MODERN MATHEMATICAL METHODS
2 (2024), No. 3, pp. 172-188
https://modernmathmeth.com/
ISSN 3023 - 5294

Research Article

Method for solving mixed boundary value problems for parabolic type equations by using modifications Lagrange-Sturm-Liouville operators

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ABSTRACT. A new method for solving a mixed boundary value problem with a parabolic type equation is obtained. Boundary conditions of the third kind are considered. The inhomogeneity of the equation and the initial conditions of the problem are arbitrary continuous functions. They are not required to satisfy boundary conditions. The solution is constructed using interpolation operators of Lagrange-Sturm-Liouville functions. A sequential approach is used to construct a generalized solution. The solution is presented as a series. The series converges uniformly on any compact set contained within the domain of definition of the solution. The coefficients of the series are linear combinations of the values of the functions from the equation and the initial conditions of the boundary value problem. A simple method for finding the coefficients of these linear combinations is proposed.

Keywords: Boundary value problem, generalized solution, method of separation of variables.

2020 Mathematics Subject Classification: 41A05, 35A35, 35K10.

1. INTRODUCTION

The paper proposes a method for obtaining a generalized solution to a mixed boundary value problem with a parabolic type equation. We consider the mixed boundary value problem

$$u_t - u_{xx} + q(x)u = f(x,t), \tag{1.1}$$

$$u(0,t)\cos\alpha + u_x(0,t)\sin\alpha = 0, \tag{1.2}$$

$$u(\pi,t)\cos\beta + u_x(\pi,t)\sin\beta = 0, \tag{1.3}$$

$$u(x,0) = \varphi(x), \tag{1.4}$$

where $x \in [0,\pi]$, $t \in [0,T]$, $T > 0$, $\alpha, \beta \in \mathbb{R}$, functions f and φ are continuous on the variable $x \in [0,\pi]$, and function f is summable on the variable t on the segment $[0,T]$, and the function q is of bounded variation. Satisfaction of the boundary conditions (1.2), (1.3) by the functions f and φ are not necessary. We obtain a generalized solution in the case where there are arbitrary continuous function (not necessarily satisfying the boundary conditions) in the initial conditions and homogeneous terms of the equations. As an intermediate approximation of the generalized solution, we use the generalized Lagrange-Sturm-Liouville operator [14]. In this method the role of Fourier coefficients of functions involved in the problem is played by the set of their values at zeros of solutions to the auxiliary Sturm-Liouville problem. The algorithm admits the use of already calculated Fourier coefficients for a countable set of auxiliary functions. When reasoning, we used ideas from works [2]-[4].

Received: 14.04.2024; Accepted: 02.12.2024; Published Online: 11.12.2024
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2. The Main Results

Unless otherwise stated, U_n is the normalized ($U_n(0) = 1$ or $U_n'(0) = 1$ if $h = \pm\infty$) n-th eigenfunction, corresponding to the eigenvalue λ_n regular Sturm-Liouville problem with potential q^* of bounded variation on $[0,\pi]$

$$U'' + [\lambda - q^*]U = 0, \tag{2.5}$$

$$U'(0) - hU(0) = 0, \tag{2.6}$$

$$U'(\pi) + HU(\pi) = 0, \tag{2.7}$$

where $h = \pm\infty$ and $H = \pm\infty$ are admitted.

We define the interpolation operator of a functions $L_n^{SL}(f,\cdot)$ sending a finite-valued function f defined at the zeros $0 \le x_{1,n} < x_{2,n} < \cdots < x_{n,n} \le \pi$ of U_n to a continuous function by the rule (cf. [14])

$$L_n^{SL}(f,x) = \sum_{k=1}^{n} f(x_{k,n}) \frac{U_n(x)}{U_n'(x_{k,n})(x - x_{k,n})} = \sum_{k=1}^{n} f(x_{k,n}) l_{k,n}^{SL}(x). \tag{2.8}$$

The value of operator (2.8) cannot approximate an arbitrary continuous function (cf. [9]). We modify this operator as follows. On the space of bounded functions defined on $[0,\pi]$ ($f \in M[0,\pi]$), we introduce the operator

$$\begin{aligned} CT_n^{SL}(f,x) =& \frac{1}{4}\sum_{k=1}^{n-1}\left(l_{k+1,n}^{SL}(x) + 2l_{k,n}^{SL}(x) + l_{k-1,n}^{SL}(x)\right) \\ &\times\left\{ f(x_{k,n}) - \frac{f(\pi)-f(0)}{\pi}x_{k,\lambda} - f(0)\right\} + \frac{f(\pi)-f(0)}{\pi}x + f(0), \end{aligned} \tag{2.9}$$

or

$$\begin{aligned} &CT_n^{SL}(f,x) \equiv \widetilde{CT}n^{SL}(f,x) \\ =&\frac{1}{4}\sum_{k=1}^{n-1} l_{k,n}^{SL}(x)\Big\{ f(x_{k+1,n}) + 2f(x_{k,n}) + f(x_{k-1,n}) \\ &-\frac{f(\pi)-f(0)}{\pi}(x_{k+1,n} + 2x_{k,n} + x_{k-1,n}) - 4f(0)\Big\} + \frac{f(\pi)-f(0)}{\pi}x + f(0), \end{aligned} \tag{2.10}$$

where the functions $l_{k,n}^{SL}$ are defined in (2.8). We introduce generalized functions in the sence of the sequential approach [7], [1, p. 1.3].

Definition 2.1. *The class of equivalent sequences of continuous functions on a compact set K, i.e., sequences converging to the same continuous function in the norm $\|f\|_{C(K)} = \max_{x\in K}|f(x)|$ is called a generalized function on K.*

By solving of a mixed boundary value problem (1.1)-(1.4), we mean a generalized function in terms of Definition 2.1. Because if the stationary sequence $f_n \equiv f$ belongs to the class of equivalent sequences in Definition 2.1, then the classical solution is a special case of the generalized solution. Making the change of variables $u(x,t) = \hat{U}(x)V(t)$ in (1.1)-(1.4) and separating the variables in Equation (1.1), we obtain the regular Sturm-Liouville problem

$$\hat{U}'' + [\hat{\lambda} - q(x)]\hat{U} = 0, \tag{2.11}$$

$$\hat{U}(0)\cos\alpha + \hat{U}'(0)\sin\alpha = 0, \tag{2.12}$$

$$\hat{U}(\pi)\cos\beta + \hat{U}'(\pi)\sin\beta = 0. \tag{2.13}$$

Properties of this problem are well studied, for example [4]. We denote by $\hat{\lambda}_m := \hat{\lambda}_m(q,\alpha,\beta)$ and $U_m := U_m(q,\alpha,\beta,x)$ $m = 0,1,2,3,\ldots$ the eigenvalues and the corresponding orthonormalized eigenfunctions of problem (2.11)-(2.13), respectively. We consider the Fourier coefficients of (3.33) calculated for the Sturm-Liouville problem (2.11)-(2.13) and linear function

$$\tau_{k,n,m} = \int_0^\pi \hat{U}_m(q,\alpha,\beta,\xi) l^{SL}_{k,n}(\xi)\, d\xi,$$

$$\tau^{(0)}_m = \int_0^\pi \hat{U}_m(q,\alpha,\beta,\xi)\, d\xi, \quad \tau^{(1)}_m = \int_0^\pi \xi \hat{U}_m(q,\alpha,\beta,\xi)\, d\xi. \tag{2.14}$$

The Fourier coefficients (2.14) are determined only by the parameters of the mixed boundary value problem and can be calculated a priori for each problem of the form (1.1)-(1.4). We consider the Fourier coefficients of the values of the operator (2.9) for arbitrary $f \in M[0,\pi]$

$$\begin{aligned}\widehat{CT}^{SL}_{n,m}[f] :=& \frac{1}{4}\sum_{k=1}^{n} (\tau_{k-1,n,m} + 2\tau_{k,n,m} + \tau_{k-1,n,m}) \\ &\times \left\{ f(x_{k,n}) - \frac{f(\pi)-f(0)}{\pi} x_{k,n} - f(0) \right\} + \frac{f(\pi)-f(0)}{\pi}\tau^{(1)}_m + f(0)\tau^{(0)}_m.\end{aligned} \tag{2.15}$$

We set

$$\nu_n = \begin{cases} -e^{-\lambda_n}\left(CT^{SL}_n(f,0)\operatorname{ctg}\alpha + CT^{SL'}_n(f,0)\right) & \text{when } \alpha \neq \pi m_1, m_1 \in \mathbb{Z}, \\ CT^{SL}_n(f,0) & \text{when } \alpha = \pi m_1, m_1 \in \mathbb{Z}, \end{cases}$$

$$\tilde{\nu}_n = \begin{cases} -e^{-\lambda_n}\left(CT^{SL}_n(f,\pi)\operatorname{ctg}\beta + CT^{SL'}_n(f,\pi)\right) & \text{when } \beta \neq \pi m_2, m_2 \in \mathbb{Z}, \\ CT^{SL}_n(f,\pi) & \text{when } \beta = \pi m_2, m_2 \in \mathbb{Z}, \end{cases}$$

$\mu_n = \frac{\sqrt{3}}{2}e^{\lambda_n}$.
We will define two operators $\eta(x,\lambda_n)[f] := \eta(x,\lambda_n)$, $\widetilde{\eta}(x,\lambda_n)[f] := \widetilde{\eta}(x,\lambda_n)$ as follows; for each continuous function f on the set $\{x_{k,n}\}_{k=1,n=1}^{n,\infty}$, we will put two functions in correspondence
(2.16)

$$\eta(x,\lambda_n) = \begin{cases} 2\sqrt{\frac{1}{3}}\nu_n\mu_n x \text{ when } x \in [0, \frac{1}{|\mu_n|}\left(\frac{\sqrt{2}}{3}\right)], \alpha \neq \pi m_1, m_1 \in \mathbb{Z}, \\ \nu_n \sin^3\left(\mu_n\left(x + \frac{1}{|\mu_n|}\left(\arcsin\sqrt{\frac{2}{3}} - \frac{\sqrt{2}}{3}\right)\right)\right) \text{ when} \\ x \in [\frac{1}{|\mu_n|}\left(\frac{\sqrt{2}}{3}\right), \frac{\pi}{|\mu_n|} - \frac{1}{|\mu_n|}\left(\arcsin\sqrt{\frac{2}{3}} - \frac{\sqrt{2}}{3}\right)], \alpha \neq \pi m_1, m_1 \in \mathbb{Z}, \\ \nu_n\left(\frac{\pi}{|\mu_n|} - \frac{1}{|\mu_n|}\left(\arcsin\sqrt{\frac{2}{3}} - \frac{\sqrt{2}}{3}\right)\right)^{-3}\left(x - \frac{\pi}{|\mu_n|} + \frac{1}{|\mu_n|}\left(\arcsin\sqrt{\frac{2}{3}} - \frac{\sqrt{2}}{3}\right)\right)^3 \\ \text{when } x \in [0, \frac{\pi}{|\mu_n|} - \frac{1}{|\mu_n|}\left(\arcsin\sqrt{\frac{2}{3}} - \frac{\sqrt{2}}{3}\right)], \alpha = \pi m_1, m_1 \in \mathbb{Z}, \\ 0 \text{ when } x \in [\frac{\pi}{|\mu_n|} - \frac{1}{|\mu_n|}\left(\arcsin\sqrt{\frac{2}{3}} - \frac{\sqrt{2}}{3}\right), \pi], \end{cases}$$

(2.17)

$$\widetilde{\eta}(x,\lambda_n)=-\begin{cases} 2\sqrt{\frac{1}{3}}\tilde{\nu_n}\mu_n(\pi-x) \text{ when } x\in[\pi-\frac{1}{\mu_n}\left(\frac{\sqrt{2}}{3}\right),\pi],\beta\neq\pi m_2, m_2\in\mathbb{Z},\\ \tilde{\nu_n}\sin^3\left(\mu_n\left(\pi-x+\frac{1}{\mu_n}\left(\arcsin\sqrt{\frac{2}{3}}-\frac{\sqrt{2}}{3}\right)\right)\right)\\ \text{when } x\in[\pi-\frac{\pi}{\mu_n}+\frac{1}{\mu_n}\left(\arcsin\sqrt{\frac{2}{3}}-\frac{\sqrt{2}}{3}\right),\pi-\frac{1}{\mu_n}\left(\frac{\sqrt{2}}{3}\right)],\beta\neq\pi m_2, m_2\in\mathbb{Z},\\ \tilde{\nu_n}\left(\frac{\pi}{\mu_n}-\frac{1}{\mu_n}\left(\arcsin\sqrt{\frac{2}{3}}-\frac{\sqrt{2}}{3}\right)\right)^{-3}\left(\pi-\frac{\pi}{\mu_n}+\frac{1}{\mu_n}\left(\arcsin\sqrt{\frac{2}{3}}-\frac{\sqrt{2}}{3}\right)-x\right)^3\\ \text{when } x\in[\pi-\frac{\pi}{\mu_n}+\frac{1}{\mu_n}\left(\arcsin\sqrt{\frac{2}{3}}-\frac{\sqrt{2}}{3}\right),\pi],\beta=\pi m_2, m_2\in\mathbb{Z},\\ 0 \text{ when } x\in[0,\pi-\frac{\pi}{\mu_n}+\frac{1}{\mu_n}\left(\arcsin\sqrt{\frac{2}{3}}-\frac{\sqrt{2}}{3}\right)]. \end{cases}$$

We set

$$\sigma_1=\begin{cases} 1 \text{ when } (\alpha=\pi m_1, m_1\in\mathbb{Z})\wedge f(0)\neq 0,\\ 0 \text{ when } (\alpha\neq\pi m_1, m_1\in\mathbb{Z})\vee f(0)=0,\end{cases}$$

(2.18)
$$\tilde{\sigma}_1=\begin{cases} 1 \text{ when } (\beta=\pi m_2, m_2\in\mathbb{Z})\wedge f(\pi)\neq 0,\\ 0 \text{ when } (\beta\neq\pi m_2, m_2\in\mathbb{Z})\vee f(\pi)=0.\end{cases}$$

We consider the Fourier coefficients of the values of the operators (2.16), (2.17) with respect to the eigenfunctions of the problem (2.11)-(2.13)

$$\hat{\eta}_{\lambda_n,m}=\langle\hat{U}_m(q,\alpha,\beta,\cdot),\eta(\cdot,\lambda_n)\rangle=\int_0^\pi \hat{U}_m(q,\alpha,\beta,\xi)\eta(\xi,\lambda_n)d\xi,$$

(2.19)
$$\hat{\tilde{\eta}}_{\lambda_n,m}=\langle\hat{U}_m(q,\alpha,\beta,\cdot),\widetilde{\eta}(\cdot,\lambda_n)\rangle=\int_0^\pi \hat{U}_m(q,\alpha,\beta,\xi)\widetilde{\eta}(\xi,\lambda_n)d\xi.$$

We introduce the operator sending $f\in C[0,\pi]$ to partial Fourier sums of $CT_n^{SL}(f,x)+\eta(x,\lambda_n)+\widetilde{\eta}(x,\lambda_n)$:

(2.20)
$$\mathbb{CT}_{n,j}^{SL}(f,x)=\sum_{m=0}^{j}\widehat{CT}_{n,m}^{SL}[f,\eta]\hat{U}_m(q,\alpha,\beta,x),$$

where (cf. (2.15) and (2.19))

$$\widehat{CT}_{n,m}^{SL}[f,\eta]:=\frac{1}{2}\sum_{k=1}^{n}(\tau_{k-1,n,m}+\tau_{k,n,m}+\tau_{k+1,n,m})\left\{f(x_{k,n})-\frac{f(\pi)-f(0)}{\pi}x_{k,n}-f(0)\right\}$$

(2.21)
$$+\frac{f(\pi)-f(0)}{\pi}\tau_m^{(1)}+f(0)\tau_m^{(0)}+\hat{\eta}_{\lambda_n,m}+\hat{\tilde{\eta}}_{\lambda_n,m}.$$

Theorem 2.1. *We assume that $T>0$, $\varepsilon>0$, functions f and φ are continuous on the variable $x\in[0,\pi]$, and function f is summable on the variable t on the segment $[0,T]$, a function q is bounded variation, and a function $j(n)$ takes integer or infinite values and satisfies the relation*

(2.22)
$$\left[n^{2(1+\varepsilon)}\right]+1\leqslant j(n)\leqslant\infty.$$

Then the generalized solution to the mixed boundary value problem (1.1)-(1.4) has the form

$$u(x,t)=\lim_{n\to\infty}\sum_{m=0}^{j(n)}\left(\widehat{CT}_{n,m}^{SL}[\varphi,\eta]e^{-\hat{\lambda}_m t}+\int_0^t e^{-\hat{\lambda}_m(t-\tau)}\widehat{CT}_{n,m}^{SL}[f(\cdot,\tau),\eta]\,d\tau\right)$$

(2.23)
$$\times U_m(q,\alpha,\beta,x).$$

The convergence in (2.23) is uniform on $[\sigma_1\tilde{\varepsilon}, \pi - \widetilde{\sigma}_1\tilde{\varepsilon}] \times [0, T]$, where $\widehat{CT}^{SL}_{n,m}[\cdot, \eta]$ are defined in (2.21), and $\sigma_1, \widetilde{\sigma}_1$ are defined in (2.18).

Comparing with the theory of difference schemes, we see that the method proposed in Theorem 2.1 has the advantages of Fourier analysis, but, unlike the classical Fourier and Krylov methods, deals with a wider set of admissible function f, φ, which means that it suffices to possess an information about these functions at the nodes $x_{k,n}$. To calculate the coefficients $\widehat{CT}^{SL}_{n,m}[f, \eta]$, one can use the Fourier coefficients $\tau_{k,n,m}$ of $l^{SL}_{k,n}$ (cf. (2.14)) independent of f and φ. To avoid calculations of integrals of rapidly oscillating functions, the following assertion can be used.

Proposition 2.1 ([13], [15]). *If $q^*(x) \equiv q(x)$ in (2.5) and (2.11), then the Fourier coefficients (2.14) of $l^{SL}_{k,n}$ (cf. (2.8)) can be found in terms of Riemann-Stieltjes integral*

$$\tau_{k,n,m} = \frac{1}{\left(\hat{\lambda}_m - \lambda_n\right)} \Big({l^{SL}_{k,n}}'(\pi) U_m(\pi) - l^{SL}_{k,n}(\pi) U'_m(\pi)$$

$$- \left({l^{SL}_{k,n}}'(0) U_m(0) - l^{SL}_{k,n}(0) U'_m(0) \right) \Big) + \frac{2}{\left(\hat{\lambda}_m - \lambda_n\right)} \int_0^\pi \frac{U_m(x)}{(x - x_{k,n})} \, d l^{SL}_{k,n}(x). \tag{2.24}$$

If the potential of the problem (1.1)-(1.4) is continuously differentiable, then the Fourier coefficients of $l^{SL}_{k,n}$ can be found in terms of the resolvent of a differential operator.

Proposition 2.2 ([13], [15]). *If $q^*(x) \equiv -q(x)$ in (2.5) and (2.11) is continuously differentiable, then the Fourier coefficients (2.14) of $l^{SL}_{k,n}$ (cf. (2.8)) can be calculated in terms of the differential Cauchy operator*

$$\begin{aligned} \Phi'''_{k,\lambda_n,m}(x) + (\lambda_n + \lambda_m)\,\Phi'_{k,\lambda_n,m}(x) =& 2 {l^{SL}_{k,n}}'(x) \left(\frac{U_m(x)}{(x - x_{k,n})} \right)' (x - x_{k,n}) \\ \Phi_{k,\lambda_n,m}(x_{k,n}) =& 0, \\ \Phi'_{k,\lambda_n,m}(x_{k,n}) =& U_m(x_{k,n}), \\ \Phi''_{k,\lambda_n,m}(x_{k,n}) =& U'_m(x_{k,n}), \end{aligned} \tag{2.25}$$

as follows:

$$\tau_{k,n,m} = \Phi_{k,\lambda_n,m}(\pi) - \Phi_{k,\lambda_n,m}(0). \tag{2.26}$$

3. Auxiliaries

If $\rho_\lambda \geq 0$, for every λ, we assume that q_λ is an arbitrary element of the ball $V_{\rho_\lambda}[0, \pi]$ of radius $\rho_\lambda = o\big(\frac{\sqrt{\lambda}}{\ln \lambda}\big)$ in the space of functions such that

$$V_0^\pi[q_\lambda] \leq \rho_\lambda, \quad \rho_\lambda = o\left(\frac{\sqrt{\lambda}}{\ln \lambda}\right), \quad \text{as } \lambda \to \infty, \quad q_\lambda(0) = 0. \tag{3.27}$$

Here, the potential q_λ depended of λ. For any potential $q_\lambda \in V_{\rho_\lambda}[0, \pi]$, all zeros of solution to the Cauchy problem

$$\begin{cases} y'' + \big(\lambda - q_\lambda(x)\big) y = 0, \\ y(0, \lambda) = 1, \\ y'(0, \lambda) = h(\lambda), \end{cases} \tag{3.28}$$

or, under the additional condition $h(\lambda) \neq 0$

$$V_0^\pi[q_\lambda] \leq \rho_\lambda,\ \rho_\lambda = o\left(\frac{\sqrt{\lambda}}{\ln\lambda}\right),\ \text{when } \lambda \to \infty,\ q_\lambda(0) = 0,\ h(\lambda) \neq 0 \tag{3.29}$$

the Cauchy problem

$$\begin{cases} y'' + \big(\lambda - q_\lambda(x)\big)y = 0, \\ y(0,\lambda) = 0, \\ y'(0,\lambda) = h(\lambda), \end{cases} \tag{3.30}$$

that belong to $[0,\pi]$ and are enumerated in ascending order will be denoted by

$$0 \leq x_{0,\lambda} < x_{1,\lambda} < \ldots < x_{n(\lambda),\lambda} \leq \pi, (x_{-1,\lambda} < 0, x_{n(\lambda)+1,\lambda} > \pi). \tag{3.31}$$

Hereinafter, for brevity, we denote $n = n(\lambda)$. We consider the function

$$s_{k,\lambda}(x) = \frac{y(x,\lambda)}{y'(x_{k,\lambda},\lambda)(x - x_{k,\lambda})}, \tag{3.32}$$

where $y(x,\lambda)$ is the solution to the Cauchy problem (3.28) or (3.30) with zeros $x_{k,\lambda}$ (cf. (3.31)). If $q_\lambda \equiv q^*$ in (3.28) or (3.30) instead of a continuously varying parameter λ, we consider $\lambda = \lambda_n$ and $h(\lambda) \equiv 1$. We obtain the identities

$$l_{k,n}^{SL} \equiv s_{k,\lambda_n}. \tag{3.33}$$

On the space of continuous functions f on $[0,\pi]$, we introduce (cf.[8]) operators

$$\begin{aligned} CT_\lambda(f,x) = \frac{1}{4}\sum_{k=1}^{n-1}&\left\{f(x_{k,\lambda}) - \frac{f(\pi)-f(0)}{\pi}x_{k,\lambda} - f(0)\right\} \\ &\times \big(s_{k-1,\lambda}(x) + 2s_{k,\lambda}(x) + s_{k+1,\lambda}(x)\big) + \frac{f(\pi)-f(0)}{\pi}x + f(0), \end{aligned} \tag{3.34}$$

which can be written in the form

$$\begin{aligned} CT_\lambda(f,x) &\equiv \widetilde{CT}_\lambda(f,x) \\ &= \sum_{k=1}^{n-1}\left\{\frac{f(x_{k+1,\lambda}) + 2f(x_{k,\lambda}) + f(x_{k-1,\lambda})}{4} - \frac{(f(\pi)-f(0))}{\pi}\frac{(x_{k+1,\lambda} + x_{k,\lambda})}{2}\right. \\ &\left. - f(0)\right\} s_{k,\lambda}(x) + \frac{f(\pi)-f(0)}{\pi}x + f(0). \end{aligned} \tag{3.35}$$

We set

$$CT_\lambda^{(1)}(f,x) = \frac{d}{dx}CT_\lambda(f,x), \tag{3.36}$$

$$CT_\lambda^{(2)}(f,x) = \frac{d^2}{dx^2}CT_\lambda(f,x). \tag{3.37}$$

The values of the operators (3.34) and (2.9), (2.10) depend only on the values of $f(x_{k,\lambda_{n-1}})$ at zeros $x_{k,\lambda_{n-1}}$ of the function $y(x,\lambda_{n-1})$.

3.1. **Asymptotic Formulas.** We will use the following statements.

Proposition 3.3 ([10, Theorem 1], [8, Proposition 2]). *Let $\rho_\lambda \geq 0, \quad \rho_\lambda = o(\sqrt{\lambda})$ when $\lambda \to \infty$, and let $V_{\rho_\lambda}[0,\pi]$ be a ball of radius ρ_λ in the space of function of bounded variation vanishing at the origin, so that*

$$V_0^\pi[q_\lambda] \leq \rho_\lambda, \quad q_\lambda(0) = 0, \text{ where } \rho_\lambda = o(\sqrt{\lambda}), \text{ when } \lambda \to \infty$$

for real λ. Then there exists $\lambda_1 > 4\pi^2\rho_\lambda^2$ such that $\lambda \geq \lambda_1$, for each potemtial $q_\lambda \in V_{\rho_\lambda}[0,\pi]$ and any $x \in [0,\pi]$ the solution of the Cauchy problem (3.28) satisfies the following inequalities:

$$\left|y(x,\lambda) - \gamma(x,\lambda,h)\cos\sqrt{\lambda}x - \beta(x,\lambda,h)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right| \leq \frac{\rho_\lambda(1+\pi\rho_\lambda)}{2\lambda}\left(1 + \frac{|h(\lambda)|}{\sqrt{\lambda}}\right), \tag{3.38}$$
$$\left|y'(x,\lambda) + \sqrt{\lambda}\gamma(x,\lambda,h)\sin\sqrt{\lambda}x - \beta(x,\lambda,h)\cos\sqrt{\lambda}x\right| \leq \frac{\rho_\lambda(1+\pi\rho_\lambda)}{2\sqrt{\lambda}}\left(1 + \frac{|h(\lambda)|}{\sqrt{\lambda}}\right),$$

where

$$\beta(x,\lambda,h) = h(\lambda) + \frac{1}{2}\int_0^x q_\lambda(\tau)d\tau,$$
$$\gamma(x,\lambda,h) = 1 - \frac{h(\lambda)}{2\lambda}\int_0^x q_\lambda(\tau)d\tau.$$

Proposition 3.4 ([10, Theorem 1′], [8, Proposition 3]). *Let $\rho_\lambda \geq 0, \quad \rho_\lambda = o(\sqrt{\lambda})$ as $\lambda \to \infty$, and let $V_{\rho_\lambda}[0,\pi]$ be the ball of radius ρ_λ in the space of functions of bounded variation vanishing at the origin. Then there exists $\lambda_1 > 4\pi^2\rho_\lambda^2$, such that $\lambda \geq \lambda_1$, for each potential $q_\lambda \in V_{\rho_\lambda}[0,\pi]$ and any $x \in [0,\pi]$ the solution of the Cauchy problem (3.30) satisfies the following inequalities:*

$$\left|y(x,\lambda) - \frac{h(\lambda)\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \delta(x,\lambda,h)\cos\sqrt{\lambda}x\right| \leq \frac{\rho_\lambda(1+\pi\rho_\lambda)|h(\lambda)|}{2\lambda\sqrt{\lambda}}, \tag{3.39}$$
$$\left|y'(x,\lambda) - h(\lambda)\cos\sqrt{\lambda}x - \sqrt{\lambda}\delta(x,\lambda,h)\sin\sqrt{\lambda}x\right| \leq \frac{\rho_\lambda(1+\pi\rho_\lambda)|h(\lambda)|}{2\lambda},$$
$$\left|y''(x,\lambda) + h(\lambda)\sqrt{\lambda}\sin\sqrt{\lambda}x - \lambda\delta(x,\lambda,h)\cos\sqrt{\lambda}x\right| \leq \frac{\rho_\lambda(1+\pi\rho_\lambda)|h(\lambda)|}{2\sqrt{\lambda}},$$

where

$$\delta(x,\lambda,h) = \frac{h(\lambda)}{2\lambda}\int_0^x q_\lambda(\tau)\,d\tau.$$

Proposition 3.5 ([10, Theorems 2, 2′], [8, Proposition 4]). *Let condition (3.27) be satisfied. Then, for each potential $q_\lambda \in V_{\rho_\lambda}[0,\pi]$, the zeros of the solutions of the Cauchy problem (3.28) lying in $[0,\pi]$ and numbered in accordance with (3.31), satisfy yhe following asymptotic formulae as $\lambda \to \infty$:*

$$x_{k,\lambda} = \frac{(k+1)\pi}{\sqrt{\lambda}} - \frac{1}{\sqrt{\lambda}}\arcsin\sqrt{\frac{\lambda}{\lambda + h^2(\lambda)}} + o\left(\frac{\lambda^{-\frac{1}{2}}}{\ln\lambda}\right),$$
$$y'(x_{k,\lambda},\lambda) = \sqrt{\lambda + h^2(\lambda)}\left((-1)^{(k+1)} + o\left(\frac{1}{\ln\lambda}\right)\right),$$

as $\lambda \to \infty$. If $h(\lambda) \neq 0$, and q_λ satisfies (3.29) the zeros of the Cauchy problem (3.31), numbered in accordance with (3.30), satisfy the asymptotic formulae

$$x_{k,\lambda} = \frac{k}{\sqrt{\lambda}}\pi + o\left(\frac{\lambda^{-\frac{1}{2}}}{\ln\lambda}\right), \text{ when } \lambda \to \infty,$$

$$y'(x_{k,\lambda},\lambda) = h(\lambda)\left((-1)^k + o\left(\frac{1}{\ln\lambda}\right)\right), \text{ when } \lambda\to\infty.$$

The convergence of symbol o to zero is uniform in $q_\lambda \in V_{\rho_\lambda}[0,\pi]$ and $k : 0 \le k \le n$.

In order to recover the conditions of the problems (3.31) from the properties of the zeros of (3.28), (3.30), we can use the results of the studies in [11], [12].

Lemma 3.1 ([8, Lemma 2]). *Let $\rho_\lambda \ge 0$, $\rho_\lambda = o\left(\frac{\sqrt{\lambda}}{\ln\lambda}\right)$, as $\lambda\to\infty$, and $V_{\rho_\lambda}[0,\pi]$ be the ball of radius ρ_λ in the space of functions of bounded variatin vanish at the origin (in the case of the Cauchy problem (3.30) it is also assumed that $h(\lambda)\neq 0$). Then there exists a positive λ_0 depending only on the rate of change of the radii of the balls ρ_λ in (3.27) or (3.29) such that for potentials $q_\lambda \in V_{\rho_\lambda}[0,\pi]$ and any $h(\lambda)$, and for arbitrary k, $0\le k\le n$, $\lambda>\lambda_0$, the functions $s_{k,\lambda}(x)$ constructed from the solutions of the Cauchy problem (3.28) or (3.30) have the following estimates:*

$$\max_{x\in[0,\pi]}|s_{k,\lambda}(x)| = \max_{x\in[0,\pi]}\left|\frac{y(x,\lambda)}{y'(x_{k,\lambda},\lambda)(x-x_{k,\lambda})}\right| \le 3 \text{ when } \lambda>\lambda_0.$$

3.2. Some Operators of the Theory of Approximations of Functions.

Proposition 3.6 ([8, Proposition 1, Remark 6]). *Let $f\in C[0,\pi]$ and let the functions q_λ and $h(\lambda)$ satisfy the condition (3.27) in the case of the Cauchy problem (3.28) or (3.29) in the case of the Cauchy problem (3.30). Then the operators (3.34), (3.35) satisfy*

$$\lim_{\lambda\to\infty} CT_\lambda(f,x) \equiv \lim_{\lambda\to\infty}\widetilde{CT}_\lambda(f,x) = f(x) \tag{3.40}$$

uniformly for x on $[0,\pi]$ and for q_λ in the balls $V_{\rho_\lambda}[0,\pi]$, for any $h(\lambda)\in\mathbb{R}$.

Proposition 3.7. *Let $\rho_\lambda \ge 0$, $\rho_\lambda = o\left(\frac{\sqrt{\lambda}}{\ln\lambda}\right)$, as $\lambda\to\infty$, and $V_{\rho_\lambda}[0,\pi]$ is the ball or radius ρ_λ in the space of functions of bounded variation that vanish at zero (additionaly, $h(\lambda)\neq 0$ for the Cauchy problem (3.30)). Then there is λ_0 such that for any potential $q_\lambda\in V_{\rho_\lambda}[0,\pi]$ and function $h(\lambda)$ and for all $\lambda>\lambda_0$ the norms of the operators (3.36) and (3.37) acting from $M[0,\pi]$ to $C[0,\pi]$ and constructed from the solution to Cauchy problem (3.28) or (3.30) are estimated from above as follows:*

$$CT_\lambda^{(1)} \le \frac{17\sqrt{\lambda}}{\pi}\ln\lambda, \tag{3.41}$$

$$CT_\lambda^{(2)} \le \frac{17\lambda}{\pi}\ln\lambda. \tag{3.42}$$

Proof. We first prove the estimate (3.41) for the operator (3.36) in the case of the Cauchy problem (3.30). Since the operator (3.35) is invariant under multiplication of $y(x,\lambda)$ by a non zero constant, we can set $h(\lambda)\equiv 1$. For an arbitrary point $x\in[0,\pi]$, we denote by k_0 the number of the nearest node to x (if there are two such nodes, then for k_0 we take any of them). From the Proposition 3.5, we obtain the estimate

$$|x-x_{k_0,\lambda}| = O\left(\frac{\pi}{\sqrt{\lambda}}\right). \tag{3.43}$$

We will consider the representation of CT_λ in the form (3.35). Then the norms of the functionals (3.36) have the estimate:

$$\begin{aligned} CT_\lambda^{(1)}(x) &\le 2\sum_{k=0}^{n}|s'_{k,\lambda}(x)| + \frac{2}{\pi} \\ &= 2\sum_{k=0}^{k_0-1}|s'_{k,\lambda}(x)| + 2|s'_{k_0,\lambda}(x)| + 2\sum_{k=k_0+1}^{n}|s'_{k,\lambda}(x)| + \frac{2}{\pi}. \end{aligned} \tag{3.44}$$

The norms of the operators (3.36) have a representation:

$$CT_\lambda^{(1)} = \max_{x\in[0,\pi]} CT_\lambda^{(1)}(x). \tag{3.45}$$

The second term on the right-hand side of (3.44) is estimated in a neighborhood of the point $x_{k_0,\lambda}$ (3.43) with the help of the Lagrange formula, and the asymptotic formulas from the Proposition 3.5 as follows:

$$2\left|s'_{k_0,\lambda}(x)\right| = 2\left|\frac{\left|y'(x,\lambda)(x-x_{k_0,\lambda}) - y(x,\lambda)\right|}{y'(x_{k_0,\lambda},\lambda)(x-x_{k_0,\lambda})^2}\right| = o\left(\frac{1}{\sqrt{\lambda}}\right).$$

Using (3.44), we obtain the estimate

$$\begin{aligned} CT_\lambda^{(1)}(x) &\le 2\sum_{k=0}^{k_0-1}\left|\frac{\left|y'(x,\lambda)\right|}{y'(x_{k,\lambda},\lambda)(x-x_{k,\lambda})}\right| + 2\sum_{k=k_0+1}^{n}\left|\frac{\left|y'(x,\lambda)\right|}{y'(x_{k,\lambda},\lambda)(x-x_{k,\lambda})}\right| \\ &+ 2\sum_{k=0}^{k_0-1}\left|\frac{\left|y(x,\lambda)\right|}{y'(x_{k,\lambda},\lambda)(x-x_{k,\lambda})^2}\right| + 2\sum_{k=k_0+1}^{n}\left|\frac{\left|y(x,\lambda)\right|}{y'(x_{k,\lambda},\lambda)(x-x_{k,\lambda})^2}\right| \\ &+ \frac{2}{\pi} + o\left(\frac{1}{\sqrt{\lambda}}\right). \end{aligned} \tag{3.46}$$

By the above-mentioned asymptotic formulas from the Proposition 3.5, for sufficiently large λ we have

$$\min_{1\le k\le n} |x_{k,\lambda} - x_{k-1,\lambda}| \ge \frac{\pi}{2\sqrt{\lambda}}, \quad \min\left(|x - x_{k_0-1,\lambda}|, |x - x_{k_0+1,\lambda}|\right| \ge \frac{\pi}{8\sqrt{\lambda}}. \tag{3.47}$$

By (3.46), (3.47) and the above-mentioned asymptotic formulas from the Proposition 3.5, there is λ_1 depending only on the change rate of the ball radii in (3.27), (3.29) such that for all $\lambda > \lambda_1$

$$\begin{aligned} CT_\lambda^{(1)}(x) &\le 2\left|y'(x,\lambda)\right|\sum_{k=0}^{n}{}'\left|\frac{1}{\left((-1)^k + o\left(\frac{1}{\ln\lambda}\right)\right)(x-x_{k,\lambda})}\right| \\ &+ 2\left|y(x,\lambda)\right|\sum_{k=0}^{n}{}'\left|\frac{1}{\left((-1)^k + o\left(\frac{1}{\ln\lambda}\right)\right)(x-x_{k,\lambda})^2}\right| + \frac{2}{\pi} + o\left(\frac{1}{\sqrt{\lambda}}\right). \end{aligned}$$

Hereinafter, the prime at the sum symbol means the abscence of terms with index $k = k_0$. If $k_0 = 0$, then the first term is missing in the sum, if $k_0 = n$, then there is no third. We have

$$\begin{aligned} CT_\lambda^{(1)}(x) &\le 2|y'(x,\lambda)|\left(1 + \left|o\left(\frac{1}{\ln\lambda}\right)\right|\right)\frac{8\sqrt{\lambda}}{\pi}\left[\int_0^{x-\frac{\pi}{8\sqrt{\lambda}}}\frac{dt}{x-t} + \int_{x+\frac{\pi}{8\sqrt{\lambda}}}^{\pi}\frac{dt}{t-x}\right] \\ &+ 2|y(x,\lambda)|\left(1 + \left|o\left(\frac{1}{\ln\lambda}\right)\right|\right)\frac{8\sqrt{\lambda}}{\pi}\left[\int_0^{x-\frac{\pi}{8\sqrt{\lambda}}}\frac{dt}{(x-t)^2} + \int_{x+\frac{\pi}{8\sqrt{\lambda}}}^{\pi}\frac{dt}{(t-x)^2}\right] \\ &+ \frac{2}{\pi} + o\left(\frac{1}{\sqrt{\lambda}}\right). \end{aligned} \tag{3.48}$$

For the sums of this integrals (3.48), we obtain the estimates

$$\int_0^{x-\frac{\pi}{8\sqrt{\lambda}}} \frac{dt}{x-t} + \int_{x+\frac{\pi}{8\sqrt{\lambda}}}^{\pi} \frac{dt}{t-x} = -\ln(x-t)\Big|_0^{x-\frac{\pi}{8\sqrt{\lambda}}} + \ln(t-x)\Big|_{x+\frac{\pi}{8\sqrt{\lambda}}}^{\pi}$$
$$\leq \begin{cases} \ln(\lambda) + \ln 16 \text{ when } x \in [\frac{\pi}{4\sqrt{\lambda}}, \pi - \frac{\pi}{4\sqrt{\lambda}}], \\ \frac{1}{2}\ln(\lambda) + \ln 8 \text{ when } x \in [0, \frac{\pi}{4\sqrt{\lambda}}] \cup [\pi - \frac{\pi}{4\sqrt{\lambda}}, \pi]. \end{cases}$$

Now, let's estimate the sum of integrals

$$\int_0^{x-\frac{\pi}{8\sqrt{\lambda}}} \frac{dt}{(x-t)^2} + \int_{x+\frac{\pi}{8\sqrt{\lambda}}}^{\pi} \frac{dt}{(t-x)^2} = \frac{16\sqrt{\lambda}}{\pi} - \frac{\pi}{x(\pi-x)}.$$

Hence, we obtain the estimates

$$\int_0^{x-\frac{\pi}{8\sqrt{\lambda}}} \frac{dt}{(x-t)^2} + \int_{x+\frac{\pi}{8\sqrt{\lambda}}}^{\pi} \frac{dt}{(t-x)^2} \leq \begin{cases} \frac{16\sqrt{\lambda}}{\pi} - \frac{\pi}{x(\pi-x)} & \text{when } x \in [\frac{\pi}{4\sqrt{\lambda}}, \pi - \frac{\pi}{4\sqrt{\lambda}}], \\ \frac{8\sqrt{\lambda}}{\pi} - \frac{1}{\pi-x} & \text{when } x \in [0, \frac{\pi}{4\sqrt{\lambda}}], \\ \frac{8\sqrt{\lambda}}{\pi} - \frac{1}{x} & \text{when } x \in [\pi - \frac{\pi}{4\sqrt{\lambda}}, \pi]. \end{cases}$$

By (3.48) and (3.45), for the norm of operator (3.36), we obtain the following estimate un with respect to $x \in [0, \pi]$:

$$CT_\lambda^{(1)} \leq \frac{16\sqrt{\lambda}}{\pi}\left[\ln(\lambda) + \frac{16 + \pi \ln 16}{\pi}\right] + o\left(\frac{\sqrt{\lambda}}{\ln \lambda}\right).$$

Therefore, in the case of the Cauchy problem (3.30), there exists a sufficiently large $\lambda_0 \geq \lambda_1$ such that for all $\lambda > \lambda_0$ the estimate (3.41) holds. Now, we show that the inequality (3.41) also holds in the case of the Cauchy problem (3.28). For this purpose, we extend the function

$$q_\lambda(x) = \begin{cases} q_\lambda(x) & \text{when } x \in [0, \pi], \\ 0 & \text{when } x \notin [0, \pi]. \end{cases} \tag{3.49}$$

We change the independent variable

$$t = \frac{\pi\left(x\sqrt{\lambda} + \arcsin\sqrt{\frac{\lambda}{\lambda + h^2(\lambda)}}\right)}{\pi\sqrt{\lambda} + \arcsin\sqrt{\frac{\lambda}{\lambda + h^2(\lambda)}}}. \tag{3.50}$$

We denote

$$\hat{y}(t, \hat{\lambda}) = y\left(\frac{\pi\sqrt{\lambda} + \arcsin\sqrt{\frac{\lambda}{\lambda + h^2(\lambda)}}}{\pi\sqrt{\lambda}}t - \frac{1}{\sqrt{\lambda}}\arcsin\sqrt{\frac{\lambda}{\lambda + h^2(\lambda)}}, \lambda\right) \tag{3.51}$$

and

$$\hat{q}_{\hat{\lambda}}(t) = \left(1 + \frac{1}{\pi\sqrt{\lambda}}\arcsin\sqrt{\frac{\lambda}{\lambda + h^2(\lambda)}}\right)^2$$
$$\times\, q_\lambda\left(\frac{\pi\sqrt{\lambda} + \arcsin\sqrt{\frac{\lambda}{\lambda + h^2(\lambda)}}}{\pi\sqrt{\lambda}}t - \frac{1}{\sqrt{\lambda}}\arcsin\sqrt{\frac{\lambda}{\lambda + h^2(\lambda)}}\right), \tag{3.52}$$

where

$$\hat{\lambda} = \left(1 + \frac{1}{\pi\sqrt{\lambda}}\arcsin\sqrt{\frac{\lambda}{\lambda + h^2(\lambda)}}\right)^2 \lambda.$$

By the Picard theorem, the functions (3.51) are solutions to the Cauchy problems

$$\hat{y}'' + \big(\hat{\lambda} - \hat{q}_{\hat{\lambda}}(t)\big)\hat{y} = 0, \quad \hat{y}\big(t(0), \hat{\lambda}\big) = y(0, \lambda) = 1,$$

$$\hat{y}'\big(t(0), \hat{\lambda}\big) = \left(1 + \frac{1}{\pi\sqrt{\lambda}} \arcsin\sqrt{\frac{\lambda}{\lambda + h^2(\lambda)}}\right) h(\lambda) \tag{3.53}$$

and

$$\hat{y}'' + \big(\hat{\lambda} - \hat{q}_{\hat{\lambda}}(t)\big)\hat{y} = 0, \; \hat{y}(0, \hat{\lambda}) = y\left(-\frac{1}{\sqrt{\lambda}} \arcsin\sqrt{\frac{\lambda}{\lambda + h^2(\lambda)}}\right) = 0,$$

$$\hat{y}'(0, \hat{\lambda}) = \sqrt{\lambda + h^2(\lambda)}\left(1 + \frac{1}{\pi\sqrt{\lambda}} \arcsin\sqrt{\frac{\lambda}{\lambda + h^2(\lambda)}}\right) = \hat{h}(\hat{\lambda}). \tag{3.54}$$

By (3.49) and (3.52), $\sqrt{\hat{\lambda}} - \frac{1}{2} \le \sqrt{\lambda} \le \sqrt{\hat{\lambda}} + \frac{1}{2}$, i.e. $\sqrt{\hat{\lambda}} \simeq \sqrt{\lambda}$. Consequently, the relation (3.27) remains valid for the problem (3.54) since (3.49) and (3.52) imply

$$q_{\hat{\lambda}}(0) = 0 \text{ and } V_0^{\pi}[\hat{q}_{\hat{\lambda}}] \le \left(1 + \frac{1}{2\sqrt{\lambda}}\right)^2 V_0^{\pi}[q_\lambda] = o\left(\frac{\sqrt{\lambda}}{\ln\lambda}\right) = o\left(\frac{\sqrt{\hat{\lambda}}}{\ln\hat{\lambda}}\right). \tag{3.55}$$

By (3.51), when $t \in [0, \pi]$ and $x \in \left[-\frac{1}{\sqrt{\lambda}} \arcsin\sqrt{\frac{\lambda}{\lambda + h^2(\lambda)}}, \pi\right]$, we have the identity

$$s_{k,\lambda}(x) \equiv \frac{y(x, \lambda)}{y'(x_{k,\lambda}, \lambda)(x - x_{k,\lambda})} \equiv \frac{\hat{y}(t, \hat{\lambda})}{\hat{y}'(t_{k,\hat{\lambda}}, \hat{\lambda})(t - t_{k,\hat{\lambda}})} \equiv \hat{s}_{k,\hat{\lambda}}(t). \tag{3.56}$$

Thus, (3.41) holds because $\hat{s}_{k,\hat{\lambda}}(t)$ are constructed from the Cauchy problem (3.54) of the form (3.30).

Let us proceed to the proof of the estimate (3.42) for the norm of the operator (3.37). Again, we will first carry out the reasoning in the case of the Cauchy problem (3.30). Since the operator (3.35) is invariant under multiplication of $y(x, \lambda)$ by a non zero constant, we can set $h(\lambda) \equiv 1$. For an arbitrary point $x \in [0, \pi]$, we denote by k_0 the number of the nearest node to x (if there are two such nodes, then for k_0 we take any of them). From the Proposition 3.5, we obtain the estimate (3.43). We will consider the representation of CT_λ in the form (3.35). Then the norms of the functionals (3.37) have the estimate:

$$\begin{aligned} CT_\lambda^{(2)}(x) \le 2\sum_{k=0}^{n} \big|s''_{k,\lambda}(x)\big| &= 2\sum_{k=0}^{k_0-1} \big|s''_{k,\lambda}(x)\big| \\ &+ 2\big|s''_{k_0,\lambda}(x)\big| + 2\sum_{k=k_0+1}^{n} \big|s''_{k,\lambda}(x)\big|. \end{aligned} \tag{3.57}$$

The norms of the operators (3.37) have a representation:

$$CT_\lambda^{(2)} = \max_{x \in [0,\pi]} CT_\lambda^{(2)}(x). \tag{3.58}$$

The second term on the right-hand side of (3.57) is estimated in a neighborhood of the point $x_{k_0,\lambda}$ (3.43) with the help of the Lagrange formula, and the asymptotic formulas from the Proposition 3.5 and the Lemma 3.1. There are constants $\lambda_1 > 0$ and $C_1 > 0$ such that for all $\lambda > \lambda_1$ and $x \in [0, \pi]$ is a fair estimate

$$2\big|s''_{k_0,\lambda}(x)\big| \le C_1\sqrt{\lambda}.$$

Hence using (3.44), we obtain the estimate

$$CT_\lambda^{(2)}(x) \le 2\sum_{k=0}^{k_0-1}\left|\frac{|y''(x,\lambda)|}{y'(x_{k,\lambda},\lambda)(x-x_{k,\lambda})}\right| + 2\sum_{k=k_0+1}^{n}\left|\frac{|y''(x,\lambda)|}{y'(x_{k,\lambda},\lambda)(x-x_{k,\lambda})}\right|$$
$$+4\sum_{k=0}^{k_0-1}\left|\frac{|y'(x,\lambda)|}{y'(x_{k,\lambda},\lambda)(x-x_{k,\lambda})^2}\right| + 4\sum_{k=k_0+1}^{n}\left|\frac{|y'(x,\lambda)|}{y'(x_{k,\lambda},\lambda)(x-x_{k,\lambda})^2}\right|$$
$$(3.59)\qquad +2\sum_{k=0}^{k_0-1}\left|\frac{|y(x,\lambda)|}{y'(x_{k,\lambda},\lambda)(x-x_{k,\lambda})^3}\right| + 2\sum_{k=k_0+1}^{n}\left|\frac{|y(x,\lambda)|}{y'(x_{k,\lambda},\lambda)(x-x_{k,\lambda})^3}\right| + O\left(\sqrt{\lambda}\right).$$

By the above-mentioned asymptotic formulas from the Proposition 3.5, for sufficiently large λ, we have (3.47). By (3.59), (3.47) and the above-mentioned asymptotic formulas from the Proposition 3.5, there is $\lambda_2 \ge \lambda_1$ depending only on the change rate of the ball radii in (3.27), (3.29) such that for all $\lambda > \lambda_2$

$$CT_\lambda^{(2)}(x) \le 2|y''(x,\lambda)|{\sum_{k=0}^{n}}'\left|\frac{1}{\left((-1)^k + o\left(\frac{1}{\ln\lambda}\right)\right)(x-x_{k,\lambda})}\right|$$
$$+4|y'(x,\lambda)|{\sum_{k=0}^{n}}'\left|\frac{1}{\left((-1)^k + o\left(\frac{1}{\ln\lambda}\right)\right)(x-x_{k,\lambda})^2}\right|$$
$$+2|y(x,\lambda)|{\sum_{k=0}^{n}}'\left|\frac{1}{\left((-1)^k + o\left(\frac{1}{\ln\lambda}\right)\right)(x-x_{k,\lambda})^3}\right| + O\left(\sqrt{\lambda}\right).$$

Let us estimate the sums as follows

$$CT_\lambda^{(2)}(x) \le 2|y''(x,\lambda)|\left(1+\left|o\left(\frac{1}{\ln\lambda}\right)\right|\right)\frac{8\sqrt{\lambda}}{\pi}\left[\int_0^{x-\frac{\pi}{8\sqrt{\lambda}}}\frac{dt}{x-t} + \int_{x+\frac{\pi}{8\sqrt{\lambda}}}^{\pi}\frac{dt}{t-x}\right]$$
$$+4|y'(x,\lambda)|\left(1+\left|o\left(\frac{1}{\ln\lambda}\right)\right|\right)\frac{8\sqrt{\lambda}}{\pi}\left[\int_0^{x-\frac{\pi}{8\sqrt{\lambda}}}\frac{dt}{(x-t)^2} + \int_{x+\frac{\pi}{8\sqrt{\lambda}}}^{\pi}\frac{dt}{(t-x)^2}\right]$$
$$+2|y(x,\lambda)|\left(1+\left|o\left(\frac{1}{\ln\lambda}\right)\right|\right)\frac{8\sqrt{\lambda}}{\pi}\left[\int_0^{x-\frac{\pi}{8\sqrt{\lambda}}}\frac{dt}{(x-t)^3} + \int_{x+\frac{\pi}{8\sqrt{\lambda}}}^{\pi}\frac{dt}{(t-x)^3}\right] + O\left(\sqrt{\lambda}\right).$$

Thus, we obtain the ratio

$$\int_0^{x-\frac{\pi}{8\sqrt{\lambda}}}\frac{dt}{x-t} + \int_{x+\frac{\pi}{8\sqrt{\lambda}}}^{\pi}\frac{dt}{t-x} = -\ln(x-t)\Big|_0^{x-\frac{\pi}{8\sqrt{\lambda}}} + \ln(t-x)\Big|_{x+\frac{\pi}{8\sqrt{\lambda}}}^{\pi}$$
$$\le \begin{cases} \ln(\lambda)+\ln 16 & \text{when } x\in\left[\frac{\pi}{4\sqrt{\lambda}}, \pi-\frac{\pi}{4\sqrt{\lambda}}\right], \\ \frac{1}{2}\ln(\lambda)+\ln 8 & \text{when } x\in\left[0,\frac{\pi}{4\sqrt{\lambda}}\right]\cup\left[\pi-\frac{\pi}{4\sqrt{\lambda}},\pi\right]. \end{cases}$$

Now, let's estimate the sums of the integrals

$$\int_0^{x-\frac{\pi}{8\sqrt{\lambda}}}\frac{dt}{(x-t)^2} + \int_{x+\frac{\pi}{8\sqrt{\lambda}}}^{\pi}\frac{dt}{(t-x)^2} = \frac{16\sqrt{\lambda}}{\pi} - \frac{\pi}{x(\pi-x)},$$
$$\int_0^{x-\frac{\pi}{8\sqrt{\lambda}}}\frac{dt}{(x-t)^3} + \int_{x+\frac{\pi}{8\sqrt{\lambda}}}^{\pi}\frac{dt}{(t-x)^3} \le \frac{8^2\lambda}{\pi^2}.$$

As a result, taking into account the asymptotic formulas of the Proposition 3.4, we get a uniform $x \in [0, \pi]$ assessment

$$CT_\lambda^{(2)}(x) \le |y''(x,\lambda)| \left(1 + \left|o\left(\frac{1}{\ln\lambda}\right)\right|\right) \frac{16\sqrt{\lambda}}{\pi} \left[\ln(\lambda) + \ln 16\right]$$
$$+ |y'(x,\lambda)| \left(1 + \left|o\left(\frac{1}{\ln\lambda}\right)\right|\right) \left[\frac{16\sqrt{\lambda}}{\pi}\right]^2$$
$$+ |y(x,\lambda)| \left(1 + \left|o\left(\frac{1}{\ln\lambda}\right)\right|\right) \frac{16\sqrt{\lambda}}{\pi} \left[\frac{8^2\lambda}{\pi^2}\right] + O\left(\sqrt{\lambda}\right).$$

The estimation of the operator norm has the form

$$CT_\lambda^{(2)} \le \frac{16\lambda}{\pi} \ln(\lambda) + O(\lambda).$$

This implies the existence of $\lambda_0 \ge \lambda_2$ so large that for all $\lambda > \lambda_0$ the estimate (3.42) is correct in the case of the Cauchy problem (3.30). The validity of the estimate (3.42) in the case of the Cauchy problem (3.28) is also established as the validity of the relation (3.41) in the case of the Cauchy problem (3.28). The Proposition 3.7 is proved completely. □

Proposition 3.8. *For any positive $\tilde{\varepsilon}$ the functions $CT_\lambda(f,x) + \eta(x,\lambda) + \widetilde{\eta}(x,\lambda)$ satisfy the relation*

$$\lim_{\lambda\to\infty} \|CT_\lambda(f,\cdot) + \eta + \widetilde{\eta} - f\|_{C[\sigma_1\tilde{\varepsilon}, \pi - \tilde{\sigma}_1\tilde{\varepsilon}]} = 0.$$

Proof. The function $\eta(x,\lambda) + \widetilde{\eta}(x,\lambda)$ is twice continuously differentiable on $[0,\pi]$, and we have the relation:

$$\operatorname{supp}(\eta + \widetilde{\eta}) \subset \left[0, \frac{\pi}{|\mu|} - \frac{1}{|\mu|}\left(\arcsin\sqrt{\frac{2}{3}} - \frac{\sqrt{2}}{3}\right)\right]$$
$$\cup \left[\pi - \frac{\pi}{|\tilde{\mu}|} + \frac{1}{|\tilde{\mu}|}\left(\arcsin\sqrt{\frac{2}{3}} - \frac{\sqrt{2}}{3}\right), \pi\right] \tag{3.60}$$

for $\lambda > \ln\frac{4}{\sqrt{3}}$, by Proposition 3.7,

$$\left[0, \frac{\pi}{|\mu|} - \frac{1}{|\mu|}\left(\arcsin\sqrt{\frac{2}{3}} - \frac{\sqrt{2}}{3}\right)\right] \cap \left[\pi - \frac{\pi}{|\tilde{\mu}|} + \frac{1}{|\tilde{\mu}|}\left(\arcsin\sqrt{\frac{2}{3}} - \frac{\sqrt{2}}{3}\right), \pi\right] = \emptyset,$$

$$\|\eta + \widetilde{\eta}\|_{C[\sigma_1\left(\frac{\pi}{|\mu|} - \frac{1}{|\mu|}\left(\arcsin\sqrt{\frac{2}{3}} - \frac{\sqrt{2}}{3}\right)\right), \pi - \tilde{\sigma}_1\left(\frac{\pi}{|\mu|} + \frac{1}{|\mu|}\left(\arcsin\sqrt{\frac{2}{3}} - \frac{\sqrt{2}}{3}\right)\right)]} \le \max(\nu, \tilde{\nu})$$
$$= O\left(e^{-\lambda}\sqrt{\lambda}\ln\lambda\right) \tag{3.61}$$

for any third kind boundary conditions. In the case of the first kind boundary conditions, the following estimate holds:

$$\|\eta + \widetilde{\eta}\|_{C[0,\pi]} \le \max(\nu, \tilde{\nu}) = \max(|CT_\lambda(f,0)|, |CT_\lambda(f,\pi)|) O(1). \tag{3.62}$$

Now, Proposition 3.8 follows from the Proposition 3.6. □

3.3. Estimate for Fourier Coefficients of $CT_\lambda(f,x)$.

Proposition 3.9. *We assume that* $\rho_{\lambda_n} \geq 0, \quad \rho_{\lambda_n} = o\left(\frac{\sqrt{\lambda_n}}{\ln \lambda_n}\right)$, *as* $n \to \infty$, *and* $V_{\rho_{\lambda_n}}[0,\pi]$ *is the ball of radius* ρ_{λ_n} *in the space of functions of bounded variation that vanish at zero (additionally,* $h(\lambda_n) \neq 0$ *for the Cauchy problem(3.30)). Let* $0 < \epsilon < 1$ *and* $j(n) := \left[\lambda_n^{\left(1+\frac{2\epsilon}{1-\epsilon}\right)}\right] + 1$. *If the function* f *is continuous for* $x \in [0,\pi]$ *and the potential* q *is of bounded variation, then*

$$\|CT_n^{SL}(f,\cdot) + \eta(\cdot) - \widetilde{\eta}(\cdot) - \sum_{m=0}^{j(n)} \widehat{CT}_{n,m}^{SL}[f,\eta]\hat{U}_m(q,\alpha,\beta,\cdot)\|_{C[0,\pi]}$$

$$=\|f\|_{C[0,\pi]} \frac{n^{-2\epsilon\left(1+\frac{2\epsilon}{1-\epsilon}\right)}}{\epsilon} O(1). \tag{3.63}$$

Proof. From the asymptotic formulas (3.3), (3.39), it follows that

$$\|U_m\|_{L^2[0,\pi]} = O(1), \text{ as } m \to \infty. \tag{3.64}$$

We first consider the case where the boundary conditions in the Sturm-Liouville problem at zero are the same as in the Cauchy problem (3.28). By (3.3) we have, up to a normalization,

$$\begin{aligned}
\left|\int_0^\pi CT_\lambda(f,x)U_m(x)\,dx\right| &\leq \left|\int_0^\pi CT_\lambda(f,x)\left(\gamma(x,\lambda_m,h)\cos\sqrt{\lambda_m}x + \beta(x,\lambda_m,h)\frac{\sin\sqrt{\lambda_m}x}{\sqrt{\lambda_m}}\right)dx\right| \\
&+ \left|\int_0^\pi CT_\lambda(f,x)\frac{\rho_{\lambda_m}(1+\pi\rho_{\lambda_m})}{2\lambda_m}\left(1+\frac{|h(\lambda_m)|}{\sqrt{\lambda_m}}\right)dx\right| \\
&= \sqrt{\gamma^2(x,\lambda_m,h) + \frac{\beta^2(x,\lambda_m,h)}{\lambda_m}} \\
&\times \left|\int_0^\pi CT_\lambda(f,x)\left(\sin\phi_{\lambda_m}\cos\sqrt{\lambda_m}x + \cos\phi_{\lambda_m}\sin\sqrt{\lambda_m}x\right)dx\right| \\
&+ \|CT_\lambda(f,\cdot)\|_{C[0,\pi]} O\left(\frac{1}{\lambda_m}\right),
\end{aligned}$$

where

$$\sin\phi_{\lambda_m} = \frac{\gamma(x,\lambda_m,h)}{\sqrt{\gamma^2(x,\lambda_m,h) + \frac{\beta^2(x,\lambda_m,h)}{\lambda_m}}}.$$

Integrating by parts (cf. [5, Ch. VIII, §6, p. 5]), we get

$$\begin{aligned}
\left|\int_0^\pi CT_\lambda(f,x)U_m(x)\,dx\right| &= \left|\int_0^\pi CT_\lambda(f,x)\sin\left(\phi_{\lambda_m}+\sqrt{\lambda_m}x\right)dx\right| + \|CT_\lambda(f,\cdot)\|_{C[0,\pi]} O\left(\frac{1}{\lambda_m}\right) \\
&= \left|-CT_\lambda(f,x)\frac{\sin'\left(\phi_{\lambda_m}+\sqrt{\lambda_m}x\right)}{\lambda_m}\right|_0^\pi + AT'_\lambda(f,x)\frac{\sin\left(\phi_{\lambda_m}+\sqrt{\lambda_m}x\right)}{\lambda_m}\Big|_0^\pi \\
&- \frac{1}{\lambda_m}\int_0^\pi AT''_\lambda(f,x)\sin\left(\phi_{\lambda_m}+\sqrt{\lambda_m}x\right)dx\Bigg| + \|CT_\lambda(f,\cdot)\|_{C[0,\pi]} O\left(\frac{1}{\lambda_m}\right).
\end{aligned}$$

We again use the asymptotic formulas from the Proposition 3.3 and return to the eigenfunctions in the firs two terms:

$$\left|\int_0^\pi CT_\lambda(f,x)U_m(x)\,dx\right| = \left|-CT_\lambda(f,x)\frac{U_m'(x)}{\lambda_m}\Big|_0^\pi + AT_\lambda'(f,x)\frac{U_m(x)}{\lambda_m}\Big|_0^\pi\right|$$
$$+\left|-\frac{1}{\lambda_m}\int_0^\pi AT_\lambda''(f,x)\sin\left(\phi_{\lambda_m}+\sqrt{\lambda_m}x\right)dx\right| + \|CT_\lambda(f,\cdot)\|_{C[0,\pi]}O\left(\frac{1}{\lambda_m}\right).$$

By (3.42),

$$\left|\int_0^\pi CT_\lambda(f,x)U_m(x)\,dx\right| = \frac{\frac{17\pi\lambda}{\pi}\ln\lambda}{\lambda_m}\|f\|_{C[0,\pi]} + \|CT_\lambda(f,\cdot)\|_{C[0,\pi]}O\left(\frac{1}{\lambda_m}\right)$$
$$= \|f\|_{C[0,\pi]}\frac{\lambda\ln\lambda}{\lambda_m}O(1).$$

The function $\eta(x,\lambda)+\widetilde{\eta}(x,\lambda)$ is twice continuously differentiable on $[0,\pi]$. Taking into account (3.61), (3.62), and the estimate $\text{mes}\,(\text{supp}(\eta+\widetilde{\eta})) = O\left(e^{-\lambda}\right)$ and arguing as above, we obtain the estimate

$$|\widehat{CT}_{\lambda,m}[f,\eta]| = \|f\|_{C[0,\pi]}\frac{\lambda\ln\lambda}{\lambda_m}O(1). \tag{3.65}$$

Let us estimate the error of partial sums of the Fourier series of $CT_\lambda(f,\cdot)$. The asymptotic behavior of the eigenvalues of the problem (2.11)-(2.13) is known (cf. [4, Ch.1, §2, (2.12)]. If for every $\lambda > 0$, we consider an eigenvalue such that

$$\frac{\lambda\ln\lambda}{\lambda_m} \cong \frac{\lambda\ln\lambda}{m^2} \le m^{-1-\epsilon}$$

for some $0<\epsilon<1$, then the approximation error for $CT_\lambda(f,\cdot)$ is uniformly majorized by using the remainder in the series $\sum_{m=1}^\infty m^{-1-\epsilon}$. We extend the estimate for sufficiently large λ and $0<\epsilon<1$. Let

$$j(\lambda) := \left[\lambda^{1+\varepsilon}\right]+1,$$

here $\varepsilon := \frac{2\epsilon}{1-\epsilon} > 0$. By (3.65), (3.64), there is $\lambda_0 > 0$ such that (3.63) holds for all $\lambda > \lambda_0$. The left boundary condition is treated in the same way as in the Cauchy problem (3.30). □

3.4. **Proof of the Main Results.**

Proof of Theorem 2.1. We fix $\varepsilon > 0$ and introduce $j(n)$ as in (2.22). By Propositions 3.8 and 3.9 for any $t\in[0,T]$, we represent the following functions via the operator (2.20)

$$\lim_{n\to\infty}\mathbb{CT}^{SL}_{n,j(n)}(f(\cdot,t),x) = \lim_{\lambda_n\to\infty}\sum_{m=0}^{j(n)}\widehat{CT}^{SL}_{n,m}[f(\cdot,t),\eta]\hat{U}_m(q,\alpha,\beta,x) = f(x,t), \tag{3.66}$$

$$\lim_{n\to\infty}\mathbb{CT}^{SL}_{n,j(n)}(\varphi,x) = \lim_{\lambda_n\to\infty}\sum_{m=0}^{j(n)}\widehat{CT}^{SL}_{n,m}[\varphi,\eta]\hat{U}_m(q,\alpha,\beta,x) = \varphi(x). \tag{3.67}$$

We consider the following family of mixed problems depending on the parameter λ_n:

$$u_{\lambda_n t} - u_{\lambda_n xx} + q(x)u_{\lambda_n} = \mathbb{CT}^{SL}_{n,j(n)}(f(\cdot,t),x), \tag{3.68}$$

$$u_{\lambda_n}(0,t)\cos\alpha + u_{\lambda_n x}(0,t)\sin\alpha = 0, \tag{3.69}$$

$$u_{\lambda_n}(\pi,t)\cos\beta + u_{\lambda_n x}(\pi,t)\sin\beta = 0, \tag{3.70}$$

$$u_{\lambda_n}(x,0) = \mathbb{CT}^{SL}_{n,j(n)}(\varphi,x). \tag{3.71}$$

The function (3.71) and the right-hand side of Equation (3.68) possess absolutely continuous derivatives with respect to x. Each of the problems (3.68)-(3.71) has a unique classical dolution. By the method of separation of variables, the classical solution satisfying the initial conditions can be represented as a uniformly convergent Fourier series on $[0,\pi]\times[0,T]$ with respect to the eigenfunctions of the Sturm-Liouville problem (2.11)-(2.13). If $j(n) < \infty$, the series becomes the finite sum

$$u_{\lambda_n}(x,t) = \sum_{m=0}^{j(n)} \Big(\widehat{CT}^{SL}_{n,m}[\varphi,\eta]e^{-\hat{\lambda}_m t} + \int_0^t e^{-\hat{\lambda}_m(t-\tau)}\widehat{CT}^{SL}_{n,m}[f(\cdot,\tau),\eta]\,d\tau\Big)U_m(q,\alpha,\beta,x). \tag{3.72}$$

By (2.21), (3.60), (3.61) and (3.62), the measure of the support of the bounded function $\eta+\widetilde{\eta}$ decreases as $O(e^{-\lambda_n})$. Consequently,

$$\int_0^t e^{-\hat{\lambda}_m(t-\tau)}\widehat{CT}^{SL}_{n,m}[f(\cdot,\tau)]\,d\tau - \int_0^t e^{-\hat{\lambda}_m(t-\tau)}\widehat{CT}^{SL}_{n,m}[f(\cdot,\tau),\eta]\,d\tau = O(e^{-\lambda_n}).$$

By the Cauchy criterion for uniform convergence of series, the perturbed data $\eta(x,\lambda_n)+\widetilde{\eta}(x,\lambda_n)$ substituted into the third term in (3.72) yield an of order $O(e^{-\frac{-\lambda_n}{2}})$. By formulas (2.16), (2.17), (3.60), (3.61), (3.62), Propositions 3.8, 3.9 and [16, §34], the solutions to the problems (3.68)-(3.71) uniformly converge on $[\sigma_1\tilde{\varepsilon}, \pi-\widetilde{\sigma}_1\tilde{\varepsilon}]\times[0,T]$ to the solution to the mixed boundary value problem (1.1)-(1.4):

$$\lim_{n\to\infty} u_{\lambda_n}(x,t) = u(x,t).$$

It remains to use (2.22) and asymptotic formula $\lambda_n \simeq n^2$ for the eigenvalues of the Sturm-Liouville problem. □

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