 0$. Hence, $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d), there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0.$$

Now, we show that $d(Tz, z) = 0$ arguing by contradiction, we assume that

$$d(Tz, z) > 0.$$

Since $x_n \to z$ as $n \to \infty$ for all $n \in \mathbb{N}$, then from Lemma 1.1 so that

$$\frac{1}{s} d(z, Tz) \leq \lim_{n\to\infty} \sup d(Tx_n, Tz) \leq s d(z, Tz). \tag{2.15}$$

Now, we are using (2.1) with $x = x_n$ and $y = z$, we have

$$F(s^2 d(Tx_n, Tz)) \leq F(M(x_n, z)) - \phi(d(x_n, z)), \forall\, n \in \mathbb{N},$$

where

$$M(x_n, z) = a\, d(x_n, z) + (1-a) \max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(z, Tx_n)\}$$

and

$$\lim_{n\to\infty} \sup \max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(z, Tx_n)\} = d(z, Tz). \tag{2.16}$$

Therefore,

$$F(s^2 d(Tx_n, Tz)) \leq F(\max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(z, Tx_n)\}) - \phi(d(x_n, z)). \tag{2.17}$$

By letting $n \to \infty$ in inequality (2.17), using (2.15), (2.16) and continuity of F, we obtain

$$\begin{aligned} F(s^2 \frac{1}{s} d(z, Tz)) &= F(sd(z, Tz)) \\ &\leq F(s^2 \lim_{n\to\infty} \sup d(Tx_n, Tz)) \\ &= \lim_{n\to\infty} \sup F(s^2 d(Tx_n, Tz)) \\ &\leq \lim_{n\to\infty} \sup F(M(x_n, z)) - \lim_{n\to\infty} \phi(d(x_n, z)) \\ &= F(d(Tz, z)) - \lim_{n\to\infty} \phi(d(x_n, z)) \\ &< F(d(z, Tz)). \end{aligned}$$

Since F is increasing, we get

$$sd(z, Tz) < d(z, Tz),$$

which implies that $d(z, Tz)(s-1) < 0$ implies $s < 1$, which is contradiction. Hence, $Tz = z$. Therefore, we have

$$d(z, u) = d(Tz, Tu) > 0.$$

Applying (2.1) with $x = z$ and $y = u$, we have

$$F(d(z, u)) = F(d(Tz, Tu)) \leq F(s^2 d(Tz, Tu)) \leq F(M(z, u)) - \phi(d(z, u)),$$

where

$$\begin{aligned} M(z, u) &= a\, d(z, u) + (1-a) \max\{d(z, u), d(z, Tz), d(u, Tu), d(u, Tz)\} \\ &= d(z, u). \end{aligned}$$

Therefore, we have

$$\begin{aligned} F(d(z, u)) &\leq F(d(z, u)) - \phi(d(z, u)) \\ &< F(d(z, u)) \end{aligned}$$

which implies that $d(z, u) < d(z, u)$, which is a contradiction. Therefore, $u = z$. □

Now, we introduce a (ϕ, F)-Gregus type quadratic contraction as follows:

Definition 2.9. *Suppose (X, d) be a complete b-rectangular metric space and $T : X \to X$ be an $(\phi - F)$-Gregus type quadratic type contraction $(\mathfrak{F})$, i.e, there exist $F \in \mathfrak{F}$ and ϕ such that for any $x, y \in X$, we have*

(2.18) $$d(Tx, Ty) > 0 \implies F[s^2 d^2(Tx, Ty) + \phi(d^2(x, y))] \leq F[M(x, y)],$$

where

$$M(x, y) = ad^2(x, y) + (1-a) \max\left\{d^2(x, Tx), d^2(y, Ty), d^2(y, Tx)\right\}.$$

Example 2.2. *Let $X = \mathbb{R}^+$ be a usual metric and $T : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by $T(x) = \frac{x}{2}$. Let $F(x) = \log x$ and $\phi(x) = \frac{1}{x}$. We need to verify the condition in (2.18). We have $d^2(Tx, Ty) = \frac{1}{4} d^2(x, y)$. Again, we have*

$$\begin{aligned} M(x, y) &= ad^2(x, y) + (1-a) \max\left\{d^2(x, y), d^2(x, Tx), d^2(y, Ty), d^2(Tx, y)\right\} \\ &= a|x-y|^2 + (1-a) \max\left\{\frac{1}{4}x^2, \frac{1}{4}y^2, \left|y - \frac{x}{2}\right|^2\right\}. \end{aligned}$$

On the other hand,

$$\begin{aligned}F[M(x,y)] &= \log\left(ad^2(x,y) + (1-a)\max\left\{\frac{1}{4}x^2, \frac{1}{4}y^2, \left|y - \frac{x}{2}\right|^2\right\}\right)\\ &\le F[M(x,y)].\end{aligned}$$

Thus, all the conditions of Definition 2.9 are satisfied. Now, we present our next result.

Theorem 2.4. *Suppose (X,d) be a complete b-rectangular metric space and $T: X \to X$ be a $(\phi - F)$-Gregus type quadratic $(\mathfrak{F})$ contraction. Then, T has a unique fixed point.*

Proof. Suppose $x_0 \in X$ be an arbitrary point in X and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then proof is finished. We can suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Substituting $x = x_{n-1}$ and $y = x_n$, from (2.1), for all $n \in \mathbb{N}$, we have

$$F[d^2(x_n,x_{n+1})] \le F[s^2d^2(x_n,x_{n+1})] + \phi(d^2(x_{n-1},x_n)) \le F(M(x_{n-1},x_n)), \tag{2.19}$$

where

$$\begin{aligned}M(x_{n-1},x_n) &= ad^2(x_{n-1},x_n)\\ &\quad + (1-a)\max\{d^2(x_{n-1},x_n), d^2(x_{n-1},x_n), d^2(x_n,x_{n+1}), d^2(x_{n+1},x_{n+1})\}\\ &= ad^2(x_{n-1},x_n) + (1-a)\max\{d^2(x_{n-1},x_n), d^2(x_n,x_{n+1})\}\\ &= d^2(x_n,x_{n+1}).\end{aligned}$$

If $M(x_{n-1},x_n) = d^2(x_n,x_{n+1})$, by (2.19), we have
$F[d^2(x_n,x_{n+1})] \le F[d^2(x_n,x_{n+1})] - \phi(d^2(x_{n-1},x_n)) < F(d^2(x_n,x_{n+1}))$. Since F is increasing, we have

$$d^2(x_n,x_{n+1}) < d^2(x_{n-1},x_n) \tag{2.20}$$

which is a contradiction. Hence, $M(x_{n-1},x_n) = d^2(x_{n-1},x_n)$. Thus,

$$F[d^2(x_n,x_{n+1})] \le F[d^2(x_{n-1},x_n)] - \phi(d^2(x_{n-1},x_n)). \tag{2.21}$$

Repeating this step, we conclude that

$$\begin{aligned}F(d^2(x_n,x_{n+1})) &\le F(d^2(x_{n-1},x_n)) - \phi(d^2(x_{n-1},x_n))\\ &\le F(d^2(x_{n-2},x_{n-1})) - \phi(d^2(x_{n-1},x_n)) - \phi(d^2(x_{n-2},x_{n-1}))\\ &\le \cdots \le F(d^2(x_0,x_1)) - \sum_{i=0}^{n}\phi(d^2(x_i,x_{i+1})).\end{aligned}$$

Since $\liminf_{\alpha\to s^+}\phi(\alpha) > 0$, we have $\liminf_{n\to\infty}\phi(d^2(x_{n-1},x_n)) > 0$, then from the definition of the limit, there exists $n_0 \in \mathbb{N}$ and $A > 0$ such that for all $n \ge n_0$, $\phi(q(x_{n-1},x_n)) > A$, hence

$$\begin{aligned}F(d^2(x_{n-1},x_{n+1})) &\le F(d^2(x_0,x_1)) - \sum_{i=0}^{n_0-1}\phi(d^2(x_i,x_{i+1})) - \sum_{i=n_0-1}^{n}\phi(d^2(x_i,x_{i+1}))\\ &\le F(d^2(x_0,x_1)) - \sum_{i=n_0-1}^{n} A\\ &= F(d^2(x_0,x_1)) - (n-n_0)A\end{aligned}$$

for all $n \ge n_0$. Taking the limit as $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} F(d^2(x_n,x_{n+1})) \le \lim_{n\to\infty}[F(d^2(x_0,x_1)) - (n-n_0)A],$$

that is, $\lim_{n\to\infty} F(d^2(x_n, x_{n+1})) = -\infty$, then from the condition (ii) of Definition 1.3, we conclude that

$$\lim_{n\to\infty} d^2(x_n, x_{n+1}) = 0. \tag{2.22}$$

Next, we shall prove that

$$\lim_{n\to\infty} d^2(x_n, x_{n+2}) = 0.$$

We assume that $x_n \neq x_m$ for every $n, m \in \mathbb{N}, n \neq m$. Indeed, suppose that $x_n = x_m$ for some $n = m + k$ with $k > 0$ and using (2.2)

$$d^2(x_m, x_{m+1}) = d^2(x_n, x_{n+1}) < d^2(x_{n-1}, x_n). \tag{2.23}$$

Continuing this process, we can that $d^2(x_m, x_{n+1}) = d^2(x_n, x_{n+1}) < d^2(x_m, x_{m+1})$ which is a contradiction. Therefore, $d^2(x_n, x_m) > 0$ for every $n, m \in \mathbb{N}, n \neq m$. Now, applying (2.1) with $x = x_{n-1}$ and $y = x_{n+1}$, we have

$$\begin{aligned} F(d^2(x_n, x_{n+2})) &= F[d^2(Tx_{n-1}, Tx_{n+1})] \\ &\leq F[s^2 d^2(Tx_{n-1}, Tx_{n+1})] \\ &\leq F(M(x_{n-1}, x_{n+1})) - \phi(d^2(x_{n-1}, x_n)), \end{aligned}$$

where

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &= ad^2(x_{n-1}, x_{n+1}) \\ &\quad + (1-a)\max\left\{d^2(x_{n-1}, x_{n+1}), d^2(x_{n-1}, x_n), d^2(x_{n+1}, x_{n+2}), d^2(x_{n+1}, x_n)\right\} \\ &= ad^2(x_{n-1}, x_{n+1}) + (1-a)\max\{d^2(x_{n-1}, x_{n+1}), d^2(x_{n-1}, x_n)\} \\ &= d^2(x_{n-1}, x_{n+1}). \end{aligned}$$

So, we get

$$F(d^2(x_n, x_{n+2})) \leq F(\max\{d^2(x_{n-1}, x_n), d^2(x_{n-1}, x_{n+1})\}) - \phi(d^2(x_{n-1}, x_{n+1})) \tag{2.24}$$

Suppose $a_n = d^2(x_n, x_{n+2})$ and $b_n = d^2(x_n, x_{n+1})$. Thus, by (2.24), one can write

$$F(a_n) \leq F(\max\{a_{n-1}, b_{n-1})\} - \phi(d^2(a_{n-1})). \tag{2.25}$$

Since F is increasing, we get

$$a_n < \max\{a_{n-1}, b_{n-1}\}.$$

By (2.2), we have

$$b_n \leq b_{n-1} \leq \max\{a_{n-1}, b_{n-1}\}$$

which implies that

$$\max\{a_n, b_n\} \leq \max\{a_{n-1}, b_{n-1}\}, \ \forall n \in \mathbb{N}.$$

Therefore, the sequence $\max\{a_{n-1}, b_{n-1}\}_{n\in\mathbb{N}}$ is decreasing sequence of real non-negative numbers. Thus, there exists $\lambda \geq 0$ such that

$$\lim_{n\to\infty} \max\{a_n, b_n\} = \lambda.$$

By (2.6), assume that $\lambda > 0$, we have

$$\lambda = \lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \sup \max\{a_n, b_n\} = \lim_{n\to\infty} \max\{a_n, b_n\}.$$

Taking the $\limsup_{n\to\infty}$ in (2.24) and applying the continuity of F and the property of ϕ, we get

$$\begin{aligned} F(\lim_{n\to\infty}\sup a_n) &\leq F(\lim_{n\to\infty}\sup \max\{a_{n-1}, b_{n-1}\}) - \lim_{n\to\infty}\sup \phi(a_{n-1}) \\ &\leq F(\lim_{n\to\infty}\sup \max\{a_{n-1}, b_{n-1}\}) - \lim_{n\to\infty}\inf \phi(a_{n-1}) \\ &< F(\lim_{n\to\infty} \max\{a_{n-1}, b_{n-1}\}). \end{aligned}$$

Therefore, $F(\lambda) < F(\lambda)$, which is a contradiction. Hence,

$$\lim_{n\to\infty} d^2(x_n, x_{n+2}) = 0. \tag{2.26}$$

Next, we shall prove that $\{x_n\}_n \in \mathbb{N}$ is a Cauchy sequence.

$$\begin{aligned} \lim_{k\to\infty} M(x_{m_k}, x_{n_k}) &= ad^2(x_{m_k}, x_{n_k}) \\ &\quad + (1-a)\max\left\{d^2(x_{m_k}, x_{n_k}), d^2(x_{m_k}, x_{m_k+1}), d^2(x_{n_k}, x_{n_k+1}), d^2(x_{n_k}, x_{m_k+1})\right\} \\ &\leq s\epsilon. \end{aligned} \tag{2.27}$$

Now, applying (2.1) with $x = x_{m_k}$ and $y = x_{n_k}$, we get

$$F[s^2 d^2(x_{m_k+1}, x_{n_k+1}] \leq F(M(x_{m_k}, x_{n_k})) - \phi(d^2(x_{m_k}, x_{n_k})). \tag{2.28}$$

Letting $k \to \infty$ the above inequality and using (2.26) and (iv), we obtain

$$\begin{aligned} F(\frac{\epsilon}{s}s^2) &= F(\epsilon s) \\ &\leq F(s^2 \lim_{k\to\infty}\sup d^2(x_{m_k+1}, x_{n_k+1})) \\ &= \lim_{k\to\infty}\sup F(s^2 d(x_{m_k+1}, x_{n_k+1})) \\ &\leq \lim_{k\to\infty}\sup F(M(x_{m_k}, x_{n_k}) - \lim_{k\to\infty}\sup \phi(d^2(x_{m_k}, x_{n_k})) \\ &= F(M(x_{m_k}, x_{n_k})) - \lim_{k\to\infty}\sup \phi(d^2(x_{m_k}, x_{n_k})) \\ &\leq F(M(x_{m_k}, x_{n_k})) - \lim_{k\to\infty}\inf \phi(d^2(x_{m_k}, x_{n_k})) \\ &< F(\lim_{k\to\infty}\sup M(x_{m_k}, x_{n_k})) \\ &\leq F(s\epsilon). \end{aligned}$$

Therefore, $F(s\epsilon) < F(s\epsilon)$. Since F is increasing, we get $s\epsilon < s\epsilon$ which is a contradiction. Then,

$$\lim_{n,m\to\infty} d^2(x_m, x_n) = 0.$$

Hence, $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d) there exists $z \in X$ such that

$$\lim_{n\to\infty} d^2(x_n, z) = 0.$$

Now, we show that $d^2(Tz, z) = 0$ arguing by contradiction, assume that

$$d^2(Tz, z) > 0.$$

Since $x_n \to z$ as $n \to \infty$ for all $n \in \mathbb{N}$, then from Lemma 1.2, we conclude that $d^2(x_n, x_m) = 0$, for all $n, m \in \mathbb{N}$. Suppose to the contrary. By Lemma 1.2, then there is $\epsilon > 0$ such that for an integer k there exists two sequences $\{m_k\}$ and $\{n_k\}$ such that

(i) $\epsilon \leq \lim_{k\to\infty} \inf d^2(x_{m_{(k)}}, x_{n_{(k)}}) \leq \lim_{k\to\infty} \sup d^2(x_{m_{(k)}}, x_{n_{(k)}}) \leq s\epsilon$,
(ii) $\epsilon \leq \lim_{k\to\infty} \inf d^2(x_{n_{(k)}}, x_{m_{(k)+1}}) \leq \lim_{k\to\infty} \sup d^2(x_{n_{(k)}}, x_{m_{(k)+1}}) \leq s\epsilon$,
(iii) $\epsilon \leq \lim_{k\to\infty} \inf d^2(x_{m_{(k)}}, x_{n_{(k)+1}}) \leq \lim_{k\to\infty} \sup d^2(x_{m_{(k)}}, x_{n_{(k)+1}}) \leq s\epsilon$,

(iv) $\frac{\epsilon}{s} \leq \lim_{k\to\infty} \inf d^2(x_{m_{(k)+1}}, x_{n_{(k)+1}}) \leq \lim_{k\to\infty} \sup d^2(x_{m_{(k)+1}}, x_{n_{(k)+1}}) \leq s^2\epsilon$.

From (2.1) and by setting $x = x_{m_k}$ and $y = x_{n_k}$, we have

$$\begin{aligned}\lim_{k\to\infty} M(x_{m_k}, x_{n_k}) &= ad^2(x_{m_k}, x_{n_k}) \\ &\quad + (1-a)\max\left\{d^2(x_{m_k}, x_{n_k}), d^2(x_{m_k}, x_{m_k+1}), d^2(x_{n_k}, x_{n_k+1}), d^2(x_{n_k}, x_{m_k+1})\right\} \\ &\leq s\epsilon. \end{aligned} \tag{2.29}$$

Now, applying (2.1) with $x = x_{m_k}$ and $y = x_{n_k}$, we get

$$F[s^2 d^2(x_{m_k+1}, x_{n_k+1}] \leq F(M(x_{m_k}, x_{n_k})) - \phi(d^2(x_{m_k}, x_{n_k})). \tag{2.30}$$

Letting $k \to \infty$ the above inequality and using (2.27) and (iv), we get

$$\begin{aligned} F(\frac{\epsilon}{s}s^2) &= F(\epsilon s) \\ &\leq F(s^2 \lim_{k\to\infty} \sup d^2(x_{m_k+1}, x_{n_k+1})) \\ &= \lim_{k\to\infty} \sup F(s^2 d(x_{m_k+1}, x_{n_k+1})) \\ &\leq \lim_{k\to\infty} \sup F(M(x_{m_k}, x_{n_k}) - \lim_{k\to\infty} \sup \phi(d^2(x_{m_k}, x_{n_k})) \\ &= F(M(x_{m_k}, x_{n_k})) - \lim_{k\to\infty} \sup \phi(d^2(x_{m_k}, x_{n_k})) \\ &\leq F(M(x_{m_k}, x_{n_k})) - \lim_{k\to\infty} \inf \phi(d^2(x_{m_k}, x_{n_k})) \\ &< F(\lim_{k\to\infty} \sup M(x_{m_k}, x_{n_k})) \\ &\leq F(s\epsilon). \end{aligned}$$

Therefore,

$$F(s\epsilon) < F(s\epsilon).$$

Since F is increasing, we get

$$s\epsilon < s\epsilon$$

which is a contradiction. Then

$$\lim_{n,m\to\infty} d^2(x_m, x_n) = 0.$$

Hence, $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d) there exists $z \in X$ such that

$$\lim_{n\to\infty} d^2(x_n, z) = 0.$$

Now, we show that $d^2(Tz, z) = 0$ arguing by contradiction, we assume that

$$d^2(Tz, z) > 0.$$

Since $x_n \to z$ as $n \to \infty$ for all $n \in \mathbb{N}$, then from Lemma 1.1, we conclude that

$$\frac{1}{s} d^2(z, Tz) \leq \lim_{n\to\infty} \sup d^2(Tx_n, Tz) \leq s\, d^2(z, Tz). \tag{2.31}$$

Now, we applying (2.1) with $x = x_n$ and $y = z$, we have

$$F(s^2 d^2(Tx_n, Tz)) \leq F(M(x_n, z)) - \phi(d^2(x_n, z)), \forall n \in \mathbb{N},$$

where

$$M(x_n, z) = a\, d^2(x_n, z) + (1-a)\max\left\{d^2(x_n, z), d^2(x_n, Tx_n), d^2(z, Tz), d^2(z, Tx_n)\right\}$$

and

$$\lim_{n\to\infty}\sup\max\left\{d^2(x_n,z),d^2(x_n,Tx_n),d^2(z,Tz),d^2(z,Tx_n)\right\}=d^2(z,Tz). \tag{2.32}$$

Therefore,

$$F(s^2d^2(Tx_n,Tz))\le F(\max\left\{d^2(x_n,z),d^2(x_n,Tx_n),d^2(z,Tz),d^2(z,Tx_n)\right\})-\phi(d^2(x_n,z)). \tag{2.33}$$

By letting $n\to\infty$ in inequality (2.33), using (2.32), (2.31) and continuity of F, we obtain

$$\begin{aligned}
F[s^2\frac{1}{s}d^2(z,Tz)] &= F[sd^2(z,Tz)]\\
&\le F[s^2\lim_{n\to\infty}\sup d^2(Tx_n,Tz)]\\
&= \lim_{n\to\infty}\sup F[s^2d^2(Tx_n,Tz)]\\
&\le \lim_{n\to\infty}\sup F(M(x_n,z))-\lim_{n\to\infty}\phi(d^2(x_n,z))\\
&= F(d^2(Tz,z))-\lim_{n\to\infty}\phi(d^2(x_n,z))\\
&< F(d^2(z,Tz)).
\end{aligned}$$

Since F is increasing, we get

$$s\,d^2(z,Tz)<d^2(z,Tz)$$

which implies that

$$d^2(z,Tz)(s-1)<0 \text{ implies } s<1$$

which is contradiction. Hence, $Tz=z$.
Therefore,

$$d^2(z,u)=d^2(Tz,Tu)>0.$$

Applying (2.2) with $x=z$ and $y=u$, we have

$$F(d^2(z,u))=F(d^2((Tz,Tu)))\le F(s^2d^2(Tz,Tu))\le F(M(z,u))-\phi(d^2(z,u)),$$

where

$$\begin{aligned}
M(z,u)&=ad^2(z,u)+(1-a)\max\left\{d^2(z,u),d^2(z,Tz),d^2(u,Tu),d^2(u,Tz)\right\}\\
&=d^2(z,u).
\end{aligned}$$

We have

$$\begin{aligned}
F(d^2(z,u))&\le F(d^2(z,u))-\phi(d^2(z,u))\\
&<F(d^2(z,u))
\end{aligned}$$

which implies that

$$d^2(z,u)<d^2(z,u)$$

which is a contradiction. Hence, $u=z$. □

Corollary 2.1. *Suppose (X,d) be a complete b-rectangular metric space and $T:X\to X$ be given mapping. Suppose that there exist $F\in\mathfrak{F}$ and $\tau\in]0,\infty[$ such that for any $x,y\in X$, we have*

$$d^2(Tx,Ty)>0 \implies F[s^2d^2(Tx,Ty)]+\tau\le[F(M(x,y))],$$

where

$$M(x,y)=a\,d^2(x,y)+(1-a)\max\left\{d^2(x,y),d^2(x,Tx),d^2(y,Ty),d^2(Tx,y)\right\}.$$

T has a unique fixed point.

If we take $a = 0$ we have the following result.

Corollary 2.2. *Suppose (X, d) be a complete b-rectangular metric space and $T : X \to X$ be given mapping. Suppose that there exist $F \in \mathfrak{F}$ and $\tau \in]0, \infty[$ such that for any $x, y \in X$, we have*

$$d^2(Tx, Ty) > 0 \implies F[s^2 d^2(Tx, Ty)] + \tau \leq [F(M(x, y))],$$

where

$$M(x, y) = (1 - a) \max \left\{d^2(x, y), d^2(x, Tx), d^2(y, Ty), d^2(Tx, y)\right\}.$$

T has a unique fixed point.

For $a = 1$ we have the following:

Corollary 2.3. *Suppose (X, d) be a complete b-rectangular metric space and $T : X \to X$ be given mapping. Suppose that there exist $F \in \mathfrak{F}$ and $\tau \in]0, \infty[$ such that for any $x, y \in X$, we have*

$$d^2(Tx, Ty) > 0 \implies F[s^2 d^2(Tx, Ty)] + \tau \leq [F(M(x, y))],$$

where

$$M(x, y) = a\, d^2(x, y).$$

T has a unique fixed point.

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