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Research Article

Notes on a class of operators with the localized single-valued extension property

SALVATORE TRIOLO*

ABSTRACT. This article concerns the permanence of the single valued extension property at a point under suitable perturbations for an unbounded operator T on a particular integrity domain. While this property is, in general, not preserved under sums and products of commuting operators.

Keywords: Localized single-valued extension property, quasi-nilpotent part, analytic core, Kato decomposition and quasi-Fredholm operators, semi-Browder operators and Riesz operators.

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1. INTRODUCTION AND BASIC DEFINITIONS

In [3] the authors P. Aiena and V. Muller have studied the stability of the localized single-valued extension property under commuting perturbations for bounded operators. Also if the ideal environment to study this type of perturbations is the algebra of bounded operators, as Aiena and Muller did, in this paper the possibility of considering unbounded operators is studied.

We consider a similar problem for unbounded operators based on the recent new works by P. Aiena et al. [4] and [5]. In this work, we shall consider the version of this property for an $(T, D(T))$ closed linear operator in $\mathcal{H}$. This property, in the case of bounded operators defined on a Banach space in this paper we extend some of the results established in the bounded case to an unbounded linear operator. Recent studies in fact go in this direction [6, 8]. First we begin with some preliminary notations and remarks.

Let $(T, D(T))$ be a (possibly unbounded) closed linear operator in $\mathcal{H}$. Clearly we define $D(T^2) := \{x \in D(T) : Tx \in D(T)\}$ and, in general, for $n \geq 2$ we put $D(T^n) := \{x \in D(T^{n-1}) : T^{n-1}x \in D(T)\}$ and $T^n(x) = T(T^{n-1}x)$. It is worth mentioning that nothing guarantees, in general, that $D(T^k)$ does not reduce to the null subspace $\{0\}$, for some $k \in \mathbb{N}$. For this reason, powers of an unbounded operator could be of little use in many occasions. Throughout this paper if $\mathcal{D}$ is linear subspace of $\mathcal{H}$ a function $f : \Omega \to D$ is analytic if $f : \Omega \to \mathcal{H}$ is analytic and $f^n(x) \in D$ for every $x \in \Omega$, and $n \in \mathbb{N}$.

Let $(T, D(T))$ be a closed linear operator in $\mathcal{H}$. As usual, the spectrum of $(T, D(T))$ is defined as the set $\sigma(T) := \{\lambda \in \mathbb{C} : \lambda I - T\}$ is not a bijection of $D(T)$ onto $\mathcal{H}$. The set $\rho(T) = \mathbb{C} \setminus \sigma(T)$ is called the resolvent set of $(T, D(T))$, while the map $R(\lambda, T) : \rho(T) \ni \lambda \mapsto (\lambda I - T)^{-1}$ is called the resolvent of $(T, D(T))$. It is well known that, if T is a bounded everywhere defined operator, $\sigma(T)$ is a compact subset of the complex plane. The viceversa is not true: there exist

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*Corresponding author: Salvatore Triolo; salvatore.triolo@unipa.it

closed unbounded operators whose spectrum is a bounded subset of $\mathbb{C}$. Thus, the spectral radius of an unbounded operator can be finite. Let $(T, D(T))$ be a closed operator in $\mathcal{H}$.

(1) A point $\lambda \in \mathbb{C}$ is said to be in the local resolvent set of $x \in \mathcal{H}$, denoted by $\rho_T(x)$, if there exist an open neighborhood $\mathcal{U}$ of λ in $\mathbb{C}$ and an analytic function $f : \mathcal{U} \to D(T)$ which satisfies

$$(\lambda I - T)f(\lambda) = x \text{ for all } \lambda \in \mathcal{U}. \tag{1.1}$$

(2) The local spectrum $\sigma_T(x)$ of T at $x \in \mathcal{H}$ is the set defined by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$ and obviously $\sigma_T(x) \subseteq \sigma(T)$, and $\sigma_T(x)$ is a closed subset of $\mathbb{C}$.

Definition 1.1. *Let $(T, D(T)), D := D(T)$ be a closed linear operator in $\mathcal{H}$ such that $T^n(D) \subseteq D$. The hyperrange of T is the subspace*

$$T^\infty(D) := \bigcap_{n \in \mathbf{N}} T^n(D) =: \mathcal{R}^\infty(T).$$

Now, let us introduce two classical quantities associated with an operator. To every linear operator T on a vector space D there corresponds the two chains:

$$\{0\} = \ker T^0 \subseteq \ker T \subseteq \ker T^2 \cdots ,$$

and

$$D = T^0(D) \supseteq T(D) \supseteq T^2(D) \cdots .$$

The ascent of T is the smallest positive integer $p = p(T)$, whenever it exists, such that $\ker T^p = \ker T^{p+1}$. If such p does not exist, we let $p = +\infty$. Analogously, the descent of T is defined to be the smallest integer $q = q(T)$, whenever it exists, such that $T^{q+1}(D) = T^q(D)$. If such q does not exist, we let $q = +\infty$.

Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$. We denote by $\mathcal{L}(\mathcal{D})$ the set of all closable linear operators from $\mathcal{D}$ to $\mathcal{D}$ and $\mathcal{L}^\dagger(\mathcal{D})$ be the space consisting of all its elements which leave, together with their adjoints, the domain $\mathcal{D}$ invariant. Then $\mathcal{L}(\mathcal{D})$ is a algebra with respect to the usual operations and $\mathcal{L}^\dagger(\mathcal{D})$ is a subalgebra of $\mathcal{L}(\mathcal{D})$ (for the definitions and in general for the details can be found [6]).

Definition 1.2. *The operator $(T, D(T))$ is said to have the single valued extension property at $\lambda_o \in \mathbb{C}$ (abbreviated SVEP at λ_o), if for every open disc $\mathbf{D}_{\lambda_o}$ centered at λ_o the only analytic function $f : \mathbf{D}_{\lambda_o} \to D(T)$ which satisfies the equation*

$$(\lambda I - T)f(\lambda) = 0 \tag{1.2}$$

is the function $f \equiv 0$.
An unbounded linear operator $(T, D(T))$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

Following [2] if $(T, D(T))$ be closed linear operator in $\mathcal{H}$ for every subset Ω of $\mathbb{C}$, the analytic spectral subspace of T associated with Ω is the set

$$X_T(\Omega) := \{x \in \mathcal{H} : \sigma_T(x) \subseteq \Omega\}.$$

Remark 1.1. *If T is globally defined ($D(T) = \mathcal{H}$) and bounded then the SVEP may be easily characterized by means of the subspace $X_T(\emptyset)$ trough the equivalence of the following statements [7]:*

(i) T has the SVEP.
(ii) If $\sigma_T(x) = \emptyset$ then $x = 0$, i.e. $X_T(\emptyset) = \{0\}$.
(iii) $X_T(\emptyset)$ is closed.

Given a (possibly unbounded) linear operator $(T, D(T))$ and a closed set $F \subseteq \mathbb{C}$, let $\mathfrak{X}_T(F)$ consist of all $x \in \mathcal{H}$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to D(T)$ that satisfies

$$(\lambda I - T)f(\lambda) = x \text{ for all } \lambda \in \mathbb{C} \setminus F. \tag{1.3}$$

Clearly, the identity $X_T(F) = \mathfrak{X}_T(F)$ holds for all closed sets $F \subseteq \mathbb{C}$ whenever T has SVEP.

The following proposition generalizes partially the result of Remark 1.1.

Theorem 1.1. *Every closed linear operator $(T, D(T))$ such that $X_T(\emptyset) = \{0\}$ has the SVEP.*

Definition 1.3. *The quasi-nilpotent part of an operator $T \in \mathcal{L}(\mathcal{D})$ is the set*

$$H_0(T) := \{x \in D : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\},$$

while the analytic core of T is the set $K(T) := X_T(\mathbb{C} \setminus \{0\})$.

Let $\mathcal{N}^\infty(T) := \bigcup_{k=1}^{\infty} \ker T^k$. For every $n \in \mathbb{N}$, we have the increasing chain of range-type subspaces

$$X_T(\emptyset) \subseteq K(T) \subseteq \mathcal{R}^\infty(T) \subseteq R(T^n) \subseteq R(T).$$

This result will be one of our principal tools.

Theorem 1.2. *For every operator $T \in \mathcal{L}(\mathcal{D})$ and $\lambda \in \mathbb{C}$, the following assertions are equivalent:*

(i) T *has SVEP at* λ.

(ii) $\ker(\lambda I - T) \cap X_T(\emptyset) = \{0\}$.

(iii) $\mathcal{N}^\infty(\lambda I - T) \cap X_T(\emptyset) = \{0\}$.

2. Results

As it is well known, it is interesting to study the preservation of localized SVEP under certain perturbations. So let us try the problem again in a different setup.

Recall that an operator $Q \in \mathcal{L}(\mathcal{D})$ is said to be quasi-nilpotent if $\sigma(Q) = \{0\}$. A quasi-nilpotent operator on an infinite-dimensional Banach space cannot be onto, since $\sigma_s(Q) \neq \emptyset$. We put

$$\mathbf{Q}_i(\mathcal{D}) := \{T \in \mathcal{L}^\dagger(\mathcal{D}) : \text{there exists an injective quasi-nilpotent operator } Q \in \mathcal{L}^\dagger(\mathcal{D}) \text{ such that } TQ = QT\}.$$

Note that a nilpotent operator $N \neq 0$ cannot be injective, since if $\ker N = \{0\}$ and $N^\nu = 0$, then $X = \ker N^\nu = \{0\}$. We start with the following result that has a central role in this paper.

Following [2] if $(T, D(T))$ be closed linear operator in $\mathcal{H}$, we consider the following:

Question 1. *Is SVEP at a point is preserved under quasi-nilpotent commuting perturbations or even under quasi-nilpotent equivalence?*

Theorem 2.3. *Suppose that $T \in \mathbf{Q}_i(\mathcal{D})$, and let $\lambda \in \mathbb{C}$. Then T has SVEP at λ.*

Proof. Since all quasi-nilpotent operators share SVEP Q has SVEP at λ. By Theorem 1.2 the condition on Q entails that

$$\mathcal{N}^\infty(\lambda I - Q) \cap X_Q(\emptyset) = \{0\},$$

while if consider an arbitrary $x \in \ker(\lambda I - T)$. Then $(\lambda I - T)^k x = 0$ for $k = 1, \ldots, n$, so that the preceding identities imply that $(\lambda I - Q)^n x = 0$. Consequently, we obtain

$$\ker(\lambda I - T) \subseteq \ker(\lambda I - Q)^n \subseteq \mathcal{N}^\infty(\lambda I - Q),$$

and therefore

$$\ker(\lambda I - T) \cap X_T(\emptyset) \subseteq N^\infty(\lambda I - Q) \cap X_Q(\emptyset) = \{0\}.$$

Hence Theorem 1.2 guarantees that T has SVEP at λ. ∎

An operator $K \in \mathcal{L}^\dagger(\mathcal{D})$ is said to be algebraic if there exists a non-trivial complex polynomial h such that $h(K) = 0$. Examples of algebraic operators are idempotent operators and operators for which some power has finite-dimensional range. If $T \in L(X)$ has SVEP at a point λ, then it may be tempting to conjecture that $T + K$ has SVEP at λ for every algebraic operator K that commutes with T. However, this cannot be true in general, since SVEP for T at λ is equivalent to SVEP for $T - \lambda I$ at 0. Nevertheless, we obtain the following result.

Theorem 2.4. *Suppose that $T \in \mathbf{Q}_i(\mathcal{D})$, and $K \in \mathcal{L}^\dagger(\mathcal{D})$, and suppose that K is algebraic, and let h be a non-zero polynomial for which $h(K) = 0$. Then $T - K$ and $T + K$ are SVEP at 0.*

Proof. Following the reasoning used in [3] for bounded operators by the classical spectral mapping theorem $h(\sigma(K)) = \sigma(h(K)) = \{0\}$, so that $\sigma(K)$ is finite, say $\sigma(K) = \{\mu_1, \dots, \mu_n\}$. For $i = 1, \dots, n$, let $P_i \in \mathcal{L}^\dagger(\mathcal{D})$ denote the spectral projection associated with K and with the spectral set $\{\mu_i\}$, and let $Y_i := R(P_i)$. From standard spectral theory, also in this case, it is known that $P_1 + \dots + P_n = I$, that $Y_1, \dots, Y_n$ are closed linear subspaces of X which are each invariant under both K and T, and that $X = Y_1 \oplus \dots \oplus Y_n$. Moreover, for arbitrary $i = 1, \dots, n$, the two restrictions $K_i := K \mid Y_i$ and $T_i := T \mid Y_i$ commute, and we have $\sigma(K_i) = \{\mu_i\}$. Because $h(K_i) = h(K) \mid Y_i = 0$, we obtain

$$h(\{\mu_i\}) = h(\sigma(K_i)) = \sigma(h(K_i)) = \{0\}.$$

Hence we may factor h in the form

$$h(\mu) = (\mu - \mu_i)^{n_i} q_i(\mu) \quad \text{for all } \mu \in \mathbb{C},$$

where $n_i \in \mathbb{N}$ and q_i is a complex polynomial for which $q_i(\mu_i) \neq 0$. We conclude that

$$0 = h(K_i) = (K_i - \mu_i I)^{n_i} q_i(K_i),$$

where $q_i(K_i) \in \mathcal{L}^\dagger(\mathcal{D})(Y_i)$ is invertible in light of $\sigma(q_i(K_i)) = q_i(\sigma(K_i)) = \{q_i(\mu_i)\}$ and $q_i(\mu_i) \neq 0$. Therefore $(K_i - \mu_i I)^{n_i} = 0$ which show that the operator $N_i := K_i - \mu_i I$ is nilpotent. Now observe that

$$T_i - K_i = (T_i - \mu_i I) - (K_i - \mu_i I) = T_i - \mu_i I - N_i.$$

Because T has SVEP at μ_i, we know that $T - \mu_i I$ has SVEP at 0. Since this condition is inherited by restrictions to closed invariant subspaces, we conclude that $T_i - \mu_i I$ has SVEP at 0, and hence, by Theorem 2.3, also $T_i - K_i = T_i - \mu_i I - N_i$ has SVEP at 0 for all $i = 1, \dots, n$. By [1, Theorem 2.9], it then follows that

$$T - K = (T_1 - K_1) \oplus \dots \oplus (T_n - K_n)$$

has SVEP at 0, as desired. An application of the main result to the operator $-K$ and $T - \lambda I$ for arbitrary $\lambda \in \mathbb{C}$ then establishes the final claim. □

For arbitrary $T \in L(\mathcal{D})$, let $\alpha(T) := \dim \ker T$ and $\beta(T) :=$ codim $R(T)$. As usual,

$$\Phi_+(\mathcal{D}) := \{T \in L(\mathcal{D}) : \alpha(T) < \infty \text{ and } T(\mathcal{D}) \text{ is closed}\}$$

denotes the class of upper semi-Fredholm operators, while

$$\Phi_-(\mathcal{D}) := \{T \in L(\mathcal{D}) : \beta(T) < \infty\}$$

stands for the class of lower semi-Fredholm operators, and $\Phi_+(\mathcal{D}) \cup \Phi_-(\mathcal{D})$ is the class of semi-Fredholm operators on $\mathcal{D}$.

Also recall that an operator $T \in L(\mathcal{D})$ is said to be semi-regular if $R(T)$ is closed and $\mathcal{N}^\infty(T) \subseteq \mathcal{R}^\infty(T)$. More generally, T is said to admit a generalized Kato decomposition if there exists a pair (M, N) of T-invariant closed subspaces of X such that $\mathcal{D} = M \oplus N$, $T|M$ is semi-regular, and $T|N$ quasi-nilpotent. If, in addition, N is of finite dimension, then T is said to be essentially semi-regular, see [1, Chapter 1] for details and properties of such operators.

Theorem 2.5. *Suppose that $T \in \mathbf{Q}_i(\mathcal{D})$, $\lambda \in \mathbb{C}$ and $\lambda I - T$ either admits a generalized Kato decomposition or is quasi-Fredholm. Then $T + Q$ has SVEP at λ.*

Proof. Under either of the two conditions on $\lambda I - T$, it is known that SVEP for T at λ is equivalent to the condition $X_T(\emptyset) \subseteq H_0(\lambda I - T) \cap K(\lambda I - T) = \{0\}$. Then establishes the final claim. $\square$

3. TWO EXAMPLES

In two recent works, [4] and [8], the following integrity domains were introduced and studied. In this example we give a family of linear operators $T_{v_k} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ with commutes with an injective quasi-nilpotent operator. Let Q_2 be the matrix:

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix},$$

where for $i, j = 1, 2, \dots,$

$$q_{ij} = \begin{cases} 0 & \text{if } i < j+1 \\ \frac{1}{j} & \text{if } i = j+1 \\ 0 & \text{if } i > j+1 \end{cases}.$$

Evidently,

$$Q(x_1, x_2, x_3, \dots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots) \quad \text{for all } (x_1, x_2, x_3, \dots) \in \ell^2(\mathbb{N}),$$

so Q is injective and quasi-nilpotent. If Q is a weighted shift with non zero weights which tend to zero, then Q is a one-to-one quasi-nilpotent operator. Put $\mathbf{e}^k := (0, \dots, 1, 0, \dots)$, with $\mathbf{e}_i^k = \delta_{ik}$, let $T_{\mathbf{e}^k}$ be the following matrix:

$$T_{\mathbf{e}^k} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{k} & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\binom{k+1}{2}} & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\binom{k+2}{3}} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix},$$

where the generic element a_{ij} is given by:

$$a_{ij} = \begin{cases} 0 & \text{if } i < j+k-1 \\ \frac{1}{\binom{i-1}{i-k}} & \text{if } i = j+k-1 \\ 0 & \text{if } i > j+k-1 \end{cases}.$$

Then, by arbitrarily choosing $k \geq 2$ and $\lambda_k \in \mathbb{R}$, we obtain a family of matrices $T_{\mathbf{v}_k} := \lambda_k T_{\mathbf{e}^k}$ that commute with Q_2: $Q_2\, T_{\mathbf{v}_k} = T_{\mathbf{v}_k}\, Q_2$. Moreover it is easy to verify that $\forall k \geq 2,\ \lambda_k \in \mathbb{R},\ T_{\mathbf{v}_k}$ is a bounded linear operator; clearly $Q_2 = T_{\mathbf{e}^2}$. More generally, if we define $Q_n := T_{\mathbf{e}^n}$, then, $\forall n \geq 2$, Q_n is quasi-nilpotent, injective, with the property that Q_n commute with $T_{\mathbf{v}_k}$, $\forall k \geq 2,\ \lambda_k \in \mathbb{R}:\ Q_n\, T_{\mathbf{v}_k} = T_{\mathbf{v}_k}\, Q_n$. We finally observe that the linear span $D := \langle T_{\mathbf{e}^1}, \ldots, T_{\mathbf{e}^k}, \ldots \rangle$ is an integral domain. If $A \in D$, then A is a matrix of the following type:

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \lambda_2 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \lambda_3 & \frac{\lambda_2}{2} & \lambda_1 & 0 & 0 & 0 & 0 & \ldots \\ \lambda_4 & \frac{\lambda_3}{3} & \frac{\lambda_2}{3} & \lambda_1 & 0 & 0 & 0 & \ldots \\ \lambda_5 & \frac{\lambda_4}{4} & \frac{\lambda_3}{6} & \frac{\lambda_2}{4} & \lambda_1 & 0 & 0 & \ldots \\ \lambda_6 & \frac{\lambda_5}{5} & \frac{\lambda_4}{10} & \frac{\lambda_3}{10} & \frac{\lambda_2}{5} & \lambda_1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix},$$

where $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots$, and the generic element a_{ij} is

$$a_{ij} = \begin{cases} 0 & \text{if } i < j \\ \frac{\lambda_{i-j+1}}{\binom{i-1}{j-1}} & \text{if } i \geq j \end{cases}.$$

Let $\mathcal{M} := \{A \in D : \sup_{i,j} |a_{ij}| < \infty\}$. Then for every $A \in \mathcal{M}$ there exists $c_A \in \mathbb{R}$, such that $|a_{ij}| \leq c_A,\ \forall i, j$. Let l^∞ be the Banach space of bounded sequences, $x \in l^\infty$ and let $c_x := \sup_i |x_i|$. Let us consider now $y = Ax$. Then

$$y_k = \sum_{j=1}^{k} \frac{\lambda_j x_{k+1-j}}{\binom{k-1}{j-1}},$$

so

$$\forall k \geq 1,\ |y_k| \leq |c_A c_x| \sum_{j=1}^{k} \frac{1}{\binom{k-1}{j-1}}.$$

Since

$$\lim_{k\to\infty} \sum_{j=1}^{k} \frac{1}{\binom{k-1}{j-1}} =: S < +\infty,$$

then $y = Ax \in l^\infty$, hence $A : l^\infty \to l^\infty$, and so A is bounded. But in general in this example $A : l^2 \to l^2$ is a unbounded operator with commutes with an injective quasi-nilpotent operator Q_2.

Example 3.1. *Let $(T, D(T))$ be the operator defined by*

$$D(T) := \{u \in L^2([0,1]) : u(x) = \int_0^x v(y)dy;\ u(1) = 0,\ v \in L^2([0,1])\}$$

$$(Tu)(x) = v(x).$$

It is easy to check that for every $u \in L^2([0,1])$ we have $\sigma_T(u) = \sigma(T) = \mathbb{C}$ and T has SVEP.

Let $(S, D(S))$ be the operator

$$D(S) := \{u \in L^2([0,1]) : u(x) = u(0) + \int_0^x v(y)dy;\ v \in L^2([0,1])\}$$

$$(Su)(x) = v(x).$$

It is easy to check that every $\lambda \in \mathbb{C}$ is an eigenvalue of S, so that the operator S does not have SVEP in every $\lambda \in \mathbb{C}$. Since $S = T^*$, this shows how deeply different the behavior of T and T^* can be with respect to SVEP.

Let $(A, D(A))$ be the operator

$$D(A) := \{u \in L^2([0,1]) : u(x) = u(0) + \int_0^x v(y)dy;\ u(1) = u(0)\, v \in L^2([0,1])\}$$

$$(Au)(x) = v(x).$$

The operator A is self-adjoint; thus has SVEP and for every $u \in L^2([0,1])$ we have $\sigma_A(u) = \sigma(A) = \{2k\pi i;\ k \in \mathbb{Z}\}$.

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Salvatore Triolo
Università di Palermo
Dipartimento di Ingegneria
90128, Palermo, Italy
Email address: salvatore.triolo@unipa.it