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Research Article

On a problem inspired by Descartes' rule of signs

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ABSTRACT. We study real univariate polynomials with non-zero coefficients and with all roots real, out of which exactly two positive. The sequence of coefficients of such a polynomial begins with m positive coefficients followed by n negative followed by q positive coefficients. We consider the sequence of moduli of their roots on the positive real half-axis; all moduli are supposed distinct. We mark in this sequence the positions of the moduli of the two positive roots. For $m = n = 2$, $n = q = 2$ and $m = q = 2$, we give the exhaustive answer to the question which the positions of the two moduli of positive roots can be.

Keywords: Real polynomial in one variable, hyperbolic polynomial, sign pattern, Descartes' rule of signs.

2020 Mathematics Subject Classification: 26C10, 30C15.

1. INTRODUCTION

The present paper treats a problem inspired by Descartes' rule of signs. The latter says that given a real univariate polynomial $Q := \sum_{j=0}^{d} a_j x^j$, $a_d \neq 0$, the number r_+ of its positive roots (counted with multiplicity) is majorized by the number $\tilde{c}$ of the sign changes in the sequence S of its coefficients and the difference $\tilde{c} - r_+$ is even. About Descartes' rule of signs see [1, 2, 3, 4, 6, 8, 9, 18, 19].

We are interested in the case when all coefficients a_j are non-zero and the polynomial Q is hyperbolic, i.e. all its roots are real. In this case one has $\tilde{c} = r_+$ and $\tilde{p} = r_-$, where $\tilde{p}$ is the number of sign preservations in the sequence S and r_- is the number of negative roots of Q. Clearly $\tilde{c} + \tilde{p} = r_+ + r_- = d$.

Definition 1.1. *A sign pattern is a vector whose components equal $+$ or $-$. The polynomial Q is said to define the sign pattern $\sigma(Q) := (\mathrm{sgn}(a_d), \mathrm{sgn}(a_{d-1}), \ldots, \mathrm{sgn}(a_0))$. We focus mainly on monic polynomials in which case sign patterns begin with a $+$.*

Consider the moduli of the roots of a hyperbolic polynomial as a sequence of d points on the positive half-axis. We study the generic case when all these moduli are distinct. One can mark in this sequence the $\tilde{p}$ positions of the moduli of negative and the $\tilde{c}$ positions of moduli of positive roots. This defines the *order of moduli* whose definition and notation should be clear from the following example:

Example 1.1. *If for the positive roots $\alpha_1 < \alpha_2 < \alpha_3$ and the moduli of the negative roots $|-\gamma_1| < \cdots < |-\gamma_4|$ of a degree 7 hyperbolic polynomial, one has*

$$\alpha_1 < |-\gamma_1| < |-\gamma_2| < \alpha_2 < |-\gamma_3| < \alpha_3 < |-\gamma_4|,$$

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then these moduli define the order $PNNPNPN$ (the letters P and N refer to the relative positions of the moduli of positive and negative roots).

Definition 1.2. *A couple (sign pattern, order of moduli) is compatible with Descartes' rule of signs if the number of letters P (resp. N) in the order equals $\tilde{c}$ (resp. $\tilde{p}$). In what follows, we consider only couples compatible with Descartes' rule of signs.*

We study the following problem:

Problem 1.1. *Consider the class of hyperbolic polynomials defining one and the same sign pattern σ. What are the possible orders of moduli for the polynomials of this class?*

The problem has been completely resolved for $\tilde{c} = 0$ and 1, see [11]. The results of this paper concern the case $\tilde{c} = 2$. We use the following notation:

Notation 1.1.

(1) *For $\tilde{c} = 1$, (resp. $\tilde{c} = 2$), we denote by $\Sigma_{m,n}$, $m + n = d + 1$, (resp. $\Sigma_{m,n,q}$, $m + n + q = d + 1$) the sign pattern consisting of m signs $+$ followed by n signs $-$ (resp. of m signs $+$ followed by n signs $-$ followed by q signs $+$).*

(2) *For a polynomial Q with $\sigma(Q) = \Sigma_{m,n}$, we denote by α its positive and by $\gamma_1 < \cdots < \gamma_{d-1}$ the moduli of its negative roots. If $\gamma_u < \alpha < \gamma_{u+1}$ (we set $\gamma_0 := 0$ and $\gamma_d := +\infty$), then we denote the given couple (sign pattern, order of moduli) by $(\Sigma_{m,n}, (u, v))$, $v = d - 1 - u$.*

(3) *For a polynomial Q with $\sigma(Q) = \Sigma_{m,n,q}$, we denote by $\beta < \alpha$ its positive and by $\gamma_1 < \cdots < \gamma_{d-2}$ the moduli of its negative roots. If $\gamma_u < \beta < \gamma_{u+1}$ and $\gamma_{u+v} < \alpha < \gamma_{u+v+1}$, $v \geq 0$ (we set $\gamma_0 := 0$ and $\gamma_{d-1} := +\infty$), then we denote the given couple (sign pattern, order of moduli) by $(\Sigma_{m,n,q}, (u, v, w))$, $w = d - 2 - u - v$.*

Definition 1.3. *If for a given sign pattern there exists a hyperbolic polynomial defining this sign pattern, then we say that the polynomial realizes the sign pattern. If, in addition, the roots of the polynomial define a given order of moduli, then we say that the polynomial realizes the given couple (sign pattern, order of moduli) or that the order of moduli is realizable with the given sign pattern.*

We can now reformulate Problem 1.1:

Problem 1.2. *For a given degree d, which couples (sign pattern, order of moduli) are realizable?*

Definition 1.4.

(1) *For a given degree d, we define the following two commuting involutions acting on the set of couples:*

$$i_m \,:\, Q(x) \mapsto (-1)^d Q(-x) \quad \text{and} \quad i_r \,:\, Q(x) \mapsto x^d Q(1/x)/Q(0).$$

The factors $(-1)^d$ and $1/Q(0)$ are introduced to preserve the set of monic polynomials. The involution i_r reads orders, sign patterns and polynomials (modulo the factor $1/Q(0)$) from the right while preserving the quantities $\tilde{c}$ and $\tilde{p}$. In particular, $i_r((\Sigma_{m,n}, (u, v))) = (\Sigma_{n,m}, (v, u))$ and $i_r((\Sigma_{m,n,q}, (u, v, w))) = (\Sigma_{q,n,m}, (w, v, u))$. The involution i_m changes the signs of the odd (resp. even) monomials for d even (resp. for d odd). It exchanges the letters P and N in the order of moduli and the quantities $\tilde{c}$ and $\tilde{p}$, therefore when answering Problem 1.2 it suffices to study the cases with $\tilde{c} \leq \tilde{p}$.

(2) *The orbits of couples under the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action are of length 4 or 2. Orbits of length 2 can occur only for sign patterns σ such that $\sigma = i_r(\sigma)$ or $\sigma = i_r i_m(\sigma)$; $\sigma = i_m(\sigma)$ is impossible. One can consider orbits also only of sign patterns or of orders of moduli. All couples of a given orbit are simultaneously (non)-realizable.*

Remarks 1.

(1) *Problem 1.2 is completely resolved for* $d \leq 6$, *see* [7] *and* [15]. *For* $\tilde{c} = 2$, *it is settled for* $n = 1$, *see* [11, Theorem 5], *and for* $q = 1$ *(hence for* $m = 1$ *as well), see* [17].

(2) *Each sign pattern is realizable with its canonical order of moduli, see* [12, Definition 2 and Proposition 1]. *For the sign pattern* $\Sigma_{m,n,q}$, *the couple with the corresponding canonical order is* $(\Sigma_{m,n,q}, (q-1, n-1, m-1))$. *There exist sign patterns (called also canonical) which are realizable only with their corresponding canonical orders, see* [13, Theorem 7]. *Among the sign patterns of the form* $\Sigma_{m,n,q}$, *canonical are only* $\Sigma_{1,n,1}$ *and* $\Sigma_{m,1,q}$. *The relative part of canonical sign patterns within the set of all sign patterns of a given degree* d, *tends to* 0 *as* d *tends to* ∞, *see* [13, Definition 9 and Proposition 10].

(3) *There exist also orders of moduli (called rigid) realizable with a single sign pattern, see* [14, Definition 6, Notation 7 and Theorem 8].

The first result of the present paper about couples $(\Sigma_{m,n,q}, (u, v, w))$ reads:

Theorem 1.1.

(1) *Suppose that* $d \geq 7$ *and* $n = q = 2$. *Then realizable can be only couples with* $w \geq m - 3$. *For each* $d \geq 7$ *fixed, there are* 15 *triples* (u, v, w) *satisfying the latter condition.*

(2) *For* $d \geq 6$, *the* 10 *triples* (u, v, w) *with* $u + v \leq 3$ *and the triple* $(0, 4, m-3)$, *are realizable with the sign pattern* $\Sigma_{m,2,2}$, $d = m + 3$.

(3) *For* $d \geq 6$, *the triples* $(4, 0, m-3)$, $(3, 1, m-3)$, $(2, 2, m-3)$ *and* $(1, 3, m-3)$ *are not realizable with the sign pattern* $\Sigma_{m,2,2}$.

The theorem is proved in Section 2. It can be automatically reformulated for the case $m = n = 2$ using the involution i_r, see Definition 1.4, and for certain couples with $\tilde{p} = 2$ with the help of the involution i_m. Our second result is formulated as follows:

Theorem 1.2.

(1) *For* $n \geq 4$, *all couples* $(\Sigma_{2,n,2}, (u, v, w))$ *with* $u \leq 2$ *and* $w \leq 2$ *are realizable.*

(2) *For* $n \geq 4$, *all couples* $(\Sigma_{2,n,2}, (u, v, w))$ *with* $u \geq 3$ *or* $w \geq 3$ *are not realizable.*

The theorem is proved in Section 3.

2. Proof of Theorem 1.1

Proof.

Part (1) We set $\alpha := 1$ which can be obtained by a linear change of the variable x. The set of monic hyperbolic polynomials defining the sign pattern $\Sigma_{d-3,2,2}$ is open and connected (see [16, Theorem 2]) and there exists a hyperbolic polynomial with this sign pattern and with $w = m - 1$, see part (2) of Remarks 1. Hence if there exists such a polynomial with $w \leq m - 4$, then for this sign pattern, there exists a hyperbolic polynomial which is of the form

$$\begin{aligned} Q :=&(x^2-1)(x-\beta)(x^{d-3}+e_1x^{d-4}+\cdots+e_{d-3}) \\ =&(x^2-1)(x^{d-2}+(e_1-\beta)x^{d-3}+(e_2-\beta e_1)x^{d-4}+(e_3-\beta e_2)x^{d-5}+\cdots \\ &+(e_{d-5}-\beta e_{d-6})x^3+(e_{d-4}-\beta e_{d-5})x^2+(e_{d-3}-\beta e_{d-4})x-\beta e_{d-3}). \end{aligned}$$

Hence one of the moduli of negative roots of the polynomial Q equals 1 and we suppose that exactly four of its moduli of negative roots are smaller than 1. The second factor in the right-hand side defines the sign pattern $\Sigma_{d-3,2}$. Indeed, this factor has exactly one positive root, so its sign pattern is of the form $\Sigma_{d-1-\nu,\nu}$, $0 < \nu < d-1$. Then the coefficient of $x^{\nu+1}$ of Q is negative, so $\nu+1\leq 3$, i.e. $\nu\leq 2$. For $\nu=1$, the coefficient of x is negative, so $\nu=2$. Thus by setting $Q:=\sum_{j=0}^{d}c_jx^j$, one obtains the conditions

$$\begin{aligned}(2.1)\qquad & e_{d-4}>\beta e_{d-5}, & e_{d-3}<\beta e_{d-4},\\ & c_4:=e_{d-4}-\beta e_{d-5}-e_{d-6}+\beta e_{d-7}>0, & c_1:=-e_{d-3}+\beta e_{d-4}>0.\end{aligned}$$

The fourth of conditions (2.1) and the inequality

$$c_3:=(e_{d-3}-\beta e_{d-4})-(e_{d-5}-\beta e_{d-6})<0$$

are corollaries of the second of conditions (2.1); for $c_3<0$, one has to use also $e_{d-5}-\beta e_{d-6}>0$ which results from the sign pattern $\Sigma_{d-3,2}$. We denote by

$$r_1>\cdots>r_{d-3}$$

the moduli of negative roots different from -1. Hence we suppose that $r_{d-7}>1>r_{d-6}$. We set

$$\begin{aligned}E_k &:= \sum_{1\leq i_1<\cdots<i_k\leq d-7} r_{i_1}\cdots r_{i_k}, && 1\leq k\leq d-7 \ \text{ and}\\ e'_s &:= \sum_{d-6\leq p_1<\cdots<p_s\leq d-3} r_{p_1}\cdots r_{p_s}, && 1\leq s\leq 4.\end{aligned}$$

We set $E_k:=0$ for $k\leq 0$ and $k\geq d-6$, and $e'_s:=0$ for $s\leq 0$ and $s\geq 5$. Using this notation, one can write

$$e_{d-6}=E_{d-7}e'_1+E_{d-8}e'_2+E_{d-9}e'_3+E_{d-10}e'_4$$

and

$$e_{d-4}=E_{d-7}e'_3+E_{d-8}e'_4.$$

It is clear that $e'_1>e'_3$, because these are elementary symmetric polynomials having the same number (namely 4) of terms and their arguments are in the interval $(0,1)$. It is also evident that $e'_2>e'_4$ (their numbers of terms are 6 and 1), $e'_3>0$ and $e'_4>0$. Therefore $e_{d-6}>e_{d-4}$. If $e_{d-5}\geq e_{d-7}$, then (see (2.1)) $c_4<0$ which contradicts the sign pattern of Q. So one has $e_{d-5}<e_{d-7}$ and conditions (2.1) imply

$$e_{d-4}/e_{d-5}>\beta>(e_{d-6}-e_{d-4})/(e_{d-7}-e_{d-5}),\quad \text{i. e.}\quad e_{d-4}e_{d-7}>e_{d-5}e_{d-6}.$$

The left and right symmetric polynomials have

$$h_1:=\binom{d}{d-4}\binom{d}{d-7}\quad\text{and}\quad h_2:=\binom{d}{d-5}\binom{d}{d-6}$$

terms respectively, where

$$h_2/h_1=h_*:=7(d-4)/5(d-6)>1,$$

i.e. $e_{d-5}e_{d-6}$ has more terms than $e_{d-4}e_{d-7}$. We show that one has $e_{d-4}e_{d-7}<e_{d-5}e_{d-6}$ which contradiction implies that the inequality $w\leq m-4$ is impossible. We set $S_j:=e_j/\binom{d}{j}$ and then use Newton's inequalities $S_j^2\geq S_{j-1}S_{j+1}$ for $j=d-5$ and $j=d-6$. Thus

$$S_{d-5}^2S_{d-6}^2\ \geq\ S_{d-4}S_{d-6}S_{d-5}S_{d-7}\,,\quad \text{i.e.}\quad S_{d-5}S_{d-6}\ \geq\ S_{d-4}S_{d-7},\quad \text{so}$$
$$e_{d-5}e_{d-6}\ \geq\ h_*e_{d-4}e_{d-7}\ >\ e_{d-4}e_{d-7}.$$

The triples (u,v,w) for which $w\geq m-3$, are all the 15 triples with $0\leq u+v\leq 4$.

Part (2) For $d = 6$, the proof can be found in [15]. Suppose that $d \geq 6$ and that the degree d polynomial Q realizes one of the 11 triples (u, v, w) of part (2) of the theorem. Then for $\varepsilon > 0$ small enough, the polynomial $(1 + \varepsilon x)Q$ realizes the triple $(u, v, w + 1)$ for degree $d + 1$.

Part (3) Suppose that the polynomial $\tilde{Q}$ realizes one of the triples $(4, 0, m - 3)$, $(3, 1, m - 3)$, $(2, 2, m - 3)$ or $(1, 3, m - 3)$ with the sign pattern $\Sigma_{m,2,2}$. Then $\gamma_1 < \beta < 1 = \alpha$. Set $A := (x - \beta)(x + \gamma_1)$ and

$$\sum_{k=0}^{d} \tilde{q}_k x^k =: \tilde{Q} := AU \ , \quad \sum_{i=0}^{d-2} u_i x^i =: U := (x - 1) \prod_{j=2}^{d-2} (x + \gamma_j).$$

The sign pattern of the product A is $(+, -, -)$. The one of U is of the form $\Sigma_{\mu,\nu}$. As $\tilde{q}_{\nu+1} = -\beta\gamma_1 u_{\nu+1} - (\beta - \gamma_1)u_\nu + u_{\nu-1} < 0$, one must have $\nu + 1 \leq 3$, i.e. $\nu = 1$ or 2. For both cases Theorem 1 of [11] states that the polynomial U has ≤ 2 negative roots with moduli smaller than 1. However one obtains that $\gamma_i \leq 1$ for $i = 2$, 3 and 4. One perturbs then these roots to obtain $\gamma_i < 1$ (without changing the signs of the coefficients of U) which brings a contradiction with [11, Theorem 1].

□

3. Proof of Theorem 1.2

3.1. **Proof of part (1).** In the proof of part (1), we use concatenation of sign patterns and of couples. The following lemma follows directly from [5, Lemma 14].

Lemma 3.1. *Suppose that for the monic polynomials P_1 and P_2 of degrees d_1 and d_2, one has $\sigma(P_i) = (+, \sigma_i)$, $i = 1, 2$, where σ_i denote what remains of the sign patterns when the initial sign + is deleted. We set $P^\dagger := \varepsilon^{d_2} P_1(x)P_2(x/\varepsilon)$. Then for $\varepsilon > 0$ small enough,*

$$\sigma(P^\dagger) := \begin{cases} (+, \sigma_1, \sigma_2) & \text{if the last position of } \sigma_1 \text{ is } + \\ (+, \sigma_1, -\sigma_2) & \text{if the last position of } \sigma_1 \text{ is } - \end{cases}.$$

Here $-\sigma_2$ is obtained from σ_2 by changing each + by − and vice versa.

Remark 3.1. *We use the symbol $*$ to denote concatenation of couples or of sign patterns. We denote by $\Sigma_{m_1,\ldots,m_s}$ the sign pattern beginning with m_1 signs + followed by m_2 signs − followed by m_3 signs + etc., so one can write*

$$\Sigma_{m_1,\ldots,m_{s-1},m_s+n_1-1,n_2,\ldots,n_\ell} = \Sigma_{m_1,\ldots,m_s} * \Sigma_{n_1,\ldots,n_\ell}.$$

When necessary we use more than two consecutive concatenations. If ε is small enough, the moduli of all roots of $P_2(x/\varepsilon)$ are smaller than the moduli of all roots of $P_1(x)$ which allows to deduce the order of the moduli of roots of $P^\dagger$.

To prove part (1), one takes into account the fact that the sign pattern $\Sigma_{2,2}$ is realizable with each of the orders $(0, 2)$, $(1, 1)$ and $(2, 0)$, see [11, Part (3) of Example 2]. Hence the cases $(\Sigma_{2,n,2}, (u, v, w))$ with $u \leq 2$, $w \leq 2$ are realizable by the triple concatenation

$$(\Sigma_{2,2}, (u, 2 - u)) * (\Sigma_{n-2}, (n - 3)) * (\Sigma_{2,2}, (2 - w, w)).$$

The concatenation factor in the middle is realizable by any hyperbolic polynomial with all coefficients positive (and with $n - 3$ negative roots).

3.2. **Plan of the proof of part (2).** We deduce part (2) from two propositions:

Proposition 3.1. *Suppose that $n \geq 4$. Then:*

(1) All couples $(\Sigma_{2,n,2}, (u, v, w))$ with either $w \geq 5$ or with $w = 4$ and $v \geq 1$, are non-realizable.
(2) All couples $(\Sigma_{2,n,2}, (0, n-2, 3))$ are non-realizable.

The proposition is proved in Subsection 3.3.

Proposition 3.2. *For $n \geq 4$, the following couples are not realizable:*

(1) $(\Sigma_{2,n,2}, (u, v, 4))$, $u + v = n - 3$
(2) $(\Sigma_{2,n,2}, (u, v, 3))$, $u + v = n - 2$, $u > 0$.

Part (2) of Proposition 3.1 and part (2) of Proposition 3.2 settle the case $w = 3$ while the first parts of these propositions resolve the case $w \geq 4$. Proposition 3.2 is proved in Subsection 3.7. In its proof three other propositions are used which are formulated below and proved in Subsections 3.4, 3.5 and 3.6 respectively.

Proposition 3.3. *The couple $(\Sigma_{2,4,2}, (1, 0, 4))$ is not realizable.*

Proposition 3.4. *The couple $(\Sigma_{2,4,2}, (2, 0, 3))$ is not realizable.*

Proposition 3.5. *The couple $(\Sigma_{2,4,2}, (1, 1, 3))$ is not realizable.*

3.3. **Proof of Proposition 3.1.**

Part (1) Denote by $\beta < \alpha$ the positive and by $\gamma_1 < \cdots < \gamma_{n+1}$ the moduli of the negative roots of a hyperbolic polynomial $Q := \sum_{j=0}^{d} q_j x^j$ supposed to realize one of the mentioned couples. Denote by e_j the elementary symmetric polynomials of the quantities γ_i. We show that $q_{d-2} > 0$ which means that the sign pattern of Q is not $\Sigma_{2,n,2}$. Clearly

$$q_{d-2} = \alpha\beta - (\alpha + \beta)e_1 + e_2.$$

Suppose that $w \geq 5$. Recall that $u + v + w = d - 2 = n + 1$. Set $\ell := n - w + 2 = u + v + 1$. Then $\gamma_{\ell-1} < \alpha < \gamma_\ell$. For $S := \sum_{j=\ell}^{n+1} \gamma_j$, one has

$$(\alpha + \beta)S < \sum_{\ell \leq i < j \leq n+1} \gamma_i \gamma_j. \tag{3.2}$$

Indeed, the left-hand side contains $2w$ products by two while the right-hand side contains $w(w-1)/2 \geq 2w$ such products. Besides, for each k, $\ell \leq k \leq n+1$, it is true that $(S - \gamma_k)\alpha < (S - \gamma_k)\gamma_k$ and $(S - \gamma_k)\beta < (S - \gamma_k)\gamma_k$. Summing up these inequalities yields

$$(w-1)(\alpha + \beta)S < 2 \sum_{\ell \leq i < j \leq n+1} \gamma_i \gamma_j. \tag{3.3}$$

For each $\nu < \ell$, it is true that $(\alpha + \beta)\gamma_\nu < (\gamma_n + \gamma_{n+1})\gamma_\nu$, so

$$(\alpha + \beta) \sum_{j=1}^{\ell-1} \gamma_j < (\gamma_n + \gamma_{n+1}) \sum_{j=1}^{\ell-1} \gamma_j. \tag{3.4}$$

Equations (3.3) and (3.4) imply

$$(\alpha + \beta)e_1 < (2/(w-1)) \sum_{\ell \leq i < j \leq n+1} \gamma_i \gamma_j + (\gamma_n + \gamma_{n+1}) \sum_{j=1}^{\ell-1} \gamma_j < e_2$$

from which $q_{d-2} > 0$ follows. Suppose that $w = 4$ and $v \geq 1$. For $\mu \leq n-4$, it is true that

$$(-\alpha - \beta + \gamma_n + \gamma_{n+1})\gamma_\mu > 0.$$

On the other hand, as $\beta < \gamma_{n-3} < \alpha < \gamma_{n-2}$, one has

$$\begin{aligned}(-\beta + \gamma_{n-3})(\gamma_{n-2} + \gamma_{n-1} + \gamma_n + \gamma_{n+1}) &> 0,\\ (-\alpha + \gamma_{n-2})(\gamma_{n-1} + \gamma_n + \gamma_{n+1}) &> 0,\\ (-\alpha - \beta)\gamma_{n-3} + (\gamma_n + \gamma_{n+1})\gamma_{n-1} &> 0,\\ -\alpha\gamma_{n-2} + \gamma_n\gamma_{n+1} &> 0\end{aligned}$$

which again proves that $q_{d-2} > 0$, because the left-hand sides of these inequalities contain all products by two of moduli in which exactly one of the factors equals α or β (but not necessarily all products $\gamma_i\gamma_j$, and not the product $\alpha\beta$; no product $\gamma_i\gamma_j$ is repeated).

Part (2) Using the same notation one can write:

$$\begin{aligned}\alpha\gamma_{n-1} &< \gamma_{n-1}\gamma_n\,, & \beta\gamma_{n-1} &< \gamma_2\gamma_n\\ \alpha\gamma_n &< \gamma_n\gamma_{n+1}\,, & \beta\gamma_n &< \gamma_2\gamma_{n+1}\\ \alpha\gamma_{n+1} &< \gamma_{n-1}\gamma_{n+1}\,, & \beta\gamma_{n+1} &< \gamma_1\gamma_{n+1}\\ \alpha\gamma_1 &< \gamma_{n-1}\gamma_1\,, & \beta\gamma_1 &< \gamma_2\gamma_1\\ \alpha\gamma_2 &< \gamma_{n-1}\gamma_2\,, & \beta\gamma_2 &< \gamma_n\gamma_1\end{aligned}$$

and for $3 \leq k \leq n-2$, $(\alpha+\beta)\gamma_k < (\gamma_n + \gamma_{n+1})\gamma_k$. All these inequalities together imply $(\alpha+\beta)e_1 < e_2$, so $q_{d-2} > 0$.

3.4. **Proof of Proposition 3.3.** Suppose that the couple is realizable by a polynomial $Q := \sum_{j=0}^{7} q_j x^j$, $q_7 = 1$, with positive roots α and β and negative roots $-y$ and $-\gamma_i$, where

(3.5) $$y < \beta < \alpha < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4.$$

As $q_6 = -\alpha - \beta + y + \gamma_1 + \cdots + \gamma_4$, one has $q_6 > 0$. With roots satisfying the above inequalities it is impossible to have $q_1 < 0$, see [11, Theorem 3]. It is impossible to have $q_1 = 0$, $q_2 < 0$ either, because one can perturb q_1 to make it negative by which action the roots remain real, distinct and satisfying the above inequalities. So $q_1 > 0$. We denote by E_j (resp. G_j) the jth elementary symmetric polynomial of the quantities γ_i (resp. $1/\gamma_i$), $i = 1, \ldots, 4$. We show that the inequalities (3.5) and

(3.6) $$\begin{aligned}q_2/\alpha\beta y E_4 &:= 1/\alpha\beta - (1/\alpha + 1/\beta)(G_1 + 1/y) + (1/y)G_1 + G_2 < 0 \quad \text{and}\\ q_5 &:= \alpha\beta - (\alpha+\beta)(E_1 + y) + yE_1 + E_2 < 0\end{aligned}$$

cannot simultaneously hold true. In what follows, we assume that $\gamma_1 = 1$ which can be achieved by a linear change of the variable x and we consider instead of the inequalities (3.5) the corresponding inequalities $\leq$. We first observe that for $\alpha + \beta$ fixed, the left-hand sides in the inequalities (3.6) are the smallest possible when the product $\alpha\beta$ is the smallest possible (i.e. when $\alpha - \beta$ is the maximal possible). For the second of these inequalities this is evident, for the first of them, one has to notice that the coefficient of $1/\alpha\beta$ in the left-hand side equals $1 - (\alpha+\beta)(G_1 + 1/y) < 0$ (because $\alpha/y > 1$). Thus it suffices to prove the proposition in the two extremal cases $\beta = y$ and $\alpha = \gamma_1 = 1$. Suppose that $\beta = y$. Then the second of inequalities (3.6) reads:

$$E_2 < \alpha E_1 + \beta^2\,.$$

This is clearly impossible, because $\alpha \leq 1$, $\beta \leq 1$, so $\alpha E_1 + \beta^2 \leq E_1 + 1$ while

$$E_2 \geq \gamma_2 + 2\gamma_3 + 3\gamma_4 \geq E_1 + 2 \, .$$

Suppose that $\alpha = \gamma_1 = 1$. We denote by e_j and g_j the elementary symmetric polynomials of the quantities γ_i and $1/\gamma_i$ respectively, where $i = 2, 3, 4$. Thus $E_1 = 1+e_1$, $E_2 = e_1+e_2$, $G_1 = 1+g_1$ and $G_2 = g_1 + g_2$. The two inequalities (3.6) read:

$$(3.7) \qquad \begin{array}{l} (q_2) : -1 + (\beta - y)g_1 - \beta y + \beta y g_2 < 0 \quad \text{and} \\ (q_5) : -1 - (\beta - y)e_1 - \beta y + e_2 < 0. \end{array}$$

Suppose first that $1-(\beta-y)g_1 \geq 0$, i. e. $\beta - y \leq 1/g_1$. Hence the left-hand side L in (q_5) satisfies the inequality

$$L \geq -1 - e_1/g_1 - \beta y + e_2 = (-g_1 - e_1 - \beta y g_1 + e_2 g_1)/g_1 \geq 0 \, ,$$

because $g_1 \leq 3$, $\beta y \leq 1$, so $-g_1 - \beta y g_1 \geq -6$, while $e_2 g_1 = 2e_1 + T$, where

$$T := \gamma_2\gamma_3/\gamma_4 + \gamma_3\gamma_4/\gamma_2 + \gamma_4\gamma_2/\gamma_3 \geq 3(\gamma_2\gamma_3\gamma_4)^{1/3} \geq 3$$

(by the inequality between the mean arithmetic and the mean geometric), so

$$e_2 g_1 \geq 2e_1 + 3 \quad \text{and} \quad L \geq (-e_1 - 6 + 2e_1 + 3)/g_1 = (e_1 - 3)/g_1 \geq 0.$$

Suppose now that $1-(\beta-y)g_1 < 0$. Then the inequality (q_2) can hold true only if $\beta y(-1+g_2) < 0$, i.e. if $-1 + g_2 < 0$. This means that for $\beta - y$ fixed, the left-hand sides of (q_2) and (q_5) are minimal when βy is maximal. This is the case when $\beta = \alpha = 1$. Hence the following inequality (derived from (q_5)) holds true:

$$(3.8) \qquad -1 - e_1 + (e_1 - 1)y + e_2 < 0.$$

The left-hand side is minimal for $y = 0$, because $e_1 \geq 3$. Our aim is to show that if $\gamma_i \geq 1$, $i = 2$, 3, 4, and if $g_2 < 1$, then $e_2 > e_1+1$; this contradiction with inequality (3.8) would finish the proof of Proposition 3.3. If $g_2 < 1$, then as $e_2 g_2 \geq 9$ (inequality between the mean arithmetic and the mean harmonic), one obtains $e_2 > 9$. So the inequality $e_2 < e_1 + 1$ is possible only for $e_1 > 8$. But for fixed quantity e_1, the quantity e_2 is minimal for $\gamma_2 = \gamma_3 = 1$, $\gamma_4 = k > 1$. For $e_1 > 8$, one should have $k > 6$. For such a choice of γ_i, one gets $e_2 - e_1 - 1 = 2k + 1 - k - 3 = k - 2 > 4 > 0$. This proves the proposition.

3.5. **Proof of Proposition 3.4.** We suppose that the polynomial $Q := \sum_{j=0}^{7} q_j x^j$, $q_7 = 1$, realizes the couple and for the moduli of its roots α, β and $-y_1$, $-y_2$, $-\gamma_1$, $-\gamma_2$, $-\gamma_3$ one has

$$(3.9) \qquad y_1 < y_2 < \beta < \alpha < \gamma_1 < \gamma_2 < \gamma_3.$$

As in the proof of Proposition 3.3, one shows that $q_6 > 0$ and $q_1 > 0$. We want to show that one cannot simultaneously have the inequalities (3.9), $q_2 < 0$ and $q_5 < 0$. We denote by E_j, G_j the elementary symmetric polynomials of the quantities γ_i and $1/\gamma_i$ respectively. We set $r := y_1 + y_2$ and $t := y_1 y_2$. Then one must have

$$(3.10) \qquad \begin{array}{r} q_2/\alpha\beta t E_3 := 1/\alpha\beta - (1/\alpha + 1/\beta)(G_1 + r/t) + G_2 + G_1 r/t + 1/t < 0 \quad \text{and} \\ q_5 := \alpha\beta - (\alpha + \beta)(E_1 + r) + rE_1 + E_2 + t < 0. \end{array}$$

For $\alpha + \beta$ fixed, the left-hand sides of the above inequalities are minimal when $\alpha\beta$ is minimal. In the equation for $q_2/\alpha\beta t E_3$ the coefficient of $1/\alpha\beta$ equals $1 - (\alpha + \beta)(G_1 + r/t) < 0$ (because $\alpha r/t > \alpha/y_1 > 1$). So it suffices to consider the cases A) $\beta = y_2$ and B) $\alpha = \gamma_1$; in both cases one

can assume that $\gamma_1 = 1$ which can be achieved by a linear change of the variable x.
Case A). If $\beta = y_2$, then these inequalities read:

$$\begin{aligned} q_2/\alpha\beta t E_3 &:= G_2 + G_1/y_1 < G_1/\alpha + 1/\alpha y_1 + 1/\beta^2 \quad \text{and} \\ q_5 &:= E_2 + y_1 E_1 < \alpha E_1 + \alpha y_1 + \beta^2 . \end{aligned} \tag{3.11}$$

One has $\alpha > 2/3$. Indeed, if the second of inequalities (3.11) holds true, then it holds true for $y_1 = 0$, because $E_1 > \alpha$. But for $y_1 = 0$ and $\alpha \leq 2/3$, it is true that $\alpha E_1 + \beta^2 < 2E_1/3 + 1 < E_2$. Next, one has $\gamma_3 < 2$. Indeed, for $\gamma_3 \geq 2$, one gets

$$\begin{aligned} E_2 &\geq 2\gamma_1 + \gamma_2 + \gamma_3 = E_1 + 1 \\ &\geq \alpha E_1 - y_1(E_1 - \alpha) + 1 = (\alpha - y_1)E_1 + \alpha y_1 + 1 \\ &\geq (\alpha - y_1)E_1 + \alpha y_1 + \beta^2 \end{aligned}$$

which is a contradiction with (3.11). Thus

$$\gamma_1 = 1 \leq \gamma_2 < \gamma_3 < 2 \quad \text{and} \quad 4 < E_1 < 5\,, \quad 2 < G_1 < 3\,, \quad 5/4 < G_2 < G_1 < 3. \tag{3.12}$$

Suppose first that $\alpha - y_1 \leq 1/2$. For E_1 fixed, consider the function

$$\Phi(\alpha, y_1) := (\alpha - y_1)E_1 + \alpha y_1$$

on the closed pentagon

$$\{(\alpha, y_1) \mid 0 \leq \alpha, y_1 \leq 1\,,\ \alpha - y_1 \leq 1/2\,\}\,.$$

As $\partial\Phi/\partial y_1 = -E_1 + \alpha < 0$, the maximal value of Φ is attained on the union of two segments $I_1 \cup I_2$, where

$$I_1 := \{y_1 = 0,\ \alpha \in [0, 1/2]\} \quad \text{and} \quad I_2 := \{y_1 = \alpha - 1/2,\ \alpha \in [1/2, 1]\}\,.$$

For $(\alpha, y_1) \in I_1$, one has $\Phi = \alpha E_1 \leq E_1/2 < E_2 - 1 < E_2 - \beta^2$ which is a contradiction with (3.11). For $(\alpha, y_1) \in I_2$, one obtains

$$\Phi = E_1/2 + \alpha(\alpha - 1/2) < E_1/2 + 1/2 < E_2 - 1 < E_2 - \beta^2$$

which is again a contradiction. Suppose now that the couple (α, y_1) belongs to the segment $I_3 := \{\alpha - y_1 = \delta,\ \alpha \in [\delta, 1],\ \delta \in [1/2, 1]\}$. Set

$$\Psi := (G_1(\alpha - y_1) - 1)/\alpha y_1\,.$$

As $G_1 > 2$ and $\alpha - y_1 \geq 1/2$, one has $\Psi > 0$. Moreover, $\Psi > (G_1\delta - 1)/(1 - \delta)$. From (3.11), one deduces that $1/\beta^2 > G_2 + \Psi$, so

$$\beta^2 < 1/(G_2 + \Psi) < 1/\left(G_2 + \frac{G_1\delta - 1}{1 - \delta}\right) = (1 - \delta)/K,$$

where

$$\begin{aligned} K &:= G_2(1 - \delta) + G_1\delta - 1 = G_2 + (G_1 - G_2)\delta - 1 \\ &> G_2 + (G_1 - G_2)/2 - 1 = (G_1 + G_2 - 2)/2 > 5/8 \end{aligned}$$

see (3.12). Thus $\beta^2 < (8/5)(1 - \delta)$. One can rewrite the second of inequalities (3.11) in the form $E_2 - \beta^2 < \Phi$. However on the segment I_3 one has

$$\Phi = E_1\delta + \alpha(\alpha - \delta) \leq E_1\delta + (1 - \delta) < E_2 - \beta^2,$$

because $E_2 = E_2\delta + E_2(1 - \delta)$ with $E_1\delta < E_2\delta$ and

$$(1 - \delta) + \beta^2 < (13/5)(1 - \delta) < 3(1 - \delta) < E_2(1 - \delta).$$

This contradiction shows that the system of inequalities (3.11) has no solution in Case A).
Case B). Suppose that $\alpha = \gamma_1 = 1$ and that there exists a polynomial $Q := (x^2-1)Y(x)$ satisfying the conditions of Case B); the roots of Y are β, $-\gamma_2$, $-\gamma_3$, $-y_1$ and $-y_2$. We consider a one-parameter family of polynomials $Q_t := Q - tU$, $U := x^2(x^2-1)(x-\beta)$, $t \geq 0$. The first and last coefficients of U are positive which means that the coefficients q_2 and q_5 of Q_t remain negative for $t \geq 0$; the coefficients q_0, q_1, q_6 and q_7 do not change. As t increases and as long as Q_t remains hyperbolic,

(1) the roots $-\gamma_3$ and $-y_2$ move to the left while $-\gamma_2$ and $-y_1$ move to the right; $-y_1$ never reaches 0, because $U(0) = 0$, while $-\gamma_2$ and $-y_2$ could reach -1;
(2) the root $-\gamma_3$ cannot go to $-\infty$, because this would mean that $q_7 = 0$;
(3) neither of the coefficients q_3 and q_4 can vanish. Indeed, for a hyperbolic polynomial without root at 0, it is impossible to have two consecutive vanishing coefficients, and when a coefficient is 0, then the two surrounding coefficients must have opposite signs ([10, Lemma 7]).

It is clear that for $t > 0$ sufficiently large, one has $Q_t(2) < 0$, so Q_t has more than 2 positive roots. This can happen only if Q_t is no longer hyperbolic, because if it is, then it keeps the sign pattern $\Sigma_{2,4,2}$ (see 3)) and hence has exactly 2 positive roots.

Loss of hyperbolicity can occur only if the following couple or triple of roots coalesce: $(-\gamma_2, -1)$, $(-y_2, -1)$ or $(-\gamma_2, -1, -y_2)$. The triple confluence is a particular case of $(-\gamma_2, -1)$.
Case B.1). We assume that for $t = t_0 > 0$, one has $-\gamma_2 = -\gamma_1 = -1$. We consider the one-parameter family $R_s := Q_{t_0} - sV$, where $V := x^2(x+1)^2(x-1)$ and $s \geq 0$. For s small enough, the sign pattern of R_s is $\Sigma_{2,4,2}$. As s increases, $-\gamma_3$ moves to the left without reaching $-\infty$, $-y_2$ and β move to the right and $-y_1$ moves to the left. Hence for some $s = s_0$, either β coalesces with 1 or $-y_1$ and $-y_2$ coalesce.
Case B.1.1). Suppose that $\beta = 1$. Then $R_{s_0} = (x^2-1)^2 W$, where $W := x^3 + Ax^2 + Bx + C$ has only negative roots, so $A, B, C > 0$. Hence

$$q_2 = A - 2C < 0, \qquad q_3 = 1 - 2B < 0, \qquad q_4 = -2A + C < 0 \quad \text{and} \quad q_5 = B - 2 < 0. \tag{3.13}$$

The discriminant set

$$\{\rho := \mathrm{Res}(W, W', x) = 0\}, \quad \rho(A,B,C) = 4A^3C - A^2B^2 - 18ABC + 4B^3 + 27C^2,$$

separates the set H_3 of hyperbolic polynomials (where $\rho < 0$) in the space $OABC$ from the set of polynomials having exactly one real root (i.e. where $\rho > 0$). We show that the discriminant set does not intersect the domain $\mathcal{D}$ in the space $OABC$ defined by conditions (3.13). As $\rho(1,1,1) = 16 > 0$, the polynomial R_{s_0} is not hyperbolic which is a contradiction.

Lemma 3.2. *The set $\{\rho = 0\}$ does not intersect the border of the domain $\mathcal{D} := \{0 < A/2 < C < 2A\ ,\ B \in (1/2, 2)\}$.*

The lemma implies that the set $\{\rho = 0\}$ does not intersect the domain $\mathcal{D}$. Indeed, the set $\{\rho = 0\}$ contains the curve $A = 3t$, $B = 3t^2$, $C = t^3$ (polynomials with a triple root at $-t$). This curve is not contained in $\mathcal{D}$, because it contains points with $C > 2A$. Hence if the set $\{\rho = 0\}$ contains a point from $\mathcal{D}$, then it contains also a point not from $\overline{\mathcal{D}}$ hence also a point from the border of $\mathcal{D}$.

Proof of Lemma 3.2. For $A = 2C$, one obtains $\rho = 32C^4 + \tau C^2 + 4B^3$, $\tau := -4B^2 - 36B + 27$, whose discriminant

$$\tau^2 - 4 \times 32 \times 4B^3 = (2B-1)(2B-9)^3$$

is negative for $B \in (1/2, 2)$. For $A = C/2$, one gets $\rho = C^4/2 + \lambda C^2 + 4B^3$, $\lambda := -B^2/4 - 9B + 27$, with discriminant

$$\Delta_0 := \lambda^2 - 4 \times (1/2) \times 4B^3 = (-2 + B)(B - 18)^3/16$$

which is positive for $B \in (1/2, 2)$. However the biquadratic in C equation $\rho = 0$ has no real solution, because $\Delta_0 - \lambda^2 = -8B^3 < 0$. Thus the discriminant set $\{\rho = 0\}$ does not intersect the sets $\{A = 2C > 0,\ B \in (1/2, 2)\}$ and $\{A = C/2 > 0,\ B \in (1/2, 2)\}$. For $B = 1/2$, making use of $C/2 < A < 2C$, one gets

$$\begin{aligned}\rho &= 27C^2 + 4A^3C - 9AC - A^2/4 + 1/2 \\ &> 27C^2 + 4A^3C - 18C^2 - C^2 + 1/2 = 8C^2 + 4A^3C + 1/2 > 0.\end{aligned}$$

For $B = 2$, one obtains $\rho = 27C^2 + 4A^3C - 36AC - 4A^2 + 32$. The derivative $\partial\rho/\partial C = 4A^3 - 36A + 54C$ takes positive values for $A \geq 3/2$, $A/2 < C < 2A$. This follows easily from the fact that the graphs of the functions $A/2$ and $(-4A^3 + 36A)/54$ intersect exactly for $A = 0$ and $A = \pm 3/2$. Hence for $A \geq 3/2$ and $A/2 \leq C \leq 2A$, ρ is minimal when $C = A/2$; in this case it equals

$$2A^4 - 61A^2/4 + 32 = 2(A^2 - 61/16)^2 + 375/128 > 0.$$

For $0 < A \leq 3/2$ and $A/2 \leq C \leq 2A$, the quantity ρ takes its minimal value for $A = 3/2$. Indeed,

$$\partial\rho/\partial A = 12A^2C - 8A - 36C \leq 27C - 8A - 36C < 0 \quad \text{for } A \leq 3/2.$$

But for $A = 3/2$, one has $\rho = 27C^2 - 81C/2 + 23 = 27(C - 3/4)^2 + 125/16 > 0$. Thus the set $\{\rho = 0\}$ does not intersect the border of the domain $\mathcal{D}$. □

Case B.1.2). Suppose that $-y_1$ and $-y_2$ coalesce. We set $c := y_1 = y_2$ and we consider the polynomial

$$\sum_{j=0}^{7} q_j x^j =: Q := (x+1)^2(x+c)^2(x-1)h(x), \quad h := x^2 + Ax - B,$$

where $A = \gamma_3 - \beta > 0$ and $B = \gamma_3\beta$. As $c \in (0, 1)$, one has

$$h(-1) = 1 - A - B < 0. \tag{3.14}$$

The conditions $q_2 < 0$ and $q_5 < 0$ read:

$$\begin{aligned}&q_2 = -Ac^2 - Bc^2 - 2Ac + 2Bc - c^2 + B < 0, && \text{i.e.} \quad B < \tfrac{(c^2+2c)A + c^2}{-c^2+2c+1} \\ &q_5 = 2Ac + c^2 + A - B + 2c - 1 < 0, && \text{i.e.} \quad B > (2c+1)A + c^2 + 2c - 1.\end{aligned} \tag{3.15}$$

We denote by (q_2), (q_5) and (g) the straight lines in the space OAB defined by the conditions $q_2 = 0$, $q_5 = 0$ and $A + B = 1$ (see (3.14)). For $c \in (0, 1)$, the slope $(c^2 + 2c)/(-c^2 + 2c + 1)$ of the line (q_2) is smaller than the slope $2c + 1$ of (q_5); both slopes are positive. The A-coordinates of the intersection points $(q_2) \cap (q_5)$ and $(q_2) \cap (g)$ equal respectively

$$A' := -(c^4 - 5c^2 + 1)/(2c^3 - 2c^2 - 2c - 1) \quad \text{and} \quad A'' := (1 + 2c - 2c^2)/(4c + 1).$$

For $c \in (0, 1)$, one has

$$A'' - A' = 9c^2(c^2 - 2c - 1)/((2c^3 - 2c^2 - 2c - 1)(4c + 1)) > 0.$$

Hence for $c \in (0, 1)$, the half-plane $\{q_5 < 0\}$ does not intersect the sector $\{q_2 < 0,\ 1 - A - B < 0\}$, so a polynomial Q as above does not exist.
Case B.2). We assume that for $t = t_0 > 0$, one has $-\gamma_1 = -y_2 = -1$. We consider the one-parameter family $T_s := Q_{t_0} - sV$, where as above $V := x^2(x-1)(x+1)^2$ and $s \geq 0$. For $s > 0$ small enough, the sign pattern of T_s is $\Sigma_{2,4,2}$. As s increases, $-\gamma_3$ moves to the left without

reaching $-\infty$, $-y_2$ and β move to the right while $-y_1$ moves to the left. Hence for some $s = s_1$, one has one of the confluences $(\beta, 1)$, $(-\gamma_2, -1)$ or $(-y_1, -1)$. In the latter two cases, there is a triple root at -1.
Case B.2.1). Suppose that $\beta = \alpha = 1$. Then one can apply the involution i_r (this does not change the sign pattern $\Sigma_{2,4,2}$) and obtain a polynomial corresponding to Case B.1.1) which was already studied.
Case B.2.2). Suppose that $-\gamma_2 = -\gamma_1 = -y_2 = -1$. Then

$$T_{s_1} = (x+1)^3(x-1)(x-\beta)(x+\gamma_3)(x+y_1) = \sum_{j=0}^{7} q_j x^j,$$

where $q_2 + 2q_5 = 3(y_1 - \beta + \gamma_3) > 0$. Hence the coefficients q_2 and q_5 cannot be both negative; the sign pattern of T_{s_1} is not $\Sigma_{2,4,2}$.
Case B.2.3). Suppose that $-\gamma_1 = -y_2 = -y_1 = -1$. Then

$$T_{s_1} = (x+1)^3(x-1)(x-\beta)(x+\gamma_2)(x+\gamma_3) = \sum_{j=0}^{7} q_j x^j,$$

where $q_2 + 2q_5 = 3(\gamma_2 + \gamma_3 - \beta) > 0$. Hence the coefficients q_2 and q_5 cannot be both negative and again the sign pattern of T_{s_1} is not $\Sigma_{2,4,2}$.

3.6. **Proof of Proposition 3.5.** Suppose that the couple is realizable by a hyperbolic polynomial $Q := \sum_{j=0}^{7} q_j x^j$ the moduli of whose positive roots $\beta < \alpha$ and of whose negative roots $-\gamma_i$ satisfy the inequalities

$$\gamma_1 < \beta < \gamma_2 < \alpha < \gamma_3 < \gamma_4 < \gamma_5.$$

For the coefficients $q_0 = \alpha\beta\gamma_1 \cdots \gamma_5$, q_2 and q_5 of the polynomial Q, it is true that

$$q_2/q_0 = \sum_{1 \le i < j \le 5} 1/\gamma_i\gamma_j - ((\alpha+\beta)/\alpha\beta) \sum_{j=1}^{5} 1/\gamma_j + 1/\alpha\beta < 0,$$

$$q_5 = \alpha\beta - (\alpha+\beta) \sum_{j=1}^{5} \gamma_j + \sum_{1 \le i < j \le 5} \gamma_i\gamma_j < 0.$$

This however is impossible. Indeed, for positive α and β and when the sum $\alpha + \beta$ is fixed, the product $\alpha\beta$ is minimal when α and β are as far apart as possible. Hence it suffices to consider the two extremal situations:
(1) $\beta = \gamma_1$. In this case

$$\begin{aligned} q_5 &= \sum_{2 \le i < j \le 5} \gamma_i\gamma_j - \alpha \sum_{j=2}^{5} \gamma_j - \gamma_1^2 \\ &= (\gamma_3 - \alpha)\gamma_2 + (\gamma_4 - \alpha)\gamma_3 + (\gamma_5 - \alpha)\gamma_4 + (\gamma_3 - \alpha)\gamma_5 \\ &+ (\gamma_2\gamma_4 + \gamma_2\gamma_5 - \gamma_1^2) > 0 \end{aligned}$$

and the sign pattern of Q is not $\Sigma_{2,4,2}$.
(2) $\alpha = \gamma_3$. We assume that $\alpha = \gamma_3 = 1$. For fixed product $\gamma_4\gamma_5$, the sum $\gamma_4 + \gamma_5$ is the minimal possible when γ_4 and γ_5 are the closest possible to one another, i.e. when they are equal. One has

$$q_2/q_0 = \frac{1}{\gamma_4\gamma_5}(1 + (\gamma_4+\gamma_5)L_2) + \psi_2, \qquad L_2 := \frac{1}{\gamma_1} + \frac{1}{\gamma_2} - \frac{1}{\beta},$$

$$q_5 = (\gamma_4+\gamma_5)L_5 + \gamma_4\gamma_5 + \psi_5, \qquad L_5 := \gamma_1 + \gamma_2 - \beta,$$

where the quantities ψ_2 and ψ_5 do not depend on γ_4 and γ_5. As the quantities L_2 and L_5 are positive, if one has $q_2 < 0$ and $q_5 < 0$ for some values of the moduli γ_i and β such that $\gamma_4\gamma_5 = \lambda > 0$, then these inequalities will hold true also for $\gamma_4 = \gamma_5 = \sqrt{\lambda}$. A similar reasoning holds true for the couples (γ_2, γ_4) and (γ_2, γ_5), but in these cases the modulus $1 = \gamma_3 = \alpha$ must remain between the two moduli of the couple, so one can only claim that the sum $\gamma_2 + \gamma_4$ or $\gamma_2 + \gamma_5$ is minimal when one of these two moduli equals 1.

We apply the reasoning to the couple (γ_2, γ_4). Hence we can assume that either $\gamma_2 = 1$ or $\gamma_4 = 1$. If $\gamma_2 = 1$, then we apply the reasoning to the couple (γ_4, γ_5) to obtain the case

$$\text{A)} \quad \gamma_2 = \gamma_3 = \alpha = 1, \quad \gamma_4 = \gamma_5.$$

If $\gamma_4 = 1$, then we apply the reasoning to the couple (γ_2, γ_5) to obtain the cases

$$\text{B)} \quad \gamma_2 = \gamma_3 = \gamma_4 = \alpha = 1 \quad \text{and}$$

$$\text{C)} \quad \gamma_3 = \gamma_4 = \gamma_5 = \alpha = 1.$$

In case A), one has to deal with the polynomial

$$(x + \gamma_1)(x - \beta)(x - 1)(x + 1)^2(x + \gamma_4)^2$$

with

$$q_2 + (1 + \gamma_1)q_5 = \beta\gamma_4^2(1 - \gamma_1) + (2\gamma_4 - \beta)(\gamma_1^2 + \gamma_1) + (2\gamma_4 - \beta - 1) + \gamma_1^2 > 0 \, ,$$

so it is impossible to have $q_2 > 0$ and $q_5 > 0$.
In case B), we consider the polynomial

$$(x + \gamma_1)(x - \beta)(x - 1)(x + 1)^3(x + \gamma_5)$$

with

$$q_2 + q_5 = \gamma_1\beta + \gamma_5(\beta - \gamma_1) + (\gamma_5 - \beta) + \gamma_1 > 0 \, ,$$

so again one cannot have $q_2 < 0$ and $q_5 < 0$ at the same time.
In case C), one considers the polynomial

$$(x + \gamma_1)(x - \beta)(x + \gamma_2)(x - 1)(x + 1)^3$$

with

$$q_2 + q_5 = \beta\gamma_2 + \beta\gamma_1 + (1 - \gamma_2)\gamma_1 + (\gamma_2 - \beta) > 0$$

and one cannot have $q_2 > 0$ and $q_5 > 0$.

3.7. **Proof of Proposition 3.2.** We prove the proposition by induction on n. We prove part (2) first. The induction base are the couples $(\Sigma_{2,4,2}, (2, 0, 3))$ and $(\Sigma_{2,4,2}, (1, 1, 3))$, see Propositions 3.4 and 3.5. Suppose that it has been proved that the couple $(\Sigma_{2,n,2}, (u, v, 3))$, $u+v = n-2$, $u > 0$, is not realizable. This means that for any positive numbers

$$y_1 < \cdots < y_u < \beta < y_{u+1} < \cdots < y_{u+v} < \alpha < \gamma_1 < \gamma_2 < \gamma_3 \tag{3.16}$$

(interpreted as the moduli of the roots of a hyperbolic polynomial, the positive roots being α and β) it is impossible to simultaneously have

$$\begin{aligned} q_2 &:= P_* \times (1/\alpha\beta - (1/\alpha + 1/\beta)G_1 + G_2) < 0 \quad \text{and} \\ q_5 &:= \alpha\beta - (\alpha + \beta)E_1 + E_2 < 0, \end{aligned} \tag{3.17}$$

where $P_* := \alpha\beta\gamma_1\gamma_2\gamma_3 y_1 \cdots y_{n-2}$, E_j (resp. G_j) denoting the corresponding elementary symmetric polynomials of the quantities y_i and γ_j (resp. of $1/y_i$ and $1/\gamma_j$). Indeed, the inequalities (3.16) provide the positive signs of q_1 and q_6. If inequalities (3.17) hold true, then as there are just two positive roots, by Descartes' rule of signs one must have $q_3 < 0$ and $q_4 < 0$;

the equalities $q_3 = 0$ and $q_4 = 0$ are impossible by virtue of [10, Lemma 7]. We set $\tilde{q}_2 := 1/\alpha\beta - (1/\alpha + 1/\beta)G_1 + G_2$. Suppose that one increases n to $n+1$, so one adds a new quantity y_j or γ_j denoted by γ. Then $P_* \mapsto P_* \times \gamma$ and the new quantities $\tilde{q}_2$ and q_{d-2} equal respectively

$$\tilde{q}_2 + (1/\gamma)(G_1 - 1/\alpha - 1/\beta) \quad \text{and} \quad q_{d-2} + \gamma(E_1 - \alpha - \beta)\ .$$

As $y_1 < \beta$ and $y_2 < \alpha < \gamma_1 < \gamma_2$, one has $E_1 - \alpha - \beta > 0$ and $G_1 - 1/\alpha - 1/\beta > 0$. This means that both $\tilde{q}_2$ and q_{d-2} increase and hence after passing from n to $n+1$ they still cannot be both negative.

To prove part (1) of the proposition, one observes first that for $u = 0$, it follows from part (1) of Proposition 3.1. So one can suppose that $u \geq 1$. Then the proof of part (2) is performed in much the same way as the proof of part (1). There are only two differences:
(1) the induction base includes also the couple $(\Sigma_{2,4,2}, (1,0,4))$ which is non-realizable, see Proposition 3.3;
(2) the inequalities (3.16) have to be replaced by

$$y_1 < \cdots < y_u < \beta < y_{u+1} < \cdots < y_{u+v} < \alpha < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4\ .$$

The rest of the reasoning is the same.

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