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Survey Article

Historical backround of wavelets and orthonormal systems: Recent results on positive linear operators reconstructed via wavelets

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ABSTRACT. This survey paper provides a historical overview of wavelets and orthonormal systems, alongside recent findings related to linear positive operators reconstructed using wavelets. The first section delves into the historical and chronological development of wavelets, highlighting some of their significant properties. The second section examines linear positive operators constructed through wavelets, discussing their structural characteristics and approximation results. While the list included in this paper is comprehensive, it is not exhaustive. We apologize to any authors whose works on wavelets and wavelet-based operators are not cited in this paper.

Keywords: Wavelets, positive linear operators, approximation, bounded variation, asymptotic properties.

2020 Mathematics Subject Classification: 00-02, 42C40, 47A58, 41A25, 41A35, 47G10.

1. INTRODUCTION

The main inspiration for this work comes from Fourier's pioneering work on Fourier series and Haar's question about the orthogonal systems presented in his PhD thesis in 1909. As everyone knows, Fourier asserted in 1807 that any 2π periodic function f is the sum

$$f(x) \approx a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

where a_0, a_k, b_k $(k = 1, 2, 3, ...)$ are Fourier coefficients calculated by

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)dx, \; a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx)\, dx,$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx)\, dx.$$

Prior to Fourier's contributions, Euler, D'Alembert and Daniel Bernoulli made attempts to represent and manipulate functions using power series. By passing from an early representation of the power series form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

to one of the trigonometrical system form

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + ...$$

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Fourier discovered a new functional universe. At the time, he derived this series representation, Fourier did not know the exact definitions of the function and the concept of integral. However, in 1873, Du Bois-Reymond constructed a continuous 2π-periodic function whose Fourier series diverged at a specific point. This discovery highlighted the existence of cases that were incompatible with Fourier's theory. These discrepancies opened up three new avenues for mathematicians of that era, each leading to significant results.

(1) They could modify or determine the notion of function that is adapted to the Fourier series.
(2) They could modify the definition of convergence of Fourier series.
(3) They could find other orthogonal systems for which the phenomenon, discovered by Du Bois Reymond in the case of trigonometric system, cannot happen.

As a solution to the first aproach, namely functional concept, was created by Henri Lebesgue, by defining the $L^2[0, 2\pi]$ space, their norm and the quadratic mean.

This approach has been studied by several authors (cf the references, in particular the books

(1) P. L. Butzer, R. J. Nessel: Fourier Analysis and Approximation, vol. 1, Academic Press, New York-London (1971),
(2) C. Bardaro, J. Musielak and G. Vinti: Nonlinear Integral Operators and Applications, De Gruyter Series in Nonlinear Analysis and Applications, vol. 9, New York-Berlin (2003)).

The solutions related with the second difficulty are the summability methods and the approximation theory.

(Cf the references, in particular the books

(3) P. P. Korovkin: Linear operators and approximation theory, Hindustan Publ., New Delhi (1960),
(4) F. Altomare, M. Campiti: Korovkin-type approximation theory and its applications, De Gruyter Studies in Mathematics, vol. 17, Walter de Gruyter and Co., Berlin (1994)).

The third route leads to the notion wavelets.

(Cf the references, in particular

(5) I. Daubechies: Orthonormal bases of compactly supported wavelets, Comm. Pure Appl. Math., 41 (1988), 909–996,
(6) I. Daubechies: Ten lectures on wavelets, CBMS-NSF Series in Appl. Math., vol. 61, SIAM Publ., Philadelphia (1992),
(7) Y. Meyer: Ondelettes: wavelets and operators, Cambridge University Press, New York (1993)).

In 1909, Haar asked himself the following question.

"Does there exist another orthonormal system $\{h_1(x), h_2(x), \ldots, h_n(x), \ldots\}$ of functions defined on $[0, 1]$ such that for any function f continuous on $[0, 1]$, the series converges to $f(x)$ uniformly on $[0, 1]$?"

The first and simplest solution to the problem was given by Alfred Haar in his PhD thesis (1909). In other words, Haar suggests an alternative system to Fourier system.

A. Haar, Dissertation: *"Zur Theorie der orthogonalen Funktionensysteme"*, Georg-August-Universität Göttingen (1909).

This system, known as the Haar system, is the first and simplest known form of wavelets and has been an important guide in the beginning and progress of the theory.

The simplest wavelet is known as the Haar wavelet defined as;

$$\varphi(x) = \begin{cases} 1 & , & 0 \le x < \frac{1}{2} \\ -1 & , & \frac{1}{2} \le x < 1 \\ 0 & , & e.w. \end{cases}$$

with the corresponding scaling function

$$\phi(t) = \begin{cases} 1 & , & 0 \le x < 1 \\ 0 & , & e.w. \end{cases} .$$

Note. Clearly, Haar wavelets constitutes an orthonormal system for the space of square-integrable functions on the real line. Unfortunately, Haar wavelets are not continuously differentiable which somewhat limits their applications. Indeed, since Haar wavelet is neither continuous nor differentiable, it is suitable for representing discrete signals not for representing smooth signals or functions.

In 1922, Hans Radamacher [31] defined another orthonormal system on the interval $[0,1]$ as follows and provided a solution similar to Haar's problem.

$$\phi(n,x) = \operatorname{sign}(\sin 2^n \pi x) = \begin{cases} 1 & , & \frac{2k-1}{2^n} \le x < \frac{2k}{2^n} \\ -1 & , & \frac{2k}{2^n} \le x < \frac{2k+1}{2^n} \end{cases} , \quad x \in [0,1].$$

The Radamecher system, which is similar to the Haar system, revealed the connection with the trigonometric system.

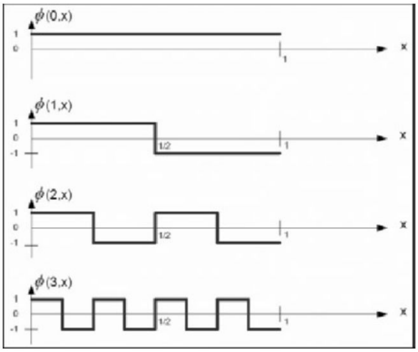


FIGURE 1. First four elements of Rademacher system

In order to eliminate this negative situation, various attempts were made by Faber [16] and Schauder [32] between 1910 and 1928, and a new system was created by taking the antiderivatives of the functions in the Haar system, thus making it possible to obtain the convergence of the series polygonally. Let $j \ge 0$ and

$$\psi_{j,k}(x) = \Delta(2^j x - k + 1), 1 \le k \le 2^j,$$

where

$$\Delta(x) = \begin{cases} 2x & , & 0 \le x \le \frac{1}{2} \\ 2 - 2x & , & \frac{1}{2} \le x \le 1 \\ 0 & , & e.w. \end{cases} ,$$

together with the function $\psi_1(x) = \psi_{0,1}(x - 1/2) + \psi_{0,1}(x + 1/2)$ on $[0,1]$ and 0 elsewhere.

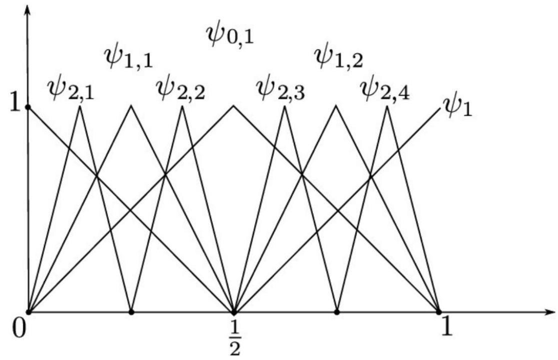


FIGURE 2. Some elements of the Faber and Schauder System

The Faber-Schauder system is the first basis defined for the $C[a,b]$ space and is generally known as the Schauder basis. The Faber-Schauder system is not an orthonormal system. As a result of applying the Gram-Schmidt orthonormalization process to the Faber-Schauder system, the Franklin system is obtained. This system was described by Philip Franklin in 1927, [17].

In 1983, Strömberg [33] defined the Strömberg wavelets, which bear his name, by using the Franklin system. If we remember that the Haar wavelet is the first known wavelet, the Strömberg wavelet is the first wavelet defined as smooth. This is a very important step for the representation of differentiable functions.

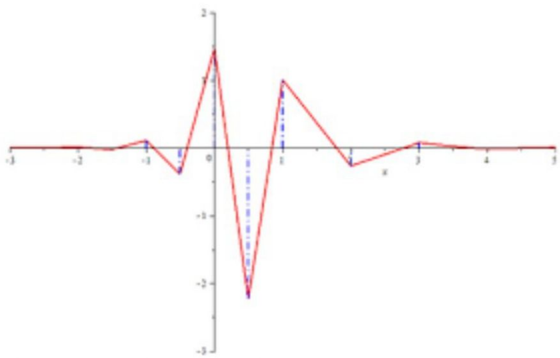


FIGURE 3. Strömberg wavelet of order zero

1.1. **Wavelet Analysis.** In this section, we outline the main notation used throughout the paper and include some general information.

It is very well-known that wavelets and wavelet expansions have the great advantage of being able to separate and identify fine details in a signal or a function. One of the main advantages of wavelets compared to the Fourier analysis and its related theories is that they offer simultaneous localization in the time and frequency domain. The second main advantage of wavelets is that they are computationally very fast and detailed when using wavelet expansions and transformations.

Unlike Fourier analysis, which uses sinusoidal functions, wavelets offer flexibility and adaptability to specific signals (target functions) and are inherently localized. Because there is no single universal wavelet, they can be tailored to suit particular applications, making them ideal for adaptive systems that adjust to match the function. Below are the graphs of a sinusoidal function and the Shannon wavelet.

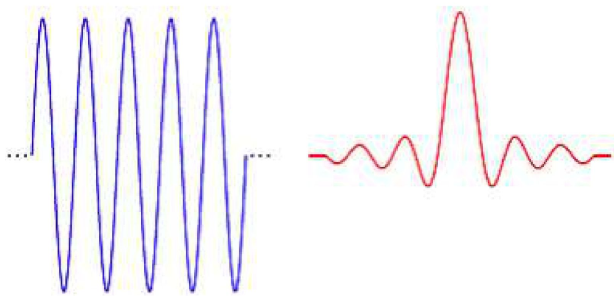


FIGURE 4. Sinusoidal function and the Shannon wavelet

Wavelet expansion, or the reconstruction of signals using wavelets, enables more precise local identification and separation of signal features. Each wavelet expansion coefficient represents a local component, making interpretation easier. Additionally, wavelets allow overlapping components of a signal to be separated in both time and frequency domains.

1.2. **Multiresolution Analysis (MRA).** A Multiresolution Analysis (MRA) is an increasing sequence $(Vj)_{j\in Z}$ of closed subspaces of $L^2(\mathbb{R})$ such that the following hold:

(i) V_j is a set of all $f \in L^2(\mathbb{R})$ which are constant on 2^{-j} length intervals and

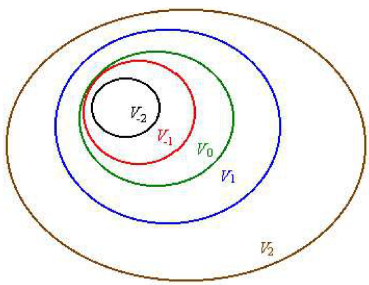
$$... \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset ... \subset V_j \subset V_{j+1} \subset .. \subset L_2(\mathbb{R}),$$

$$\overline{\bigcup_j V_j} = L_2(\mathbb{R}), \bigcap_j V_j = \{0\}\,.$$

(ii)

$$\forall j,k \in \mathbb{Z},\ f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1},$$
$$\forall k \in \mathbb{Z},\ f(x) \in V_0 \Leftrightarrow f(x-k) \in V_0$$
$$\forall j,k \in \mathbb{Z},\ f(x) \in V_j \Leftrightarrow f(x-2^{-j}k) \in V_j,$$

(see [14]-[15]).

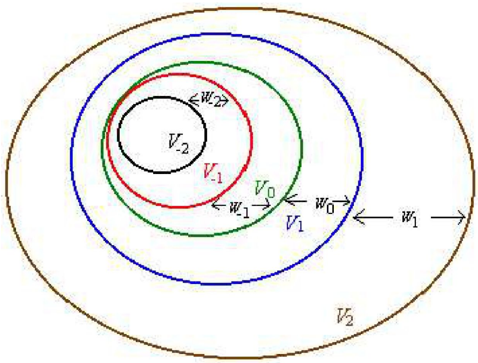


Since $V_j \subset V_{j+1}$ (for all $j \in \mathbb{Z}$), then there exist subspaces W_j of $L_2(\mathbb{R})$ satisfying

$$\begin{aligned} V_1 =& V_0 \oplus W_0, \\ V_2 =& V_1 \oplus W_1, \\ &\vdots \\ V_{j+1} =& V_j \oplus W_j, \end{aligned}$$

and

$$L_2(\mathbb{R}) = ... \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus ... \oplus W_j \oplus W_{j+1} \oplus ...$$

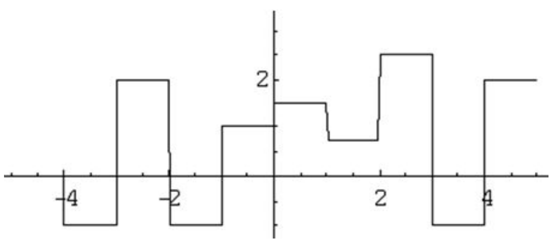


Let

$$\phi(x) = \begin{cases} 1 & , \quad 0 \leq x < 1 \\ 0 & , \quad d.y. \end{cases}$$

be a scale function and then the subspaces are

$$\begin{aligned} V_0 : f(x) &= \sum_k a_k \phi(x-k) \\ V_1 : f(x) &= \sum_k a_k \phi(2x-k) \\ V_2 : f(x) &= \sum_k a_k \phi(2^2x-k) \\ (j \in \mathbb{Z}),\ V_j : f(x) &= \sum_k a_k \phi(2^jx-k). \end{aligned}$$



Definition 1.1 (Wavelet). *A wavelet is a small wave which oscillates and decays in the time domain. In other words, functions called "wavelets" are generated from one single function ϕ (scale function) by dilations and translations as*

$$\phi_{n,k}(x) = 2^{n/2}\phi\left(2^n x - k\right), \;\; n,k \in Z.$$

The fundamental idea behind wavelets is to analyze the signals and functions according to scale. A wavelet basis set starts with two orthogonal functions: the scaling function (or father wavelet) $\phi(t)$ and the wavelet function (or mother wavelet) $\varphi(t)$. By scaling and translation of these two orthogonal functions, we obtain a complete basis set. The scaling and wavelet functions, respectively, satisfy

$$\int_{-\infty}^{\infty} \phi(t)dt = 1, \int_{-\infty}^{\infty} \varphi(t)dt = 0.$$

These two functions have finite energy, namely $\phi, \varphi \in L^2(\mathbb{R})$, and orthogonal. In general, the wavelets refers to the set of family of orthonormal functions of the form

$$\varphi_{a,b}(t) = \frac{1}{\sqrt{a}}\varphi\left(\frac{t-b}{a}\right), \quad a > 0, b \in \mathbb{R}, \tag{1.1}$$

where φ is the basic wavelet.

The wavelet analysis procedure is to adopt a wavelet prototype function, called an analyzing wavelet (father wavelet) or mother wavelet. Some of the special cases of a and b, one can obtain from (1.1) different type of wavelets, such as Haar wavelet, Shannon wavelet, Franklin system, Meyer wavelets, etc. (see [9]).

In the present study, we consider orthonormal bases of wavelets in $L^2(\mathbb{R})$, and assume that there is a scaling function (father wavelet) $\phi(t)$ whose translates $\{\phi(t-n)\}$ are orthogonal and the mother wavelet $\varphi(t)$ based on the father wavelet $\phi(t)$ gives rise to the orthonormal basis $\varphi_{j,k}(t)$ of $L^2(\mathbb{R})$, where

$$\varphi_{j,k}(t) = 2^{j/2}\varphi(2^j t - k). \tag{1.2}$$

Hence, by using a multiresolution analysis (MRA), each $f \in L^2(\mathbb{R})$ has the following wavelet expansion

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} b_{j,k}\varphi_{j,k}(x),$$

where $b_{j,k}$ are wavelet coefficients defined by

$$b_{j,k} = \langle f(x), \varphi_{j,k}(x)\rangle = 2^{j/2}\int_{\mathbb{R}} f(x)\overline{\varphi(2^j x - k)}dx.$$

1.3. **Daubechies Wavelets.** Assume that the scale function (or father wavelet) $\psi \in L_\infty(\mathbb{R})$ and satisfies:

(i) ψ is a compactly supported, namely there is a real constant $0 < \lambda \leq 1$ such that $\operatorname{supp}\psi \subset [0, \lambda]$,

(ii) $\int_{\mathbb{R}} \psi(x)dx = 1$,

(iii) the first N moments of the father wavelet ψ satisfy

$$m_j^w(\psi) := \int_{\mathbb{R}} x^j\psi(x)dx = 0, \quad j = 1, ..., N.$$

Obviously, the absolute moments of the father wavelet ψ

$$M_j^w(\psi) := \int\limits_{\mathbb{R}} |x|^j \, |\psi(x)| \, dx < +\infty$$

for every $j \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Actually, Daubechies wavelets have strong relations with the properties of continuity and differentiability. Namely, for an arbitrary fixed integer $N \geq 1$, compactly supported Daubechies wavelet ψ is supported with $[0, 2N-1]$, in addition there exists a constant $r > 0$ such that for $N \geq 2$, $\psi \in C^{rN}(\mathbb{R})$ and to have a given number of vanishing moments. In particular, when $N = 1$, then the first Daubechies wavelet ψ will be the classical Haar basis. As N increases, the regularity of the wavelets increase (see [14]-[15]).

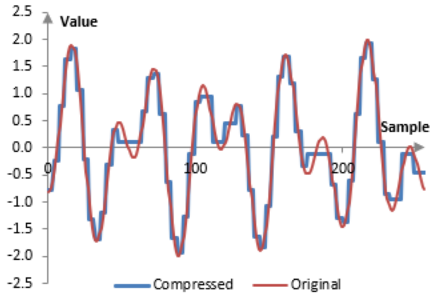


FIGURE 5. Approximation by Haar wavelet

This means that if we want to use Daubechies wavelets to reconstruct a function, it is more convenient to choose or construct wavelets based on the continuity or differentiability properties of the given function.

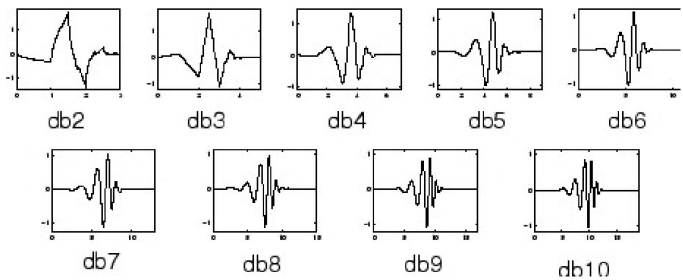


FIGURE 6. Daubechies wavelets (N=2,3,...)

2. Recent Results on Positive Linear Operators Reconstructed via Wavelets

In this section, we deal with the very recent studies about the Bernstein type operators and Neural Network (NN) operators reconstructed via wavelets.

For a bounded real valued function f defined on the interval $[0,1]$ ($f \in B[0,1]$), the Bernstein operators B_n, $n \geq 1$ are defined by

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \quad , \quad n \geq 1, \tag{2.3}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis $(0 \le x \le 1)$.

2.1. **Bernstein Operators Constructed Using Wavelets.** Owing to the above definitions, first of all we will recall the wavelet type Bernstein operators WB_n introduced by the author [24].

Definition 2.2 (2023, [24]). *Let $f \in B[0,1]$, and let $w \in L_\infty(R)$ be a father wavelet satisfying (i)-(iii). Then the wavelet type Bernstein operators are defined by:*

$$(WB_n f)(t) := n \sum_{k=0}^{n} p_{n,k}(t) \int_0^1 f(x)\, w(nx-k) dx,$$

with $t \in [0,1]$, specifying that $supp(w) \subseteq [0,\lambda]$, $0 < \lambda \le 1$.

Remark 2.1 (2023, [24]). *If we choose the father wavelet w as the Haar scaling function, namely $w(x) = \chi_{[0,1]}(x)$, then clearly our wavelet type operators reduce to the Kantorovich form of the Bernstein operators. Indeed:*

$$\begin{aligned}(WB_n f)(t) &= n \sum_{k=0}^{n} p_{n,k}(t) \int_0^1 f(x)\, w(nx-k) dx \\ &= \sum_{k=0}^{n} p_{n,k}(t) \int_0^1 f\left(\frac{u+k}{n}\right) w(u) du \\ &= n \sum_{k=0}^{n} p_{n,k}(t) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(z)\, dz = (K_n f)(t).\end{aligned}$$

This shows that the wavelet type Bernstein operators (2.9) are a natural extension of the Kantorovich type of the Bernstein operators. As presented and proved in [24], we have the followings.

Theorem 2.1 (2023, [24]). *Let $f \in B[0,1]$ and let $w \in L_\infty(R)$ be a father wavelet satisfies (i)-(iii). Then the moments of wavelet type Bernstein operators, constructed by using the compactly supported Daubechies wavelets (2.9) and the Bernstein operators (2.3) are the same, namely*

$$(WB_n x^s)(t) = (B_n x^s)(t), \quad s = 0, 1, ..., K$$

holds true.

Remark 2.2 (2023, [24]). *By the properties (ii) and (iii), one gets*

$$\begin{aligned}(WB_n (x-t)^\beta)(t) &= \frac{1}{n^\beta} \sum_{k=0}^{n} p_{n,k}(t) (k-nt)^\beta \\ &= (B_n (x-t)^\beta)(t).\end{aligned}$$

Throughout this work, the first two central moments of the wavelet type Bernstein operators (2.9) satisfy

$$\begin{aligned}\mu_1(t) &:= \frac{1}{n} \sum_{k=0}^{n} p_{n,k}(t) (k-nt) = 0, \\ \mu_2(t) &:= \frac{1}{n^2} \sum_{k=0}^{n} p_{n,k}(t) (k-nt)^2 = \frac{t(1-t)}{n} \le \frac{1}{4n}\end{aligned} \tag{2.4}$$

for every $t \in [0,1]$.

It is also well-known that for each $s \in \mathbb{N}_0$ there is a constant A_s only depending upon s such that

$$0 \leq \mu_{2s}(t) \leq \frac{A_s}{n^s} < \infty$$

hold. Moreover, for every $t \in [0,1]$ and for some $\beta > 0$, the discrete absolute moments of order β satisfy

$$\widetilde{\mu}_\beta(t) := (B_n |x-t|^\beta)(t) \leq 2\Gamma\left(\frac{\beta}{2}+1\right)\frac{1}{n^{\beta/2}} < \infty, \tag{2.5}$$

where $\Gamma(\cdot)$ stands for the Gamma function (see [3]). In [24], we have also proved the following:

Theorem 2.2 (2023, [24]). *Let $f \in B[0,1]$ and let $\psi \in L_\infty(R)$ be a father wavelet satisfying (i)-(iii). Then*

$$\lim_{n\to\infty} (WB_n f)(t_0) = f(t_0)$$

holds true at each point t_0 of continuity of f.

As a consequence of the Theorem 2.2, we have also the following uniform convergence result.

Corollary 2.1 (2023, [24]). *The same arguments of Theorem 2.2 apply to the case when $f \in C[0,1]$. In this case the convergence is uniform with respect to $t \in [0,1]$, and hence one has*

$$\lim_{n\to\infty} \|(WB_n f) - f\|_{C[0,1]} = 0.$$

Theorem 2.3 (2023, [24]). *Let $f \in C[0,1]$ and let $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfies (i)-(iii). Then*

$$\|WB_n f\|_\infty \leq K \|f\|_\infty,$$

holds true, where $K = \lambda \|\psi\|_\infty$.

Theorem 2.4 (2023, [24]). *Let $f \in C[0,1]$ and let $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfies (i)-(iii). Then*

$$\lim_{n\to\infty} (WB_n f)(x) = f(x),$$

and

$$|(WB_n f)(x) - f(x)| \leq (K+1) K_2\left(f; \frac{m_2(\varphi)+\lambda^2}{n^2}\right),$$

where $K = \lambda \|\psi\|_\infty$ and $K_2(f;\delta)$ is the Peetre's K-functional.

Theorem 2.5 (2023, [24]). *Let $f \in C[0,1]$, $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfies (i)-(iii) and $\alpha \in (0,2)$ be fixed real number. Then*

$$\omega_2(f;t) = \mathcal{O}(t^\alpha) \Rightarrow |(WS_n f)(x) - f(x)| = \mathcal{O}(1/n)^\alpha$$

holds true.

Theorem 2.6 (2023, [24]). *Let $f \in L^1[0,1]$ and let $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfies (i)-(iii). Then*

$$\|WB_n f\|_1 \leq K \|f\|_1,$$

holds true, where $K = nh\|\psi\|_\infty \|p_{n,k}\|_1$ and $h := \lfloor \lambda \rfloor + 1$. Here $\lfloor x \rfloor$ denotes the floor function of the real number x.

Theorem 2.7 (2023, [24]). *Let $f \in L^p[0,1]$ $(1 \leq p \leq \infty)$ and let $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfies (i)-(iii). Then*

$$\|WS_n f\|_p \leq K_p \|f\|_p,$$

holds true, where $K_p = n\|\psi\|_\infty \|p_{n,k}\|_1^{1/p} h^{1/p} > 0$, and $h := \lfloor \lambda \rfloor + 1$.

2.2. **Extension of the Generalized Bézier Operators by Wavelets.** Let $p_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$ be the Bernstein basis and let $J_{n,k}\left(t\right) = \sum_{j=k}^{n} p_{n,j}\left(t\right),\ t \in [0,1]$, be the Bézier basis functions introduced in [8]. For a bounded real valued function f defined on the interval $[0,1]$ ($f \in B[0,1]$) and $\alpha \geq 1$, the Bézier modification $B_{n,\alpha}$ of the well-known Bernstein operators are defined as

$$(2.6) \qquad \left(B_{n,\alpha} f\right)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) Q_{n,k}^{(\alpha)}(x) \quad \text{for } x \in [0,1],$$

where $Q_{n,k}^{(\alpha)}(t) = J_{n,k}^{\alpha}(t) - J_{n,k+1}^{\alpha}(t)$ for $t \in [0,1]$ ($J_{n,l}(x) \equiv 0$ if $l > n$). If $\alpha = 1$, then $B_{n,\alpha}$ reduce to the classical Bernstein operators.

In 2004, Gupta [19] considered the generalized Kantorovich type operators as;

$$(2.7) \qquad \left(K_{n,c} f\right)(x) := n \sum_{k=0}^{\infty} p_{n,k,c}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)du$$

where f is locally integrable on the interval $[0,\infty)$ and are of polynomial growth as $u \to \infty$, whose particular cases reduce to the well-known Szász-Kantorovich and Baskakov-Kantorovich operators. Indeed, here

$$p_{n,k,c}(x) = (-1)^k \frac{x^k}{k!} \phi_{n,c}^{(k)}(x),$$

and as special cases, $\phi_{n,c}(x) = (1+cx)^{-n/c}$ for $c = 1$, and $\phi_{n,c}(x) = \exp(-nx)$ for $c = 0$, then the generalized Kantorovich type operators (2.7) turns out to be Baskakov-Kantorovich and Szász-Kantorovich operators, respectively.

As a generalization, Gupta [19] defined the Bézier variant of the aforementioned generalized Kantorovich type operators as follows:

$$(2.8) \qquad \left(K_{n,c}^{\alpha} f\right)(t) := n \sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(t) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)du,$$

where $Q_{n,k,c}^{(\alpha)}(t) = J_{n,k,c}^{\alpha}(t) - J_{n,k+1,c}^{\alpha}(t)$, $\alpha \geq 1$ and

$$J_{n,k,c}(t) = \sum_{j=k}^{\infty} p_{n,j,c}(t)$$

be the Baskakov basis and Szász basis for $c = 1$ and $c = 0$, respectively. Clearly, if $\alpha = 1$ then the operators $K_{n,c}^{\alpha}$ (2.8) reduce to operators (2.7). Now, we construct the generalized Bézier operators by using the compactly supported Daubechies wavelets.

Definition 2.3 (2022, [23]). *Let $f \in L_1[0,\infty)$ and $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfies (i)-(iii). Then the wavelet type generalized Bézier operators, constructed by using the compactly supported*

Daubechies wavelets, are defined by:

$$(WB^{\alpha}_{n,c}f)(t) := n\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t) \int_{\mathbb{R}} f(x)\,\psi(nx-k)dx \quad (t\in\mathbb{R}) \tag{2.9}$$
$$= \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t) \int_{\mathbb{R}} f\left(\frac{x+k}{n}\right)\psi(x)dx \quad (t\in\mathbb{R})$$
$$= \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t) \int_{0}^{\lambda} f\left(\frac{x+k}{n}\right)\psi(x)dx \quad (t\in\mathbb{R}).$$

Remark 2.3 (2022, [23]). *If we choose the father wavelet ψ as the Haar scaling function, namely $\psi(x) = \chi_{[0,1]}(x)$, then clearly our wavelet type operators reduce to the Kantorovich form of the generalized Bézier operators (2.8) considered and investigated by Gupta in* [1], [19] *and* [20]. *Indeed;*

$$(WB^{\alpha}_{n,c}f)(t) = n\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t) \int_{\mathbb{R}} f(x)\,\psi(nx-k)dx$$
$$= \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t) \int_{\mathbb{R}} f\left(\frac{u+k}{n}\right)\psi(u)du$$
$$= \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t) \int_{0}^{1} f\left(\frac{u+k}{n}\right)\psi(u)du.$$

This means that our new operators constructed by father wavelets are a natural extension of the Kantorovich type of the generalized Bézier operators and also its Durrmeyer type operators.

Theorem 2.8 (2022, [23]). *Let $\psi \in L_{\infty}(R)$ be a father wavelet satisfies (i)-(iii). Then the moments of wavelet type generalized Bézier operators constructed by using the compactly supported Daubechies wavelets (2.9) are given by*

$$(WB^{\alpha}_{n,c}x^s)(t) = \sum_{k=0}^{\infty} \frac{k^s}{n^s} Q^{(\alpha)}_{n,k,c}(t), \quad s = 0, 1, ..., M,$$

where M is a positive integer.

Remark 2.4 (2022, [23]). *Moreover, the central moments of the wavelet type generalized Bézier operators (2.9) are given by,*

$$(WB^{\alpha}_{n,c}(x-t)^{\beta})(t) = \frac{1}{n^{\beta}} \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t)\,(k-nt)^{\beta}.$$

Throughout this work, as in the case of the generalized Bézier operators, we assume that the first two central moments of the generalized Bézier operators, constructed by using the compactly supported Daubechies wavelets (2.9) satisfy for $c = 1$ and $c = 0$,

$$w_0(t) := (WB^{\alpha}_{n,c}1)(t) = 1,$$
$$w_1(t) := (WB^{\alpha}_{n,c}(x-t))(t) = 0,$$
$$w_2(t) := (WB^{\alpha}_{n,c}(x-t)^2)(t) = \frac{\alpha t(1+ct)}{n},$$

and in addition for each $x \in [0,\infty)$

$$(WB^{\alpha}_{n,c}(x-t)^m)(t) = O\left(n^{-[(m+1)/2]}\right), n \to \infty$$

(see [19]).

Theorem 2.9 (2022, [23]). *Let $f \in L_1[0,\infty)$ and let $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfies (i)-(iii). Then*

$$\lim_{n\to\infty} (WB^{\alpha}_{n,c}f)(t_0) = f(t_0)$$

holds true at each point t_0 of continuity of f.

Corollary 2.2 (2022, [23]). *The same arguments of Theorem 2.9 apply to the case when $f \in C[0,\infty) \cap L_\infty(\mathbb{R})$. In this case the convergence is uniform with respect to $x \in [0,\infty)$, and hence one has*

$$\lim_{n\to\infty} \left\| \left(WB^{\alpha}_{n,c}f\right) - f \right\|_\infty = 0.$$

Theorem 2.10 (2022, [23]). *Let $f \in C[0,\infty) \cap L_\infty(\mathbb{R})$ and let $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfies (i)-(iii). Then*

$$\left\| WB^{\alpha}_{n,c}f \right\|_\infty \le K \left\| f \right\|_\infty,$$

holds true, where $K = \lambda \left\| \psi \right\|_\infty$.

Theorem 2.11 (2022, [23]). *Let $f \in C[0,\infty) \cap L_\infty(\mathbb{R})$ and let $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfies (i)-(iii). Then*

$$\lim_{n\to\infty} \left(WB^{\alpha}_{n,c}f\right)(x) = f(x),$$

and

$$\left| \left(WB^{\alpha}_{n,c}f\right)(x) - f(x) \right| \le (K+1)\, K_2\left(f; \frac{n\alpha t(1+ct)+\lambda^2}{n^2}\right),$$

where $K = \lambda \left\| \psi \right\|_\infty$ and $K_2(f;\delta)$ is the Peetre's K-functional.

Theorem 2.12 (2022, [23]). *Let $f \in C[0,\infty) \cap L_\infty(\mathbb{R})$, $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfies (i)-(iii) and $\eta \in (0,2)$ be fixed real number. Then*

$$\omega_2(f;t) = \mathcal{O}(t^\eta) \Rightarrow \left| \left(WB^{\alpha}_{n,c}f\right)(x) - f(x) \right| = \mathcal{O}(1/n)^\eta$$

holds true.

2.3. **Some Properties of the Generalized Bézier Operators Constructed by Wavelets in BV Spaces.** We first recall the following main definitions.

Definition 2.4. *Let g be a bounded function on a compact interval $I = [a,b]$. The modulus of variation $\nu_n(g;[a,b])$ of the function g is defined for nonnegative integers n as follows:*

$$\nu_0(g;[a,b]) := 0$$

and for $k \ge 1$

$$\nu_k(g;[a,b]) := \sup_{\Pi_k} \sum_{j=0}^{k-1} \left| g(x_{2j+1}) - g(x_{2j}) \right|,$$

where Π_k is an arbitrary system of k disjoint intervals (x_{2j}, x_{2j+1}), $j = 0,1,...,k-1$ i.e., $a \le x_0 < x_1 \le x_2 < x_3 ... \le x_{2k-2} < x_{2k-1} \le b$ (see [12]).

Some properties of the modulus of variation and its applications can be found e.g. in [12]. In particular, $\nu_k(g;I)$ is a non-decreasing function of $k \in \mathbb{N}$,

$$\nu_k(g;I) \leq 2k \sup_{t\in I} |g(t)|\,,$$
$$\nu_k(g;Z) \leq \nu_k(g;I)$$

for any compact interval Z contained in I. Let $p \geq 1$. The pth power variation of a function g on a compact interval $I=[a,b]$ is denoted by $V_p(g,I)$ and is defined as the upper bound of the set of numbers

$$\left(\sum_j |g(s_j)-g(t_j)|^p\right)^{1/p}$$

over all finite systems of non-overlapping intervals $(s_j,t_j) \subset I$. If f is of bounded pth power variation on I, then for every $k \in \mathbb{N}$,

$$\nu_k(g;I) \leq k^{1-1/p} V_p(g,I). \tag{2.10}$$

The class of all functions of bounded pth power variation on every compact interval contained in $[0,\infty)$ will be denoted by $BV^p_{loc}[0,\infty)$.

Theorem 2.13 (2023, [25])**.** *Let f be a bounded variation on every finite subinterval of $[0,\infty)$ and are of polynomial growth as $t\to\infty$, i.e., for some $r>1$ and some absolute constant M, $|f(t)| \leq Mt^r$ holds true for $t \in [0,\infty)$. Let $\psi \in L_\infty(R)$ be a father wavelet satisfies (i)-(iii) and let $x \in (0,\infty)$ the one-sided limits $f(x+)$, $f(x-)$ exist at a fixed point $x \in (0,\infty)$. Then, for all sufficiently large integers n, one has*

$$\begin{aligned}
&\left|\left(WB_{n,\alpha}f\right)(x) - \frac{1}{2^\alpha}f(x+) - \left(1-\frac{1}{2^\alpha}\right)f(x-)\right| \\
\leq& 2v_1(g_x;H_x(x/\sqrt{n})) \\
&+\frac{16n\alpha m_2(\psi)}{x^2}\left[\sum_{j=1}^{m-1}\frac{v_j(g_x;H_x(jx/\sqrt{n}))}{j^3}+\frac{v_m(g_x;H_x(x))}{m^2}\right] \\
&+\frac{\alpha\Delta(n,x,c)}{\sqrt{nx\,(1+cx)}}\,|f(x+)-f(x-)| + \frac{\alpha M m_2(\psi)}{x^{2r}n^r},
\end{aligned}$$

where $m := [\sqrt{n}]$,

$$\Delta(n,x,c) := \left\{\begin{array}{lll} 2^\alpha\left[3(1+x)+2C_1\right] & , & c=1 \\ 2^{\alpha-1}\left[2\sqrt{1+3x}+1\right] & , & c=0 \end{array}\right.,$$

and

$$g_x(t) := \left\{\begin{array}{ccc} f(t)-f(x+) & if & t>x \\ 0 & if & t=x \\ f(t)-f(x-) & if & 0\leq t<x \end{array}\right.. \tag{2.11}$$

Using the notations used in Theorem 2.13 and using inequality (2.10), we get:

Theorem 2.14 (2023, [25]). *Let $f \in BV_{loc}^{p}[0,\infty)$, $p \geq 1$ and let $x \in (0,\infty)$. Then for all sufficiently large integers n, one has*

$$\left|(WB_{n,\alpha}f) - \frac{1}{2^{\alpha}}f(x+) - \left(1-\frac{1}{2^{\alpha}}\right)f(x-)\right|$$
$$\leq \left(2+\frac{16n\alpha m_2(\psi)}{x^2}\right)V_p(g_x;H_x(x/\sqrt{n}))$$
$$+\frac{16n\alpha m_2(\psi)}{x^2}\left(\frac{1}{\sqrt{n}}\right)^{1+1/p}\sum_{k=1}^{n}\frac{V_p(g_x;H_x(x/\sqrt{k})}{\left(\sqrt{k}\right)^{1-1/p}}$$
$$+\frac{\alpha\Delta(n,x,c)}{\sqrt{nx\,(1+cx)}}\left|f(x+)-f(x-)\right| + \frac{\alpha M m_2(\psi)}{x^{2r}n^{r}}.$$

So, we get the following approximation theorem.

Corollary 2.3 (2023, [25]). *Suppose that $f \in M_{loc}[0,\infty)$ (in particular, $f \in BV_{loc}^{p}[0,\infty),\ p \geq 1$) and it satisfies the growth condition as in Theorem 2.13. Then at every point $x \in (0,\infty)$ at which the limits $f(x+)$, $f(x-)$ exist, we have*

$$\lim_{n\to\infty}(WB_{n,\alpha}f)(x) = \frac{1}{2^{\alpha}}f(x+) + \left(1-\frac{1}{2^{\alpha}}\right)f(x-).$$

Obviously, the above relations hold true for every measurable function f bounded on $[0,\infty)$, in particular for every function f of bounded pth power variation $(p \geq 1)$ on the whole interval $[0,\infty)$.

Remark 2.5 (2023, [25]). *For the particular value $c = 1$ and $c = 0$, our theorems improve the main results of* [1], [19] *and* [20].

2.4. Asymptotic Properties and Quantitative Results of the Wavelet Type Bernstein Operators.

Theorem 2.15 (2024, [28]). *Let $f : [0,1] \to \mathbb{R}$ be a bounded function. Moreover, we assume that $f'(t)$ exists at a fixed point t. Then the following asymptotic formula holds:*

$$(WB_n f)(t) = f(t) + o\left(n^{-1/2}\right),\ \ (n \to \infty).$$

Theorem 2.16 (2024, [28]). *Let $f \in B[0,1]$ and let $t \in [0,1]$ be fixed. If for a certain $r \in N$, $f \in C^{r}$ locally at the point t, then the following asymptotic formula holds:*

$$(WB_n f)(t) = f(t) + \sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\mu_i(t) + o\left(n^{-r/2}\right),\ \ (n \to \infty),$$

where μ_i is the $i-th$ order algebraic moment.

As a consequence of the aforementioned theorems, we can establish the following first and second order Voronovskaya type theorems, respectively.

Theorem 2.17 (2024, [28]). *Let $f \in B[0,1]$ and let $t \in [0,1]$ be fixed. If $f \in C^{1}$ locally at the point t, then we have*

$$\lim_{n\to\infty} n^{1/2}\left[(WB_n f)(t) - f(t)\right] = 0.$$

Theorem 2.18 (2024, [28])**.** *Let $f \in B[0,1]$ and let $t \in [0,1]$ be fixed. If $f \in C^2$ locally at the point t, then we have*

$$\begin{aligned}\lim_{n\to\infty} n\left[(WB_n f)(t) - f(t)\right] &= \frac{1}{2} f''(t)\,\mu_2(t)\\ &= \frac{t(1-t)}{2} f''(t)\,.\end{aligned}$$

Here, we study quantitative estimates of the convergence results given in the previous theorems.

Theorem 2.19 (2024, [28])**.** *Let f be a bounded function $f : [0,1] \to R$. Moreover, we also assume that $f \in C^1$ locally at a fixed point t. Then, there holds*

$$\left| n\left((WB_n f)(t) - f(t)\right)\right| \le 2D_1 K_1\left(f'; \frac{D_2}{4nD_1}\right),$$

where

$$\begin{aligned}D_1 &= M_1^w(w) + M_0^w(w)\widetilde{\mu}_1(t)\\ D_2 &= M_2^w(w) + M_0^w(w)\widetilde{\mu}_2(t) + 2M_1^w(w)\widetilde{\mu}_1(t)\end{aligned}$$

and $M_i^w(\cdot)$ ($i = 0,1,2$) are the absolute moments.

Similarly we have the following quantitative estimates for the $r-th$ order asymtotic expansion.

Theorem 2.20 (2024, [28])**.** *We also assume that $f \in C^r$ locally at a fixed point t, then we have*

$$\left| n^r\left((WB_n f)(t) - f(t) - \sum_{i=1}^{r} \frac{f^{(i)}(t)}{i!}\mu_i(t)\right)\right| \le \frac{2L_r}{r!} K_1\left(f^{(r)}; \frac{J_r}{2n(r+1)L_r}\right),$$

where

$$\begin{aligned}L_r &= \sum_{k=0}^{r}\left(\begin{array}{c} r\\ k\end{array}\right) M_k^w(w)\widetilde{\mu}_{r-k}(t)\\ J_r &= \sum_{k=0}^{r+1}\left(\begin{array}{c} r+1\\ k\end{array}\right) M_k^w(w)\widetilde{\mu}_{r+1-k}(t).\end{aligned}$$

2.5. **Neural Network Operators Described Using Wavelets.** In 1989, Cybenko [13] gave an answer to the superposition problem on $C[a,b]$ with his famous density theorem, which states that every continuous function defined on $[a,b]$ can be approximated by a sequence constructed by a linear combination of sigmoidal functions. In other words, Cybenko confirmed that a neural network with solely one hidden-layer is capable of always approximating to a continuous function.

Based on the idea developed by Cybenko, the theory of the mathematical models of the neural network (NN) operators arise since 1992 with the pioneer work of Cardaliaguet and Euvrard [11], and then in the next years, they have been largely studied by several authors under different aspects. Especially in the last two decades, there are many new version of artificial neural networks has been introduced and widely studied.

In particular, in 1997 Anastassiou [5] pointed out and obtained that the compactly supported bell-shaped functions used in the Density Theorem of Cybenko and the Cardaliaguet and Euvrard (NN) operators can be obtained from sigmoidal functions used effectively in Artificial Neural Networks, serious relations have emerged between the Cybenko convergence theorem and the Theory of Approximations.

In this section, we shall recall some notation and background material of the theory of Neural Networks. We denote by $C[a,b], B[a,b]$ and $L_\infty(\mathbb{R})$ the sets of continuous, bounded and essentially bounded functions with their usual norms, respectively.

Definition 2.5 (**Centered bell-shaped function**). *A function $b : \mathbb{R} \to \mathbb{R}$ is said to be centered bell-shaped if b belongs to $L^1(\mathbb{R})$ and its integral is nonzero, if it is nondecreasing on $(-\infty, 0)$ and nonincreasing on $[0, +\infty)$.*

Definition 2.6 (**Sigmoidal Function**). *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfies*

$$\lim_{x\to-\infty} \sigma(x) = 0 \quad \textit{and} \quad \lim_{x\to\infty} \sigma(x) = 1,$$

then it is called sigmoidal function.

As an example of a sigmoidal and its corresponding bell-shaped function, we can consider the ramp function, which is very useful and important for the neural network.

Definition 2.7 (**Ramp Function**). *A ramp function is a special sigmoidal function defined as*

$$R(x) = \begin{cases} 0 & , \quad x \le -1/2 \\ x + 1/2 & , \quad -1/2 < x < 1/2 \\ 1 & , \quad x \ge 1/2 \end{cases} .$$

Clearly, a bell-shaped function can be define by using Sigmoidal (or Ramp) function, namely

$$b_\sigma(x) = \frac{\sigma(x+1) - \sigma(x-1)}{2},$$

and

$$b_R(x) = R(x + 1/2) - R(x - 1/2).$$

Moreover, $b_R : \mathbb{R} \to \mathbb{R}$ is a bell-shaped kernel function obtained by ramp function $R(x)$ that satisfies following assumptions:

b_R is a continuous function on $\mathbb{R}$,

$$b_R \in L^1(\mathbb{R}), \ \sum_{k=0}^{n} b_R(u-k) = 1 \text{ for every } u \in \mathbb{R},$$

and

$$O_R := \sup_{u\in\mathbb{R}} \sum_{k=0}^{n} b_R(u-k) < \infty, \tag{2.12}$$

where the convergence of the series (2.12) is uniform on each compact subintervals of $\mathbb{R}$. Some other sigmoidal functions are Gompertz function, Logistic function, Error function and Hyperbolic tangent functions, etc. Now, we will give some definitions of the Neural Network (NN) Operators.

Definition 2.8 (**Cardaliaguet and Euvrard (NN) Operators**). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function and $n \in \mathbb{N}^+$, the Cardaliaguet and Euvrard (NN) Operators are defined as:*

$$(F_n f)(x) = \sum_{k=-n^2}^{n^2} \frac{f\left(\frac{k}{n}\right)}{Bn^\alpha} b(n^{-\alpha}(nx-k),$$

where $0 < \alpha < 1$, b is a bell-shaped function with compact support $\subset [-T, T]$, and

$$B := \int_{-T}^{T} b(x)dx.$$

The Cardaliaguet and Euvrard (NN) Operators and its different modifications were intensively studied by Anastassiou, Spigler, Costarelli, Vinti. In [5], Anastassiou defined the Cardaliaguet and Euvrard (NN) Operators as follows;

Definition 2.9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function, $T > 0, n \in \mathbb{N}^+$ and $n \geq \max\{T + |x|, T^{-1/\alpha}\}$, the Cardaliaguet and Euvrard (NN) Operators are given by*

$$\left(F_n f\right)(x) = \sum_{k=\lceil nx - Tn^\alpha \rceil}^{\lfloor nx + Tn^\alpha \rfloor} \frac{f\left(\frac{k}{n}\right)}{Bn^\alpha} b(n^{-\alpha}(nx-k),$$

where again $0 < \alpha < 1, b$ is a bell-shaped function with compact support $\subset [-T, T]$, and

$$B := \int_{-T}^{T} b(x)dx.$$

Definition 2.10 (Neural Network (NN) Operators). *Let $f : [0,1] \to \mathbb{R}$ be a bounded function, and $n \in \mathbb{N}^+$. The positive linear neural network operators activated by the ramp function R, are defined as.*

$$\left(N_n f\right)(x) = \frac{\sum\limits_{k=0}^{n} f\left(\frac{k}{n}\right) b_R\left(nx-k\right)}{\sum\limits_{k=0}^{n} b_R\left(nx-k\right)}, \quad x \in [0,1], \tag{2.13}$$

where b_R is the bell-shaped function obtained by the ramp function R.

Moreover, we will examine and analyse various properties of the wavelet type extension of the neural network operators.

Definition 2.11 (2023, [26], **Wavelet type Neural Network (NN) Operators).** *Let $f : [0,1] \to \mathbb{R}$ be a bounded measurable function, $n \in \mathbb{N}^+$. and let $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfying (i)-(iii). Then the wavelet type Neural Network (NN) operators activated by the ramp function R, constructed by using the compactly supported Daubechies wavelets, are defined by:*

$$\left(WN_n f\right)(t) = n\frac{\sum\limits_{k=0}^{n} b_R\left(nt-k\right) \int\limits_{\mathbb{R}} f\left(x\right) \psi(nx-k)dx}{\sum\limits_{k=0}^{n} b_R\left(nt-k\right)}, \quad t \in [0,1], \tag{2.14}$$

where b_R is the bell-shaped function obtained by the ramp function R.

Remark 2.6 (2023, [26]). *If we choose the father wavelet ψ as the Haar scaling function, namely $\psi(x) = \chi_{[0,1]}(x)$, then clearly our wavelet type operators reduce to the Kantorovich form of the Neural Network (NN) operators. Indeed;*

$$(WN_n f)(t) = n\frac{\sum\limits_{k=0}^{n} b_R\left(nt-k\right) \int\limits_{\mathbb{R}} f\left(x\right) \psi(nx-k)dx}{\sum\limits_{k=0}^{n} b_R\left(nt-k\right)}$$

$$= \frac{\sum\limits_{k=0}^{n} b_R\left(nt-k\right) \int\limits_{0}^{1} f\left(\frac{u+k}{n}\right) \psi(u)dx}{\sum\limits_{k=0}^{n} b_R\left(nt-k\right)}.$$

This means that our operators constructed by wavelets are a natural extension of the Kantorovich type of the NN operators and also its Durrmeyer type operators.

Remark 2.7 (2023, [26]). *Moreover, the central moments of the wavelet type NN operators (2.14) are the same as of the classical NN operators (2.13). Indeed, we get*

$$\begin{aligned}(WN_n\,(x-t)^{\beta})(t) &= \frac{1}{n^{\beta}}\frac{\sum\limits_{k=0}^{n} b_R(nt-k)\,(k-nt)^{\beta}}{\sum\limits_{k=0}^{n} b_R(nt-k)} \\ &= (N_n\,(x-t)^{\beta})(t).\end{aligned}$$

Throughout this work, for every $u\in\mathbb{R}$ and for some $\beta>0$, we assume that the algebraic and discrete absolute moment of order β are given by, i.e.,

$$m_{\beta}(b_R) := \sup_{u\in\mathbb{R}}\sum_{k=0}^{n} b_R\,(u-k)\,(u-k)^{\beta}\,,$$

$$M_{\beta}(b_R) := \sup_{u\in\mathbb{R}}\sum_{k=0}^{n} b_R\,(u-k)\,|u-k|^{\beta}<\infty.$$

Theorem 2.21 (2023, [26]). *Let $f\in B[0,1]$ and let $\psi\in L_{\infty}(R)$ be a father wavelet satisfying (i)-(iii). Then the moments of wavelet type NN operators, constructed by using the compactly supported Daubechies wavelets (2.14) and the NN operators (2.13) are the same, namely*

$$(WN_n x^s)(t) = (N_n x^s)(t), \quad s=0,1,...,K$$

holds true.

Theorem 2.22 (2023, [26]). *Let $f\in B[0,1]$ be a measurable function and let $\psi\in L_{\infty}(\mathbb{R})$ be a father wavelet satisfying (i)-(iii). Then*

$$\lim_{n\to\infty}(WN_n f)(t_0) = f(t_0)$$

holds true at each point t_0 of continuity of f.

Theorem 2.23 (2023, [26]). *Let $f\in C[0,1]$ and let $\psi\in L_{\infty}(\mathbb{R})$ be a father wavelet satisfies (i)-(iii). Then*

$$|(WN_n f)\,(x)-f(x)| \le 4\omega\,(f;1/n)$$

holds true.

Corollary 2.4 (2023, [26]). *The same arguments of Theorem 2.22 apply to the case when $f\in C[0,1]$. In this case the convergence is uniform with respect to $x\in[0,1]$, and hence one has*

$$\lim_{n\to\infty}\|(WN_n f)-f\|_{C[0,1]} = 0.$$

Theorem 2.24 (2023, [26]). *Let $f\in C[0,1]$ and let $\psi\in L_{\infty}(\mathbb{R})$ be a father wavelet satisfying (i)-(iii). Then*

$$\lim_{n\to\infty}(WN_n f)\,(x) = f(x),$$

and

$$|(WN_n f)\,(x)-f(x)| \le (K+1)\,K_2\left(f;\frac{M_2(b_R)+\lambda^2 O_R+2\lambda M_1(b_R)}{n^2}\right),$$

where $K=\lambda\,\|\psi\|_{\infty}$ and $K_2(f;\delta)$ is the Peetre's K-functional.

Theorem 2.25 (2023, [26]). *Let $f \in C[0,1]$, $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfying (i)-(iii) and $\alpha \in (0,2)$ be fixed real number. Then*

$$\omega_2(f;t) = \mathcal{O}(t^\alpha) \Rightarrow |(WN_n f)(x) - f(x)| = \mathcal{O}(1/n)^\alpha$$

holds true.

2.6. **Graphical Representations.** Now, we will give some graphical examples for these approach, namely convergence to functions by means of wavelet type Neural Network operators $(WN_n f)(x)$. We note that in all the following Figures, the graph with the red line belongs to the target function.

Example 2.1. *Let $f(x) = x - x^2$, and take the activation function as a Ramp function for the neural network operators. We consider a special case of the wavelet type Neural Network operators $(WN_n f)(x)$, namely Kantorovich type Neural Network operators.Then one has for $n = 3, 5$ and for $n = 20$.*

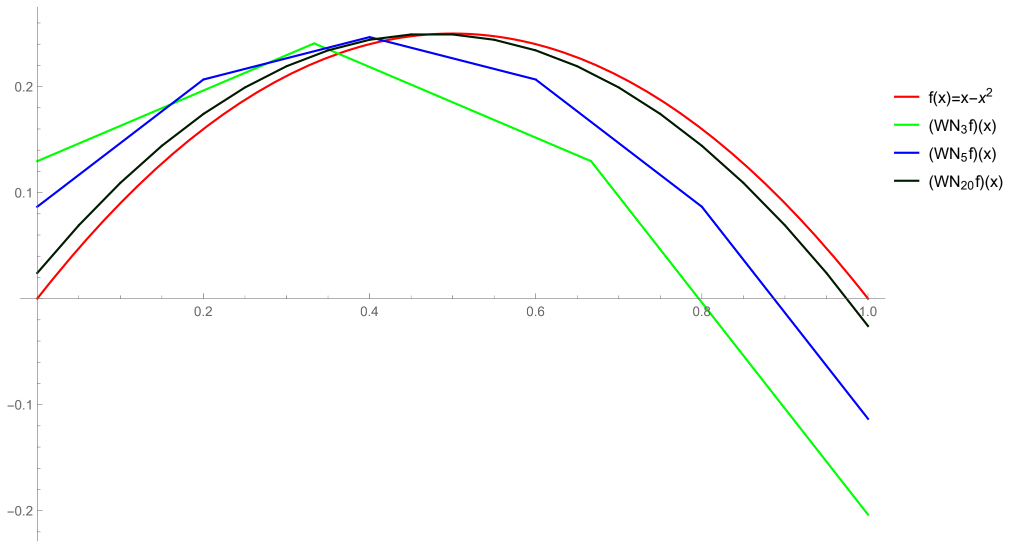


FIGURE 7. Approximation of $f(x) = x - x^2$ by Kantorovich type NN operator activated by Ramp function, for $n = 3, 5$ and $n = 20$.

Example 2.2. *Let $f(x) = x - x^2$, and take the activation function as a Ramp function for the neural network operators. We consider the wavelet type Neural Network operators $(WN_n f)(x)$ constructed by using Haar scaling function. Then one has for $n = 3, 6$ and for $n = 15$.*

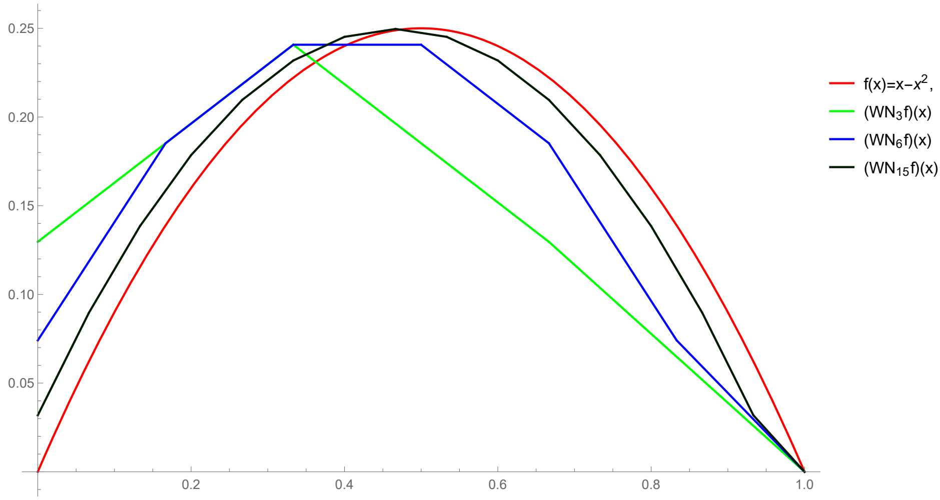


FIGURE 8. Approximation of $f(x) = x - x^2$ by Haar Wavelet type NN operator activated by Ramp function, for $n = 3, 6$ and $n = 15$.

Example 2.3. *Let $f(x) = x - x^2$, and take the activation function as a Ramp function for the neural network operators. We consider the wavelet type Neural Network operators $(WN_n f)(x)$ constructed by using Shannon wavelet function. Then one has for $n = 30, 60$ and for $n = 150$.*

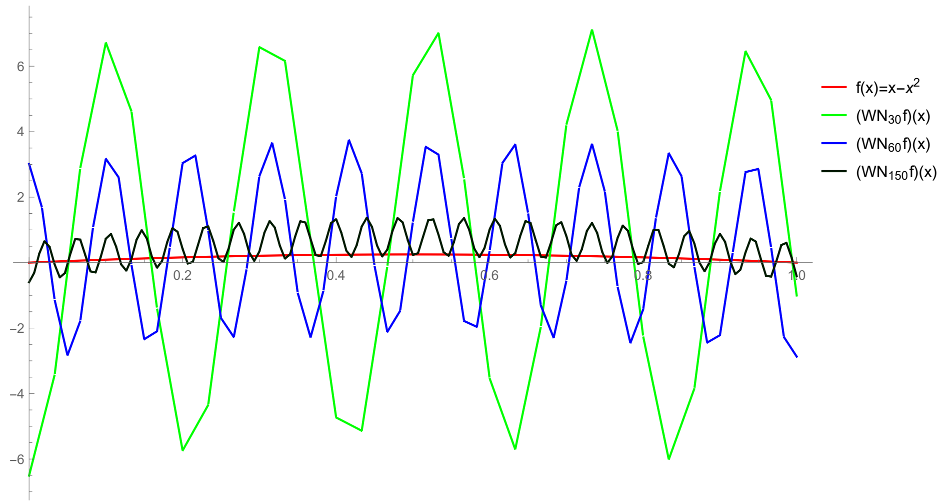


FIGURE 9. Approximation of $f(x) = x - x^2$ by Shannon Wavelet type NN operator activated by Ramp function, for $n = 30, 60$ and $n = 150$.

3. Conclusion and Future Work

In contrast to Fourier analysis, which provides information about frequencies but not their timing, wavelets offer insights into both the frequency content and the specific time intervals at which these frequencies occur. This dual capability makes wavelets particularly effective for analyzing dynamic signals, as they achieve higher resolution in both the time and frequency domains.

Furthermore, wavelets possess properties that enable their application to approximation problems in L_p spaces. Given their advantages for approximation tasks in these spaces, there is significant potential for leveraging wavelets in machine learning and neural network contexts. Future research will focus on adapting the insights and methodologies developed for wavelets to these emerging fields and theories.

Additionally, considering the effects and advantages of multivariate forms of operators in image processing and data analysis, the use of these multivariate forms of operators, such as sampling operators, which can be reconstructed with appropriate wavelets, can be identified as an open problem and an area for future research in image processing and data analysis. For the one dimensional case of the sampling operators reconstructed using wavelets please see the recent paper of the author [27].

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