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Research Article

On prescribed mass solutions for a kind of nonlinear elliptic systems

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ABSTRACT. This paper is devoted to studying the following nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + \lambda u = \mu f(u) + h(x), \ \ x \in \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \int_{\mathbb{R}^2} u^2 dx = \sigma, \end{cases}$$

where $\sigma > 0$ is given, $\mu > 0$, $h(x)$ acts as a perturbation, f satisfies an exponential critical growth, $\lambda \in \mathbb{R}$ is a Lagrange multiplier. Without taking into account the Ambrosetti-Rabinowitz condition, we prove the existence of normalized ground state solutions in two cases.

Keywords: Normalized ground state solutions, nonlinear Schrödinger equations, exponential critical growth.

2020 Mathematics Subject Classification: 34C37, 35A15, 37J45, 47J30.

1. INTRODUCTION

The nonlinear Schrödinger equation is an important class of mathematical and physical problems, and a great deal of work has been done in the last few decades on the existence of solutions to its ground state. Li [20] proved the existence of normalized ground states by introducing the Sobolev subcritical approximation, and reprocess the Sobolev subcritical problem by using the Pohožaev constraint, the Schwarz symmetry rearrangement, and various scale transformations. Chen and Zou [12] proved the existence of normalized ground state solutions for the Schrödinger equation with a perturbation, this is the first contribution to the normalized solution equation with a perturbation. Thomas Bartsch and Sébastien de Valeriola [3] proved that there are infinitely many solutions to the nonlinear Schrödinger equation. For more results on the existence of ground state solutions to the combinatorial nonlinear Schrödinger equation, we refer to [24, 25, 26, 28] and references therein.

It is natural to ask what is the normalized ground state solution of the nonlinear Schrödinger equation for the combination of these factors? Therefore, we investigate normalized ground state solution for the following nonlinear Schrödinger equation with a perturbation

$$\begin{cases} -\Delta u + \lambda u = \mu f(u) + h(x), \ \ x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} u^2 dx = \sigma, \end{cases} \tag{1.1}$$

where $\sigma > 0$ is given, $\mu > 0$, $\lambda \in \mathbb{R}$ is a Lagrange multiplier and f satisfies an exponential critical growth when $N = 2$, and $f(u) = |u|^{2^*-2}u$ when $N \geq 3$ and $2^* = \frac{2N}{N-2}$. Solving the

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normalized ground state solution of equation (1.1), we consider the following energy functional $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ constrained on $S(\sigma)$

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^N} F(u)dx - \int_{\mathbb{R}^N} hudx, \tag{1.2}$$

where $F(u) := \int_0^u f(t)dt$ for $N = 2$, $F(u) := \frac{1}{2^*}|u|^{2^*}$ for $N \geq 3$, and

$$S(\sigma) := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 dx = \sigma \right\}.$$

Within this framework, every solution to (1.1) is a critical point of J on $S(\sigma)$. For the L^2 subcritical growth case of f, i.e., f has a growth $|u|^{p-1}$ with $p < 2_* := 2 + \frac{4}{N}$, J on $S(\sigma)$ is bounded below and the global minimum can be found by minimization methods [6, 9, 16, 27]. For the L^2 supercritical growth case for f, i.e., $p > 2_*$, J on $S(\sigma)$ is unbounded from below. For this case, in 1997, Jeanjean [18] used a mountain pass structure and $s \star u(\cdot) := e^{\frac{Ns}{2}} u(e^s \cdot)$ to study the L^2 supercritical problems. In [17], Ikoma and Tanaka established a deformation result for the L^2 normalized solutions and gave alternative proofs from the results of [3, 18]. We note the importance in these studies of the following conditions on f.

- (H_1) $f : \mathbb{R} \to \mathbb{R}$ is continuous and odd.
- (H_2) There exist constants $\gamma_1, \gamma_2 \in \mathbb{R}$ with $2_* < \gamma_1 < 2^*$, $2^* := +\infty$ for $N = 1, 2$ and $2^* := \frac{2N}{N-2}$ for $N \geq 3$, such that

$$0 < \gamma_1 F(t) \leq f(t)t \leq \gamma_2 F(t),\ \forall t \in \mathbb{R}\backslash\{0\}. \tag{1.3}$$

In [19], Jeanjean and Lu worked the normalized solutions of nonlinear Schrödinger equation by replacing the Ambrosetti-Rabinowitz conditions with the following conditions:

- (H_3) $\lim\limits_{|t|\to+\infty} \frac{F(t)}{|t|^{2_*}} = +\infty$.
- (H_4) $\frac{G(t)}{|t|^{2_*}}$ is strictly increasing on $(0, +\infty)$ and strictly decreasing on $(-\infty, 0)$, where $G(t) := f(t)t - 2F(t)$ and $g(t) = G^{'}(t)$.
  In particular, they studied the case of an exponential subcritical growth of f when $N = 2$, i.e., for all $\alpha > 0$.

$$\lim_{|t|\to+\infty} \frac{|f(t)|}{e^{\alpha t^2}} = 0. \tag{1.4}$$

In [7], Bieganowski and Mederski studied the existence of normalized ground state solutions to nonlinear Schrödinger equation when $N \geq 3$ and without Ambrosetti-Rabinowitz conditions, the results were determined with the following conditions:

- (H_5) $g(t)t \succ 2_* G(t)$.
  In other words, $g(t)t \geq 2_* G(t)$ for any $t \in \mathbb{R}$, and for any $\rho > 0$ there exists $|t| < \rho$ such that $g(t)t > 2_* G(t)$.

Recently, Jeanjean in [18] worked on space $H^1_{\mathrm{r}}(\mathbb{R}^N)$ and obtained some compactness results which enable to get over the lack of compactness of the Sobolev embedding in whole $\mathbb{R}^N$ when $q > 4$ if $N = 2$. Not assuming the Ambrosetti-Rabinowitz conditions, the existence of normalized ground state solution to the stationary nonlinear Schrödinger equation when $N = 2$ for any $\rho > 0$ is proved by Chang [11] using the Trudinger-Moser inequality in $\mathbb{R}^2$ and the constrained minimization method.

Inspired by the Sobolev critical situation and previous studies, in this paper we focuses on the exponential critical growth for $N = 2$ in (1.1) and without assuming the Ambrosetti-Rabinowttz conditions, which is a novelty for this type of problems. Combining the above papers and conditions, we make several assumptions on f:

(f_1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to+\infty} \frac{|f(t)|}{e^{\alpha t^2}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases}$$

(f_2) $\lim\limits_{|t|\to 0} \frac{f(t)}{|t|^3} = 0$.

(f_3) $g(t)t \geq 4G(t)$, $\forall t \in \mathbb{R}$.

(f_4) there exist $p > 4$ and $\eta > 0$ such that

$$\operatorname{sgn}(t) f(t) \geq \eta |t|^{p-1}, \ \forall t \in \mathbb{R},$$

where $\operatorname{sgn} : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ and

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{if} \quad t > 0, \\ -1 & \text{if} \quad t < 0. \end{cases}$$

Using some of the minimax arguments in [18, 19] and the conditions of (f_2) and (f_4), Alves et al. [1, 2] discussed the multiplicity and existence of normalized solutions to nonlinear Schrödinger equation when $N = 2$ and f is allowed to grow as an exponential critical, i.e., (f_1): there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to+\infty} \frac{|f(t)|}{e^{\alpha t^2}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases} \tag{1.5}$$

In particular, in [2], the authors showed the existence of normalized ground state solutions for nonlinear Schrödinger equation under some appropriate conditions on f and $\alpha_0 = 4\pi$ for $\rho \in (0, 1)$ of (1.5). In [4, 5], Bartsch and Soave proved the multiplicity and existence of normalized ground state solutions for nonlinear Schrödinger equation by the ideas of [15]. With the exception of (H_1) and (H_2), they also make the following assumption:

(H_6) G is of class C^1 and $g(t)t \geq 2_* G(t)$; for $N = 2$, $g(t)t \geq 4G(t)$, $\forall t \in \mathbb{R}$.

For the perturbation $h(x)$, we require it to have higher regularity. We introduce an auxiliary function

$$H(x) := \nabla h(x) \cdot x.$$

$H \in L^2(\mathbb{R}^2)$. Define the Nehari-Pohozaev-type functional as follows:

$$\mathcal{P}(u) := \int_{\mathbb{R}^2} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^2} G(u) dx + 2 \int_{\mathbb{R}^2} hu dx - \int_{\mathbb{R}^2} \langle \nabla h, x \rangle u dx, \ \forall u \in H^1(\mathbb{R}^2).$$

Set

$$\mathcal{B} := \{u \in H^1(\mathbb{R}^2)\backslash\{0\} : \mathcal{P}(u) = 0\}. \tag{1.6}$$

The assumption on h and H are needed:

(h_1) $h \in L^2(\mathbb{R}^2)$ is radial, $h(x) > 0$ on a set with positive measure and $H(x) > 2h(x)$.

(h_2) $h \in H^1(\mathbb{R}^2)$ and $h(x) = h(|x|) \geq 0$.

We first consider the mass-subcritical case: $1 < p < 4$. By the G-N inequality, the lower limit of functional J on $S(\sigma)$ is bounded. Note that $\mathcal{B}$ includes all the nontrivial solutions of (1.1). To obtain the normalized ground state solution of (1.1), as in [7, 23], we convert finding the minimizers for J on $S(\sigma)$ into finding the minimizers for J on $S(\sigma)\cap\mathcal{B}$. Therefore, we simply examine the following minimization problems:

$$c(\sigma):=\min_{u\in S(\sigma)\cap\mathcal{B}} J(u), \tag{1.7}$$

$$c(\sigma)_r:=\min_{u\in S(\sigma)\cap\mathcal{B}\cap H_r^1(\mathbb{R}^2)} J(u). \tag{1.8}$$

Definition 1.1. *The solutions of (1.1) are the minimizers of (1.7). If the solution $u_\sigma\in S(\sigma)\cap\mathcal{B}$ of (1.7) satisfies*

$$J(u_\sigma)=\inf\{J(u):u\in S(\sigma)\cap\mathcal{B},(J|_{S(\sigma)\cap\mathcal{B}})'(u)=0\}, \tag{1.9}$$

we call that it is the normalized ground state solution to (1.1).

We prove that u_σ can be found for any positive perturbation $h\in L^2(\mathbb{R}^2)$, and we have the following theorem.

Theorem 1.1. *Assume $N=2,p<4$, if f satisfies $(f_1)-(f_4)$ and h satisfies (h_1). Then $c(\sigma)$ is attained for any $\sigma>0$. Thus there exists a normalized ground state solution u to (1.1) and $u\in S(\sigma)\cap\mathcal{B}$. Furthermore, $u>0$.*

As for the mass-supercritical case: $p>4$, the functional J is boundless on $S(\sigma)$. We will prove that there still exists a mountain-pass structure after a small radial perturbation $h(x)$. We have the following theorem.

Theorem 1.2. *Assume $N=2,p>4$, if f satisfies $(f_1)-(f_4)$. Let $\sigma>0$ be fixed and let $\mathcal{L}(\mu,\sigma)>0$ be defined by*

$$\mathcal{L}(\mu,\sigma)<\frac{1}{2}\sigma^{-\frac{1}{2}}\Big(m_\infty(\sigma)+\frac{\mu}{2}\big(7|F(u_n)|_2-2|f(u_n)u_n|_2\big)\Big), \tag{1.10}$$

where $m_\infty(\sigma)$ define by Lemma 2.3. If h satisfies (h_2) and moreover

$$\max\{|h|_2,|H|_2\}<\mathcal{L}(\mu,\sigma), \tag{1.11}$$

then (1.1) has a mountain pass solution u and $u>0$.

This paper is structured as follows. In Section 2, we present some preliminary results. In Section 3 and Section 4, we give the proof of Theorem 1.1 and Theorem 1.2, respectively. Throughout this paper, we use the symbol $||\cdot||$ and $|\cdot|_p$ to indicate the H^1 norm and $L^p(\mathbb{R}^2)$ norm. Let $u^\star$ be the symmetric decreasing rearrangement of $u\in H^1$, and for $p\in[1,\infty)$

$$|\nabla u^\star|_2\le|\nabla u|_2, |u^\star|_p=|u|_p, \text{ and } \int_{\mathbb{R}^2} undx\le\int_{\mathbb{R}^2} u^\star n^\star dx, \tag{1.12}$$

and let $H^\star:=\{u\in H^1,u=u^\star\}$. For simplicity, in the following, we use C_1, C_2,..., which may be different positive constants in different places.

2. Preliminaries

In this section, we present some preliminary results.

Lemma 2.1 (Trudinger-Moser inequality by [8])**.** *If $\alpha>0$ and $u\in H^1(\mathbb{R}^2)$, we have the following inequality in $\mathbb{R}^2$*

$$\int_{\mathbb{R}^2}(e^{\alpha u^2}-1)dx<\infty.$$

Moreover, if $|u|_2 \leq D_1 < +\infty$ *with* $D_1 > 0$ *and* $\alpha < 4\pi$, $|\nabla u|_2 \leq 1$, *then there exists a constant* $C > 0$, *which depends just on* D_1 *and* α, *such that*

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)dx < C(D_1, \alpha).$$

Lemma 2.2 ([29]). *Assume that* $N = 2$, *for any* $u \in H^1(\mathbb{R}^2)$ *and* $p > 1$, *we have the following Gagliardo-Nirenberg inequality*

$$|u|_p \leq \mathcal{C}_{2,p}|\nabla u|_2^{\beta_p}|u|_2^{(1-\beta_p)}, \quad \beta_p := \frac{p-2}{p},$$

and the Sobolev inequality

$$\mathcal{S}_p|u|_p^2 \leq ||u||_{H^1}^2,\ u \in H^1,$$

where $\mathcal{S}_p$ *is the best embedding constant for the Sobolev inequality.*

Let $\sigma,\ \mu > 0$ and $N = 2$. The pair $(\lambda, u) \in \mathbb{R} \times H^1$ is the solution to the following equation:

$$\begin{cases} -\Delta u + \lambda u = \mu f(u) \ \text{ in } \ \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} u^2 dx = \sigma, \end{cases} \tag{2.13}$$

where f satisfies an exponential critical growth. The critical points of $J_\infty : H^1 \to \mathbb{R}$ can be considered as solutions of (2.13), and

$$J_\infty(u) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^2} F(u)dx, \tag{2.14}$$

restricted on $S(\sigma)$. For the L^2 subcritical growth case of f, i.e., $p < 4$, then J on $S(\sigma)$ is bounded below. For the L^2 supercritical growth case of f, i.e., $p > 4$, then J on $S(\sigma)$ is boundless. We introduce the Nehari-Pohozaev-type constraint for individual equations

$$P_\infty := \left\{u \in H^1 \backslash \{0\} : \mathcal{P}_\infty(u) = 0\right\}, \tag{2.15}$$

where

$$\mathcal{P}_\infty(u) := \int_{\mathbb{R}^2} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^2} G(u)dx. \tag{2.16}$$

We discuss the following minimization problems:

$$c_\infty(\sigma) = \inf_{u \in S(\sigma)} J_\infty(u),$$
$$m_\infty(\sigma) = \inf_{u \in S(\sigma) \cap P_\infty} J_\infty(u).$$

We have the following lemma.

Lemma 2.3. *Suppose* $\sigma > 0$, $p > 1$ *and* $p \neq 4$, *then* (2.13) *has only one solution* u_∞ *and is positive. Furthermore,*

(i) if $1 < p < 4$, *then*

$$c_\infty(\sigma) = \inf_{u \in S(\sigma)} J_\infty(u) = J_\infty(u_\infty) < 0, \tag{2.17}$$

more precisely, $c_\infty(\sigma)$ *and hence* $c_\infty(\sigma)$ *is strictly decreasing for* $\sigma > 0$.

(ii) if $p > 4$, *then*

$$m_\infty(\sigma) = \inf_{u \in S(\sigma) \cap P_\infty} J_\infty(u) = J_\infty(u_\infty) > 0, \tag{2.18}$$

more precisely, $m_\infty(\sigma)$ *and hence* $m_\infty(\sigma)$ *is strictly decreasing for* $\sigma > 0$.

Jeanjean [18] and Chang [11] et al. have studied the existence of a ground state solution to equation (2.13). The expression of $c_\infty(\sigma)$ and $m_\infty(\sigma)$ in Lemma 2.3 can also be found in Lemma 2.1 in [21].

Lemma 2.4. *Assume that* $(f_1)-(f_3)$ *hold. Then*

(i) $\frac{G(t)}{t^4}$ *is non-decreasing on* $(0,+\infty)$ *and non-increasing on* $(-\infty,0)$.

(ii) $I(t) := \frac{f(t)t-4F(t)}{t^2}$ *is non-decreasing on* $(0,+\infty)$ *and non-increasing on* $(-\infty,0)$.

(iii) $f(t)t \geq 4F(t)$, *for all* $t\in\mathbb{R}$.

Proof. Let $I_1(t) := \frac{G(t)}{t^4}$. Since $G(t) = f(t)t - 2F(t)$ is C^1 and $g(t) = G^{'}(t)$, we can see that $I_1^{'}(t) = \frac{g(t)t-4G(t)}{t^5}$. Since $f\in C^1$, by (f_3), we have $I_1^{'}(t)\leq 0$ for $t<0$, $I_1^{'}(t)\geq 0$ for $t>0$, hence (i) holds. Identical, $I^{'}(t) = \frac{g(t)t-4G(t)}{t^3}$, we have $I^{'}(t)\leq 0$ for $t<0$, $I^{'}(t)\geq 0$ for $t>0$, hence we obtain (ii). Let

$$I_2(t) = \begin{cases} \dfrac{f(t)t-4F(t)}{t^2} & \text{for } t\neq 0 \\ 0 & \text{for } t<0 \end{cases}.$$

By $(f_1)-(f_3)$, it can be shown that $I_2(t)$ is a continuous function on $\mathbb{R}$. It follows from (ii) that $I_2(t)\geq 0$ for all $t\in\mathbb{R}$, this means that (iii) holds. □

Lemma 2.5. *Assume* $(f_1)-(f_4)$ *hold and* $h>0$. *Set* $s\star u(\cdot) = e^s u(e^s\cdot)$. *Then, for any* $u\in H^1(\mathbb{R}^2)\backslash\{0\}$, *we conclude*

(i) $J(s\star u)\to 0^+$ *as* $s\to-\infty$.

(ii) $J(s\star u)\to -\infty$ *as* $s\to+\infty$.

Proof. By (f_1) and (f_2), there exist $l>p>4$ and $\alpha>\alpha_0$, such that there exists $C_\varepsilon>0$ for any $\varepsilon>0$, we have

$$|F(t)|\leq \varepsilon|t|^4 + C_\varepsilon|t|^{l+1}\left(e^{\alpha|t|^2}-1\right), \forall t\in\mathbb{R}.$$

Then, by the Hölder inequality and $(e^r-1)^p\leq e^{rp}-1$ for $p>1$, $r\geq 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^2}|F(u)|dx &\leq \varepsilon\int_{\mathbb{R}^2}|u|^4dx + C_\varepsilon\int_{\mathbb{R}^2}|u|^{l+1}\left(e^{\alpha|u|^2}-1\right)dx \\ &\leq \varepsilon\int_{\mathbb{R}^2}|u|^4dx + C_\varepsilon\left(\int_{\mathbb{R}^2}|u|^{2l+2}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}\left(e^{2\alpha|u|^2}-1\right)dx\right)^{\frac{1}{2}}.\end{aligned}$$

Define $w(x) := \sqrt{\frac{\alpha}{\pi}}e^s u(e^s x)$. Then

$$\int_{\mathbb{R}^2}|\nabla w|^2dx = e^{2s}\frac{\alpha}{\pi}\int_{\mathbb{R}^2}|\nabla u|^2dx \to 0 \text{ as } s\to-\infty$$

and

$$\int_{\mathbb{R}^2}|w|^2dx = \frac{\alpha}{\pi}\int_{\mathbb{R}^2}|u|^2dx.$$

By Lemma 2.1, we have that there exists $|s_0|$ sufficiently large and $s_0<0$ such that for all $s\leq s_0$,

$$\int_{\mathbb{R}^2}\left(e^{2\alpha|e^s u(e^s x)|^2}-1\right)dx = \int_{\mathbb{R}^2}\left(e^{2\pi|w|^2}-1\right)dx\leq C_1$$

for some $C_1>0$. Thus, for $s\leq s_0$,

$$\int_{\mathbb{R}^2}|F(e^s u(e^s x))|dx \leq \varepsilon e^{2s}\int_{\mathbb{R}^2}|u(x)|^4dx + C_\varepsilon C_1^{\frac{1}{2}}e^{ls}\left(\int_{\mathbb{R}^2}|u(x)|^{2l+2}dx\right)^{\frac{1}{2}}. \tag{2.19}$$

Then

$$
\begin{aligned}
J(s\star u) &= \frac{1}{2}\int_{\mathbb{R}^2}|\nabla(e^s u(e^s x))|^2dx - \mu\int_{\mathbb{R}^2}F(e^s u(e^s x))dx - e^s\int_{\mathbb{R}^2}hudx\\
&\geq \frac{1}{2}e^{2s}\int_{\mathbb{R}^2}|\nabla u|^2dx - \mu\varepsilon e^{2s}\int_{\mathbb{R}^2}|u(x)|^4dx - \mu C_\varepsilon C_1^{\frac{1}{2}}e^{ls}\left(\int_{\mathbb{R}^2}|u(x)|^{2l+2}dx\right)^{\frac{1}{2}} - e^s\int_{\mathbb{R}^2}hudx.
\end{aligned}
$$

By $l > p > 4$, taking $\varepsilon > 0$ small when necessary, we obtain that there exists $s_1 \leq s_0$ such that $J(s\star u) \geq 0$ for $s \leq s_1$. Moreover, according to (2.19), it can be derived

$$
\int_{\mathbb{R}^2} F\left(e^s u(e^s x)\right)dx \to 0 \text{ as } s \to -\infty,
$$

which means that $J(s\star u) \to 0^+$ for $s \to -\infty$, hence (i) holds.

For (ii), by (f_1), (f_2) and (f_4), we have that there exist $p > 4$, $t_0 \in (0,1]$, $C_2, C_3 > 0$ such that

$$
\begin{aligned}
&F(t) \geq C_2|t|^p, \forall |t| \geq t_0,\\
&F(t) \leq C_3|t|^2, \forall |t| \leq t_0.
\end{aligned}
$$

Then, for some $C_4 > 0$,

$$
\begin{aligned}
\int_{\mathbb{R}^2}F(u)dx &= \int_{\{|u(x)|\geq t_0\}}F(u)dx + \int_{\{|u_n|<t_0\}}F(u)dx\\
&\geq C_2\int_{\{|u(x)|\geq t_0\}}|u|^pdx - C_3\int_{\{|u(x)|<t_0\}}|u|^2dx\\
&= C_2\int_{\mathbb{R}^2}|u|^pdx - \int_{\{|u(x)|<t_0\}}\left[C_2|u|^p + C_3|u|^2\right]dx\\
&\geq C_2\int_{\mathbb{R}^2}|u|^pdx - C_4\int_{\mathbb{R}^2}|u|^2dx,
\end{aligned}
\tag{2.20}
$$

which means that

$$
J(s\star u) \leq \frac{1}{2}e^{2s}\int_{\mathbb{R}^2}|\nabla u|^2dx + \mu C_4\int_{\mathbb{R}^2}|u|^2dx - \mu C_2 e^{(p-2)s}\int_{\mathbb{R}^2}|u|^pdx - e^s\int_{\mathbb{R}^2}hudx.
$$

By $p > 4$, it can be seen that $J(s\star u) \to -\infty$ for $s \to +\infty$, hence we get (ii). □

Lemma 2.6. *Assume $(f_1)-(f_3)$ hold and $h \geq 0$. We have*

(i) for any $u \in H^1(\mathbb{R}^2)\setminus\{0\}$, we have that there exists $s_u \in \mathbb{R}$ such that $\mathcal{P}(s_u\star u) = 0$ and

$$
J(s_u\star u) \geq J(s\star u),\ \forall s \neq s_u,
\tag{2.21}
$$

furthermore, if $u \in \mathcal{B}$, then $J(u) = \max_{s\in\mathbb{R}} J(s\star u)$,

(ii) $S(\sigma)\cap\mathcal{B} \neq \emptyset$,

(iii) there exists $\rho_0 > 0$ such that $\inf_{u\in s(\sigma)\cap\mathcal{B}} ||\nabla u||_2 \geq \rho_0$.

Proof. By Lemma 2.5, we can prove that $J(s\star u)$ has a global maximum at some $s_u \in \mathbb{R}$. By $\frac{d}{ds}J(s\star u) = \mathcal{P}(s\star u)$, we get (i). Take $u \in S(\sigma)$, by (i), we have $s_u \in \mathbb{R}$ such that $s_u\star u \in \mathcal{B}$. Recall that $s_u\star u \in S(\sigma)$, then (ii) holds.

For the proof of (iii), we can paradoxically suppose that there exists a sequence $\{u_n\} \subset S(\sigma)\cap\mathcal{B}$ such that $||\nabla u_n||_2 \to 0$ as $n \to +\infty$ and then, according to Lemma 2.1, there exists $C_1 > 0$ such that

$$
\int_{\mathbb{R}^2}\left(e^{2\alpha|u_n|^2} - 1\right)dx \leq C_1.
$$

Using arguments similar to those in Lemma 2.5, by Lemma 2.2, we have that there exists $l > p > 4$, $C_2 > 0$, and $C_\varepsilon > 0$ for any $\varepsilon > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^2} |f(u_n)u_n|dx \leq & \varepsilon \int_{\mathbb{R}^2} |u_n|^4 dx + C_\varepsilon \left(\int_{\mathbb{R}^2} |u|^{2l+2}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} e^{2\alpha|u_n|^2} - 1\right)^{\frac{1}{2}} \\ \leq & \varepsilon C_2 \rho \int_{\mathbb{R}^2} |\nabla u_n|^2 dx + C_\varepsilon C_1^{\frac{1}{2}} C_2 \rho^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\nabla u_n|^2 dx\right)^{\frac{l}{2}}.\end{aligned}$$

Then taking $\varepsilon = \frac{1}{2C_2\rho}$, in view of $F(t) \geq 0, \forall t \in \mathbb{R}$, we infer that

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla u_n|^2 dx = & \int_{\mathbb{R}^2} [f(u_n)u_n - 2F(u_n)]\, dx \\ \leq & \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_n|^2 dx + C_\varepsilon C_1^{\frac{1}{2}} C_2 \rho^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\nabla u_n|^2 dx\right)^{\frac{l}{2}},\end{aligned}$$

which implies that $\frac{1}{2} \leq C_\varepsilon C_1^{\frac{1}{2}} C_2 \rho^{\frac{1}{2}} ||\nabla||_2^{l-2}$. Since $l > 4$, we get a contradiction. □

3. Proof of Theorem 1.1

In this section, we shall give the proof of Theorem 1.1.

Lemma 3.7. *Assume that* $2 < p < 4$, $(f_1) - (f_3)$ *hold and* $h(x)$ *satisfying* (h_1). *Then* $c(\sigma) = c(\sigma)_r$.

Proof. For any $s \in \mathbb{R}$ and $u \in S(\sigma)$, according to Lemma 2.4 (ii), we obtain

$$\begin{aligned}& J(s_u \star u) - \frac{1}{2}\mathcal{P}(s \star u) \\ =& \frac{\mu}{2} \int_{\mathbb{R}^2} \left[f\left(e^s u(e^s x)\right) e^s u(e^s x) - 4F\left(e^s u(e^s x)\right)\right] dx \\ & -2\int_{\mathbb{R}^2} h e^s u(e^s x) dx + \int_{\mathbb{R}^2} \langle \nabla h \cdot x\rangle e^s u(e^s x) dx \\ =& \frac{\mu}{2} \int_{\mathbb{R}^2} \frac{f(e^s u)e^s x - 4F(e^s u)}{e^{2s}u^2} u^2 dx - 2e^s \int_{\mathbb{R}^2} hu dx + e^s \int_{\mathbb{R}^2} \langle \nabla h \cdot x\rangle u dx.\end{aligned} \tag{3.22}$$

Clearly, $J(s_u \star u) - \frac{1}{2}\mathcal{P}(s \star u)$ is non-decreasing for $s \in \mathbb{R}$. Denote by $\{u_n\}$ a sequence of minima of J on $S(\sigma) \cap \mathcal{B}$ and by $\{u_n^*\}$ the Schwarz symmetrization of $\{u_n\}$. Then

$$\int_{\mathbb{R}^2} |\nabla u_n^*|^2 dx \leq \int_{\mathbb{R}^2} |\nabla u_n|^2 dx,$$

$$\int_{\mathbb{R}^2} |u_n^*|^2 dx = \int_{\mathbb{R}^2} |u_n|^2 dx,$$

$$\int_{\mathbb{R}^2} F(u_n^*) dx = \int_{\mathbb{R}^2} F(u_n) dx, \int_{\mathbb{R}^2} f(u_n^*)u_n^* dx = \int_{\mathbb{R}^2} f(u_n)u_n dx.$$

$$\int_{\mathbb{R}^2} h u_n^* dx = \int_{\mathbb{R}^2} h u_n dx, \int_{\mathbb{R}^2} \langle \nabla h \cdot x\rangle u_n^* dx = \int_{\mathbb{R}^2} \langle \nabla h \cdot x\rangle u_n dx.$$

It can be seen that $u_n^* \in S(\sigma) \cap H_r^1(\mathbb{R}^2)$ and $\mathcal{P}(u_n^*) \le \mathcal{P}(u_n) = 0$. By **(f₃)**, there exists $s_u^* := s_u^*(u_n^*) \le 0$ such that $\mathcal{P}(s_u^* \star u_n^*) = 0$. Hence, according to Lemma 2.6, we get

$$\begin{aligned} c(\sigma) &\le c(\sigma)_r \\ &\le J(s_u^* \star u_n^*) \\ &= J(s_u^* \star u_n^*) - \frac{1}{2}\mathcal{P}(s_u^* \star u_n^*) \\ &\le J(u_n^*) - \frac{1}{2}\mathcal{P}(u_n^*) \\ &= \frac{\mu}{2}\int_{\mathbb{R}^2} [f(u_n^*)u_n^* - 4F(u_n^*)]\,dx - 2\int_{\mathbb{R}^2} hu_n^* dx + \int_{\mathbb{R}^2} \langle \nabla h \cdot x\rangle u_n^* dx \\ &= \frac{\mu}{2}\int_{\mathbb{R}^2} [f(u_n)u_n - 4F(u_n)]\,dx - 2\int_{\mathbb{R}^2} hudx + \int_{\mathbb{R}^2} \langle \nabla h \cdot x\rangle udx \\ &= J(u_n) = c(\sigma) + o_n(1), \end{aligned} \tag{3.23}$$

which means that $c(\sigma) = c(\sigma)_r$. □

Lemma 3.8. *Suppose $2 < p < 4$, $(f_1) - (f_3)$ hold and $h(x)$ satisfying (h_1). Assume that $\{u_n\} \subset H_r^1(\mathbb{R}^2)$ is a sequence of bounded minimisations of J on $S(\sigma)$ and there exists $u_0 \in H_r^1(\mathbb{R}^2)$ such that $u_n(x) \to u_0(x)$, $x \in \mathbb{R}^2$ and*

$$\int_{\mathbb{R}^2} F(u_n)dx \to \int_{\mathbb{R}^2} F(u_0)dx, \tag{3.24}$$

$$\int_{\mathbb{R}^2} f(u_n)u_n dx \to \int_{\mathbb{R}^2} f(u_0)u_0 dx. \tag{3.25}$$

Then from $c(\sigma) > 0$ and u_0, $c(\sigma)$ is attained.

Proof. First, we claim that the following result holds under $(f_1) - (f_3)$,

$$\tau(u) := \int_{\mathbb{R}^2} [f(u)u - 4F(u)]dx > 0,\ \forall u \in H_r^1(\mathbb{R}^2)\setminus\{0\}. \tag{3.26}$$

Indeed, according to the Strauss radial lemma in [6], we can suppose that u is continuous. Because of $u \in H_r^1(\mathbb{R}^2)$, we obtain $|u(x)| \to 0$ for $|x| \to +\infty$. According to Lemma 2.4 (iii), we obtain $\tau(u) \ge 0$. If $\tau(u) = 0$, then for all $x \in \mathbb{R}^2$, $f(u(x))u(x) - 4F(u(x)) = 0$. Then, there exists an open interval I such that $0 \in \bar{I}$ and $f(u)u - 4F(u) = 0$ for $u(x) \in \bar{I}$. By direct computations, we obtain that $F(u) = C|u|^4$ for some $C > 0$ and $u \in \bar{I}$. But by (f_1) and (f_2), we know that this is a paradox. Hence (3.26) holds.

Since $\{u_n\} \subset S(\sigma)$, by (3.24)-(3.25) and the Fatou lemma, we have

$$\int_{\mathbb{R}^2} u_0^2 dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^2} u_n^2 dx \le \sigma$$

and

$$\int_{\mathbb{R}^2} |\nabla u_0|^2 dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla u_n|^2 dx = \liminf_{n\to+\infty} \int_{\mathbb{R}^2} T(u_n)dx = \int_{\mathbb{R}^2} T(u_0)dx,$$

where $T(u) = \mu G(u) - 2hu + \langle \nabla h \cdot x\rangle u$. If $\int_{\mathbb{R}^2} |\nabla u_0|^2 dx < \int_{\mathbb{R}^2} T(u_0)dx$, defining $t_0 := \left(\frac{\int_{\mathbb{R}^2} |\nabla u_0|^2 dx}{\int_{\mathbb{R}^2} T(u_0)dx}\right)^{\frac{1}{2}}$, we have $t_0 \in (0,1)$. Note that

$$\mathcal{P}(u_0(\frac{x}{t_0})) = \int_{\mathbb{R}^2} |\nabla u_0|^2 dx - t_0^2 \int_{\mathbb{R}^2} T(u_0)dx = 0$$

and

$$\int_{\mathbb{R}^2} \big(u_0(\frac{x}{t_0})\big)^2 dx = t_0^2 \int_{\mathbb{R}^2} u_0^2 dx < \sigma,$$

it follows that $u_0(\frac{\cdot}{t_0}) \in S(\sigma) \cap \mathcal{B}$. By (3.24)-(3.26) and Fatou lemma, we obtain

$$\begin{aligned} c(\sigma) \leq J\big(u_0(\frac{x}{t_0})\big) &= \frac{1}{2}\int_{\mathbb{R}^2} \big|\nabla\big(u_0(\frac{x}{t_0})\big)\big|^2 dx - \mu \int_{\mathbb{R}^2} F\big(u_0(\frac{x}{t_0})\big)dx - \int_{\mathbb{R}^2} hu_0(\frac{x}{t_0})dx \\ &= \frac{1}{2}\int_{\mathbb{R}^2} \big|\nabla u_0\big|^2 dx - \mu t_0^2 \int_{\mathbb{R}^2} F(u_0)dx - t_0^2 \int_{\mathbb{R}^2} hu_0 dx \\ &= \frac{1}{2}t_0^2 \int_{\mathbb{R}^2} T(u_0)dx - \mu t_0^2 \int_{\mathbb{R}^2} F(u_0)dx - t_0^2 \int_{\mathbb{R}^2} hu_0 dx \\ &= \frac{1}{2}t_0^2 \int_{\mathbb{R}^2} \big[T(u_0) - 2\mu F(u_0) - 2hu_0\big]dx \\ &< \frac{1}{2}\int_{\mathbb{R}^2} \big[\mu\big(f(u_0) - 4F(u_0)\big) - 4hu_0 + \langle \nabla h \cdot x\rangle u_0\big]dx \\ &= \liminf_{n\to+\infty} \frac{1}{2}\int_{\mathbb{R}^2} \big[\mu\big(f(u_n) - 4F(u_n)\big) - 4hu_n + \langle \nabla h \cdot x\rangle u_n\big]dx \\ &\leq \liminf_{n\to+\infty} J(u_n) = c(\sigma), \end{aligned} \tag{3.27}$$

which gives a contradiction. Hence $\int_{\mathbb{R}^2} |\nabla u_0|^2 dx = \int_{\mathbb{R}^2} T(u)dx = \int_{\mathbb{R}^2} \big[\mu G(u) - 2hu + \langle \nabla h \cdot x\rangle u\big]dx$. Thus $u_0 \in S(\sigma) \cap \mathcal{B}$ and $\int_{\mathbb{R}^2} |\nabla u_n|^2 dx \to \int_{\mathbb{R}^2} |\nabla u_0|^2 dx$. Using Lemma 3.7, we get $J(u_0) = c(\sigma)$. Now, we prove $c(\sigma) > 0$. In fact, by $(f_1)-(f_3)$ and $u_0 \in S(\sigma) \cap \mathcal{B} \cap H_r^1(\mathbb{R}^2)$, using (3.26), Lemma 3.7 and (h_1), we get

$$c(\sigma) = J(u_0) = \frac{\mu}{2}\int_{\mathbb{R}^2} \big(f(u_0)u_0 - 4F(u_0)\big)dx - 2\int_{\mathbb{R}^2} hu_0 dx + \int_{\mathbb{R}^2} \langle \nabla h \cdot x\rangle u_0 dx > 0.$$

□

Lemma 3.9 ([12]). *Assume that $p < 4$, $(f_1)-(f_3)$ hold and $h(x)$ satisfying (h_1). Let $\{u_n\}_{n\geqslant 1} \subset S(\sigma) \cap \mathcal{B}$ be a sequence for $c(\sigma)$ such that $u_n \rightharpoonup u_0$ in H^1 and let $\sigma_1 := |u_0|_2^2$. If $\sigma_1 < \sigma$, then there exists $y_n \subseteq \mathbb{R}^2$ and $\mu_0 \in H^1\backslash\{0\}$ such that*

$$|y_n| \to \infty, u_n(\cdot + y_n) \rightharpoonup \mu_0 \text{ in } H^1 \tag{3.28}$$

$$\lim_{n\to\infty} |u_n - u_0 - \mu(\cdot - y_n)|_2^2 = 0, \tag{3.29}$$

and $\sigma = \sigma_1 + \sigma_2$, where $\sigma_2 := |\mu_0|_2^2$. Moreover, the following hold

$$J(u_0) = c(\sigma_1), J_\infty(\mu_0) = c_\infty(\sigma_2) \tag{3.30}$$

and

$$c(\sigma) = c(\sigma_1) + c_\infty(\sigma_2). \tag{3.31}$$

Since $\{u_n\} \subset S(\sigma) \cap \mathcal{B}$ is the minimizing sequence of $c(\sigma)$, we have $dJ|_{S(\sigma)\cap\mathcal{B}}(u_n) \to 0$ and there exists a real numbers sequence of $\{\lambda_n\}$ such that

$$J^{'}(u_n)[\eta] + \lambda_n \int_{\mathbb{R}^2} u_n \eta dx \to 0 \tag{3.32}$$

for every $\eta \in H^1$. Now, we have the following lemma.

Lemma 3.10. *Under the assumptions of Lemma 3.9, if there exists $\sigma_1 < \sigma$ and $\sigma_1 = \int_{\mathbb{R}^2} u_0^2 dx$, then u_0 and μ_0 satisfy*

$$(3.33)\qquad \begin{cases} -\Delta u_0 + \bar{\lambda} u_0 = \mu f(u_0) + h \ \ in\ \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} u_0^2 dx = \sigma_1, \end{cases}$$

and

$$(3.34)\qquad \begin{cases} -\Delta \mu_0 + \bar{\lambda} \mu_0 = \mu f(\mu_0) \ \ in\ \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} \mu_0^2 dx = \sigma - \sigma_1. \end{cases}$$

Moreover, $\mu_0 > 0$ in $\mathbb{R}^2$.

Proof. By the assumption of Lemma 3.9, there exists a bounded sequence $\{\lambda_n\}$, and there is a subsequence of $\{\lambda_n\}$ which converges to $\bar{\lambda}$. Using u_n as the test functions in (3.32), one can be found the values of λ_n and

$$(3.35)\qquad -\lambda_n \sigma = \int_{\mathbb{R}^2} |\nabla u_n|^2 dx - \int_{\mathbb{R}^2} F(u_n) dx - \int_{\mathbb{R}^2} h(u_n) dx + o(1)$$

with $o(1) \to 0$ as $n \to \infty$. Therefore, the boundedness of $\{\lambda_n\}$ comes out of the boundedness of $\{u_n\}$ in H^1. Up to a subsequence, we assume that $\lambda_n \to \bar{\lambda}$. Since (3.32), if there exists $\sigma_1 < \sigma$, by [12], it follows that

$$(3.36)\qquad \begin{cases} -\Delta \mu_0 + \bar{\lambda} \mu_0 = \mu f(\mu_0) \ \ \text{in}\ \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} \mu_0^2 dx = \sigma - \sigma_1. \end{cases}$$

Since $\mu_0 > 0$ and (H_2), we have

$$-\Delta \mu_0 + (\bar{\lambda})_+ \mu_0 \geq -\Delta \mu_0 + \bar{\lambda} \mu_0 = \mu f(\mu_0) \geq 0.$$

Using the strong maximum principle and $\sigma - \sigma_1 > 0$, we obtain that $\mu_0 > 0$. By a similar proof, it can be shown that u_0 satisfies (3.33). □

Proof of Theorem 1.1. By Lemma 3.8, the minimizing sequence $\{u_n\}$ satisfy $u_n \to u_0$ and $c(\sigma) = J(u_0)$, so according to (3.32), we obtain that

$$(3.37)\qquad \begin{cases} -\Delta u_0 + \bar{\lambda} u_0 = \mu f(u_0) + h \ \ \text{in}\ \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} u_0^2 dx = \sigma. \end{cases}$$

Observe that $h(x) \geq 0$, then according to the maximum principle, $u_0 > 0$ and we complete the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

In this section, for $p > 4$, we shall give the prove of Theorem 1.2. In the following, we will prove that J on $S(\sigma)$ has a sort of mountain-pass geometry.

Define $\varphi_0 := \frac{\sigma}{\pi} e^{-|x|^2}$, $\forall x \in \mathbb{R}^2$. Clearly, $\int_{\mathbb{R}^2} |\varphi_0|^2 dx = \sigma$, which implies that $\varphi_0 \in S(\sigma)$. By (f_4), for any $s \in \mathbb{R}$ and $\eta > 0$, we obtain

$$(4.38)\qquad J(s \star \varphi_0) \leq \frac{1}{2} e^{2s} \int_{\mathbb{R}^2} |\nabla \varphi_0|^2 dx - \frac{\eta}{p} e^{(p-2)s} \mu \int_{\mathbb{R}^2} |\varphi_0|^p dx - e^s \int_{\mathbb{R}^2} h \varphi_0 dx.$$

To understand the geometry of the generalized function J on $S(\sigma)$, it will be useful to consider the function $\zeta : \mathbb{R}^+ \to \mathbb{R}$ defined by the following equation:

$$\zeta(t) := \frac{1}{2}A_1 t^2 - \frac{\eta}{p}A_2 t^{p-2} - |h|_2 \sigma^{\frac{1}{2}} t,$$

where

$$A_1 := \int_{\mathbb{R}^2} |\nabla \varphi_0|^2 dx, \ A_2 := \mu \int_{\mathbb{R}^2} |\varphi_0|^p dx.$$

Since $0 < 1 < 2 < p-2$, we obtain that $\zeta(0^+) = 0^-$ and $\zeta(+\infty) = -\infty$. The role of assumption (1.11) is clarified by the following lemma.

Lemma 4.11. *Suppose that* $(f_1)-(f_4)$ *hold. By* (h_2)*, if*

$$\max\{|h|_2, |H|_2\} < \mathcal{L}(\mu, \sigma),$$

then the function $\zeta(t)$ *has a global strict maximum at positive level and a local strict minimum at negative level. Moreover, there exist* $0 < T_1 < T_2$*, both depending on* σ*, such that* $\zeta(T_1) = 0 = \zeta(T_2)$ *and* $\zeta(t) > 0$ *if and only if* $t \in (T_1, T_2)$*.*

Proof. For $t > 0$, we see that $\zeta(t) > 0$ if and only if

$$\xi(t) > |h|_2 \sigma^{\frac{1}{2}},$$

where $\xi(t) := \frac{1}{2}A_1 t - \frac{\eta}{p}A_2 t^{p-3}$. Clearly, $A_1, A_2 > 0$. Then

$$\xi'(t) = \frac{1}{2}A_1 - \frac{\eta(p-3)}{p}A_2 t^{p-4}.$$

Observe that $p-3 > 1 > 0$, then $\xi(t)$ has a unique critical point $\bar{t}$ on $(0, +\infty)$, which is a global maximum point at positive level. Indeed, the expression of $\bar{t}$ is

$$\bar{t} = \left(\frac{A_1 p}{2\eta(p-3)A_2}\right)^{\frac{1}{p-4}} \tag{4.39}$$

and the maximum value of $\xi(t)$ is

$$\xi(\bar{t}) = \eta^{\frac{-1}{p-4}} A_3 > 0$$

with

$$A_3 := A_1^{\frac{p-3}{p-4}} \left(\frac{p}{2(p-3)A_2}\right)^{\frac{1}{p-4}} \frac{p-4}{2(p-3)} > 0.$$

Consequently, if (1.11) holds, then $\xi(\bar{t}) > |h|_2 \sigma^{\frac{1}{2}}$, thus the equation $\zeta = 0$ has two roots T_1, T_2 and ζ is positive on (T_1, T_2). Moreover, ζ has a global maximum point t_2. Based on the expression of ζ, we can derive that ζ also has a local minimum point t_1 in $(0, T_1)$ with a negative level. □

Set

$$R_\vartheta := \{u \in S(\sigma) : |\nabla u|_2 < \vartheta\},$$
$$J^c := \{u \in S(\sigma) : J(u) < c\}.$$

By Lemma 2.5 and Lemma 4.11, there exists a $\vartheta_1 > 0$ small enough, such that

$$J(u) < \frac{1}{2}\zeta(t_2), \text{for any } u \in R_{\vartheta_1}. \tag{4.40}$$

Moreover, $J^{\zeta(t_1)} \subset \{|\nabla u|_2 > T_2\}$ since $J(u) \geq \zeta(|\nabla u|_2)$. Now, a mountain pass structure of J on manifold $S(\sigma)$ is obtained. Let

$$\Gamma := \left\{ \gamma \in C([0,1], S(\sigma) : \gamma(0) \in R_{\vartheta_1}, \gamma(1) \in J^{\zeta(t_1)} \right\}, \tag{4.41}$$

and the mountain pass value is

$$m(\sigma) := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)). \tag{4.42}$$

The classical mountain-pass theorem shows that there exists a P-S sequence $\{u_n\}_{n \geq 1}$ which satisfy

(i) $J(u_n) \to m(\sigma)$,
(ii) $J^{'}|_{S(\sigma)}(u_n) \to 0$.

However, we cannot prove that $\{u_n\}_{n \geq 1}$ is bounded in H^1. To resolve this difficulty, we introduce an auxiliary function

$$\tilde{J}(t, u) := J(t \star u).$$

The corresponding minimax of $\tilde{J}$ on $\mathbb{R} \times S(\sigma)$ is structured as follows

$$\tilde{\Gamma} := \left\{ \gamma = (\gamma_1, \gamma_2) \in C([0,1], \mathbb{R} \times S(\sigma) : \gamma(0) \in (0, R_{\vartheta_1}), \gamma(1) \in (0, J^{\zeta(t_1)}) \right\},$$

and its minimax value is

$$\tilde{m}(\sigma) := \inf_{\gamma \in \tilde{\Gamma}} \max_{t \in [0,1]} \tilde{J}(\gamma(t)). \tag{4.43}$$

Using the method introduced by Jeanjean in [18], one can derive that $\tilde{m}(\sigma) = m(\sigma)$. Recall that $h(x) \geq 0$ and $h(x) = h(|x|)$, we have

$$\tilde{J}(t, |u|^*) \leq \tilde{J}(t, u). \tag{4.44}$$

Thus,

$$\tilde{m}(\sigma) = \tilde{m}_r(\sigma) := \inf_{\gamma \in \tilde{\Gamma}_r} \max_{t \in [0,1]} \tilde{J}(\gamma(t)), \tag{4.45}$$

where

$$\tilde{\Gamma}_r := \left\{ \gamma = (\gamma_1, \gamma_2) \in C([0,1], \mathbb{R} \times S_r(\sigma) : \gamma(0) \in (0, R_{\vartheta_1}), \gamma(1) \in (0, J^{\zeta(t_1)}) \right\},$$

and $S_r(\sigma) = S(\sigma) \cap H_r^1$. Let $\mathcal{F} = \{\gamma([0,1]) : \gamma \in \Gamma\}$. Then, $\mathcal{F}$ is a cohomology stable family of compact subsets of $\mathbb{R} \times S_r(\sigma)$ with extended closed boundary $\{0\} \times \bar{R}_{\vartheta_1} \cup \{0\} \times J^{\zeta(t_1)}$ (by the terminology in [15, Sect.5]), and the superlevel set $\left\{ \tilde{J} \geq m(\sigma) \right\}$ is a dual set for $\mathcal{F}$, which implies that the $\mathcal{F} = \{\gamma([0,1]) : \gamma \in \Gamma\}$ satisfies the assumptions of [15, Theorem 5.2]. Hence, combining with (4.44), we obtain a minimisation sequence $\{\gamma_n([0,1]), \gamma_n(t) = (\gamma_{1,n}(t), \gamma_{2,n}(t))\}$ for $m(\sigma)$ such that $\gamma_{1,n}(t) = 0, \gamma_{2,n}(t) \in S(\sigma) \cap H^\star, \forall\, t \in [0,1]$, there exists a sequence $(s_n, \tilde{u}_n) \subset \mathbb{R} \times S(\sigma) \cap H^\star$ such that when $n \to \infty$, $\tilde{J}(s_n, \tilde{u}_n) \to m(\sigma)$ and

$$\partial_s \tilde{J}(s_n, \tilde{u}_n) \to 0, \ ||\partial_n \tilde{I}(s_n, \tilde{u}_n)||_{T_{u_n} S_r(\sigma)} \to 0, \tag{4.46}$$

$$|s_n| + dist(\tilde{u}_n, \gamma_{2,n}[0,1]) \to 0. \tag{4.47}$$

Let $u_n = s_n \star \tilde{u}_n \in S_r(\sigma)$. By (4.47), it follows that $\{s_n\}$ is bounded and $u_n^- \to 0$ a.e. in $\mathbb{R}^2$. Moreover, (4.46) implies that

$$\mathcal{P}(u_n) = \partial_s \tilde{J}(s_n, \tilde{u}_n) \to 0,$$

and for any $\phi \in T_{u_n} S_r(\sigma)$,

$$J^{'}(u_n)[\phi] = \partial_u \tilde{J}(s_n, \tilde{u}_n)[(-s_n) \star \phi] = o(1)||(-s_n) \star \phi||_{H_r^1} = o(1)||\phi||_{H_r^1}. \tag{4.48}$$

In summary, we can get the following lemma.

Lemma 4.12. *Assume that h satisfies (1.11), then there exists a radial P-S sequence $\{u_n\}$ of $J|_{S_r(\sigma)}$, such that*

$$J(u_n) \to c(\sigma), \tag{4.49}$$

$$J^{'}|_{S_r(\sigma)} \to 0, \tag{4.50}$$

$$\mathcal{P}(u_n) \to 0, \tag{4.51}$$

as $n \to \infty$, where

$$\mathcal{P}(u) := \int_{\mathbb{R}^2} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^2} G(u) dx + 2 \int_{\mathbb{R}^2} h u dx - \int_{\mathbb{R}^2} \langle \nabla h \cdot x \rangle u dx, \ \forall u \in H^1(\mathbb{R}^2).$$

We note that (4.50) implies that there exists $\{\lambda_n\}_{n \geq 1}$, such that

$$J^{'}(u_n)[\eta] + \lambda_n \int_{\mathbb{R}^2} u_n \eta dx \to 0. \tag{4.52}$$

For any $\phi \in C_{0,r}^{\infty}(\mathbb{R}^2)$. By the assumption (1.11), we will show that the Lagrange multipliers $\{\lambda_n\}$ and P-S sequence are bounded in the following.

Lemma 4.13. *Assume that h satisfies (1.11). Let $\{u_n\} \subset S(\sigma)$ be a radial non-negative P-S sequence satisfies (4.49) $-$ (4.51), then $\{u_n\}$ is bounded in H^1. Furthermore, $\{\lambda_n\}$ in (4.52) are bounded.*

Proof. According to (4.49), $\{u_n\} \subset S(\sigma)$ as a P-S sequence, we have

$$m(\sigma) = J(u_n) + o(1) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_n|^2 dx - \mu \int_{\mathbb{R}^2} F(u_n) dx - \int_{\mathbb{R}^2} h u_n dx + o(1). \tag{4.53}$$

Therefore $\{u_n\}$ is bounded in H^1 since $h \in H^1$ and $|\nabla h \cdot x|_2 < \infty$. We evaluate $\{\lambda_n\}$. Using $\{u_n\}$ as the test function for (4.52), we can conclude that

$$o(1)||u_n||_{H^1} = \int_{\mathbb{R}^2} |\nabla u_n|^2 dx - \mu \int_{\mathbb{R}^2} F(u_n) dx - \int_{\mathbb{R}^2} h u_n dx + \lambda_n \sigma. \tag{4.54}$$

So

$$|\lambda_n| = \frac{1}{\sigma} \left| o(1)||u_n||_{H^1} - \int_{\mathbb{R}^2} |\nabla u_n|^2 dx + \mu \int_{\mathbb{R}^2} F(u_n) dx + \int_{\mathbb{R}^2} h u_n dx \right| < +\infty. \tag{4.55}$$

Therefore, $\{\lambda_n\}$ are also bounded. □

Proof of Theorem 1.2. We show the compactness of the P-S sequence. We first show that $\{\lambda_n\}$ has lower bounded. Indeed, under (4.51) and (4.52), we have

$$\lambda_n \sigma = \lambda_n \int_{\mathbb{R}^2} u_n^2 dx = - \int_{\mathbb{R}^2} |\nabla u_n|^2 dx + \mu \int_{\mathbb{R}^2} F(u_n) dx + \int_{\mathbb{R}^2} h u_n dx$$

$$= \mu \int_{\mathbb{R}^2} \big(3F(u_n) - f(u_n)u_n\big)dx + 3\int_{\mathbb{R}^2} hu_n dx - \int_{\mathbb{R}^2} \langle \nabla h \cdot x\rangle\, u_n dx + o(1). \tag{4.56}$$

We also have that

$$\begin{aligned} m(\sigma) &= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2 dx - \mu\int_{\mathbb{R}^2} F(u_n)dx - \int_{\mathbb{R}^2} hu_n dx + o(1) \\ &= \frac{1}{2}\Big[\mu\int_{\mathbb{R}^2} \big(f(u_n)u_n - 4F(u_n)\big)dx - 4\int_{\mathbb{R}^2} hu_n dx + \int_{\mathbb{R}^2} \langle \nabla h \cdot x\rangle\, u_n dx\Big] + o(1). \end{aligned} \tag{4.57}$$

We prove that for any $\sigma > 0$, there holds

$$m(\sigma) \geq m_\infty(\sigma) - |h|_2\sigma^{\frac{1}{2}} \tag{4.58}$$

by (1.11), we can conclude that

$$\begin{aligned} \lambda_n\sigma + o(1) &= 2m(\sigma) + \mu\int_{\mathbb{R}^2} \big(7F(u_n) - 2f(u_n)u_n\big)dx + 7\int_{\mathbb{R}^2} hu_n dx - 2\int_{\mathbb{R}^2} \langle\nabla h \cdot x\rangle u_n dx \\ &\geq 2(m_\infty(\sigma) - |h|_2\sigma^{\frac{1}{2}}) + \mu\big(7|F(u_n)|_2 - 2|f(u_n)u_n|_2\big) - 2|\nabla h\cdot x|_2\sigma^{\frac{1}{2}} \\ &\geq 2m_\infty(\sigma) + \mu\big(7|F(u_n)|_2 - 2|f(u_n)u_n|_2\big) - 4\mathcal{L}(\mu,\sigma)\sigma^{\frac{1}{2}} \\ &\geq 2\sigma^{\frac{1}{2}}\left(\sigma^{-\frac{1}{2}}\left(m_\infty(\sigma) + \frac{\mu}{2}\big(7|F(u_n)|_2 - 2|f(u_n)u_n|_2\big)\right) - 2\mathcal{L}(\mu,\sigma)\right). \end{aligned} \tag{4.59}$$

Since

$$\mathcal{L}(\mu,\sigma) < \frac{1}{2}\sigma^{-\frac{1}{2}}\left(m_\infty(\sigma) + \frac{\mu}{2}\big(7|F(u_n)|_2 - 2|f(u_n)u_n|_2\big)\right). \tag{4.60}$$

Note that for any $\gamma(t) \subset \Gamma$, we have that there exists $t^* \in \mathbb{R}^+$ such that

$$J_\infty(t^* \star \gamma(1)) = \max_{t>0} J_\infty(\gamma(t)). \tag{4.61}$$

Furthermore, because of the structure of Γ in (4.41), we derive that $t^* < 1$. Hence if $J(\gamma(\bar{t})) = \max_{t\in[0,1]} J(\gamma(t))$, then

$$\begin{aligned} \max_{t\in[0,1]} J(\gamma(t)) &= J(\gamma(\bar{t})) \\ &\geq J(t^* \star \gamma(1)) \\ &\geq J_\infty(t^* \star \gamma(1)) - |h|_2\sigma^{\frac{1}{2}} \\ &= \max_{t>0} J_\infty(\gamma(t)) - |h|_2\sigma^{\frac{1}{2}}. \end{aligned} \tag{4.62}$$

Because of the γ is arbitrary, we obtain that (4.58) holds. By the constructions of $\{u_n\}$, we have that $\{u_n\} \in S(\sigma) \cap H^\star$, therefore, together with Lemma 4.13, there exists $\tilde{u} \in S(\sigma)$ such that

$$u_n \rightharpoonup \bar{u} \text{ in } H^1,\ u_n \to \bar{u} \text{ in } L^p(\mathbb{R}^2). \tag{4.63}$$

Notice that $\{\lambda_n\}$ has lower bound and is positive, then up to a subsequence, we suppose that $\lambda_n \to \bar{\lambda} > 0$. Because of the weak convergence of $\{u_n\}$ in H^1 and the strong convergence in $L^p(\mathbb{R}^2)$, using (4.52), we obtain

$$\begin{aligned} o(1) &= (dJ(u_n) - dJ(\bar{u}))[u_n - \bar{u}] + \bar{\lambda}\int_{\mathbb{R}^2} |u_n - \bar{u}|^2 dx \\ &= \int_{\mathbb{R}^2} |\nabla u_n - \nabla\bar{u}|^2 dx + \bar{\lambda}\int_{\mathbb{R}^2} |u_n - \bar{u}|^2 dx + o(1) \end{aligned} \tag{4.64}$$

with $o(1) \to 0$ as $n \to \infty$. Since $\bar{\lambda} > 0$, thus we have that $\int_{\mathbb{R}^2} |\nabla u_n - \nabla\bar{u}|^2 dx \to 0$ and $\int_{\mathbb{R}^2} |u_n - \bar{u}|^2 dx \to 0$. Thus we get that in H^1, u_n strongly converges to $\bar{u}$. □

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References

[1] C. O. Alves, C. Ji and O. H. Miyagaki: *Multiplicity of normalized solutions for a Schrödinger equation with critical growth in* $\mathbb{R}^N$, arXiv:2103.07940 (2021).

[2] C. O. Alves, C. Ji and O. H. Miyagaki: *Normalized solutions for a Schrödinger equation with critical growth in* $\mathbb{R}^N$, Calc. Var. Partial Differ. Equ., **61** (18) (2022), 1–24.

[3] T. Bartsch, S. De Valeriola: *Normalized solutions of nonlinear Schrödinger equations*, Arch. Math., **100** (2013), 75–83.

[4] T. Bartsch, N. Soave: *A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems*, J. Funct. Anal., **272** (2017), 4998–5037.

[5] T. Bartsch, N. Soave: *Correction to "A natural constraint approach to normalized solutions on nonlinear Schrödinger equations and systems"*, J. Funct. Anal., **272** (2017), 4998–5037.

[6] H. Berestycki, P. L. Lions: *Nonlinear scalar field equations I: existence of a ground state*, Arch. Rat. Mech. Anal., **82** (1983), 313–345.

[7] B. Bieganowski, J. Mederski: *Normalized ground states of the nonlinear Schrödinger equation with at least mass critical growth*, J. Funct. Anal., **280** (11) (2021), Article ID: 108989.

[8] D. M. Cao: *Nontrivial solution of semilinear elliptic equation with critical exponent in* $\mathbb{R}^2$, Comm. Partial Differential Equations **17** (1992), 407-435.

[9] T. Cazenave, P. L. Lions: *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Commun. Math. Phys. **85** (1982), 549–561.

[10] T. Cazenave: *Semilinear Schrödinger equations*, American Mathematical Society, Providence (2003).

[11] X. Chang, M. Liu and D. Yan: *Normalized ground state solutions of nonlinear Schrödinger equations involving exponential critical growth*, J. Geom. Anal., **33** (3) (2023), Article ID: 83.

[12] Z. Chen, W. Zou: *Existence of normalized positive solutions for a class of nonhomogeneous elliptic equations*, J. Geom. Anal., **33** (2023), Article ID: 147.

[13] J. M. do Ó: *N-Laplacian equations in* R^N *with critical growth*, Abstr. Appl. Anal., **2** (1997), 301–315.

[14] A. J. Fernández, L. Jeanjean, R. Mandel and M. Maris: *Non-homogeneous gagliardo-nirenberg inequalities in* $\mathbb{R}^N$ *and application to a biharmonic non-linear Schrödinger equation*, J. Differ. Equ., **330** (5) (2022), 1–65.

[15] N. Ghoussoub: *Duality and Perturbation Methods in Critical Point Theory*, Cambridge Tracts in Mathematics and Mathematical Physics, vol. 107. Cambridge University Press, Cambridge (1993).

[16] H. Hajaiej, C. Stuart: *Existence and non-existence of Schwarz symmetric ground states for elliptic eigenvalue problems*, Ann. Math. Pura Appl., **184** (2005), 297–314.

[17] N. Ikoma, K. Tanaka: *A note on deformation argument for* L^2 *normalized solutions of nonlinear Schrödinger equations and systems*, Adv. Differ. Equ., **24** (2019), 609–646.

[18] L. Jeanjean: *Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlinear Anal., **28** (1997), 1633–1659.

[19] L. Jeanjean, S. S. Lu: *A mass supercritical problem revisited*, Calc. Var. Partial Differ. Equ., **59** (174) (2020), 1–43.

[20] X. Li: *Existence of normalized ground states for the Sobolev critical Schrödinger equation with combined nonlinearities*, Calc. Var., **60** (169) (2021), 1–14.

[21] H. Li, W. Zou: *Normalized ground states for semilinear elliptic systems with critical and subcritical nonlinearities*, J. Fixed Point Theory Appl., **23** (3) (2021), 1–30.

[22] Z. Ma, X. Chang: *Normalized ground states of nonlinear biharmonic Schrödinger equations with Sobolev critical growth and combined nonlinearities*, Appl. Math. Lett., **135** (2023), 1–7.

[23] J. Mederski, J. Schino: *Least energy solutions to a cooperative system of Schrödinger equations with prescribed* $\mathbb{R}^2$*-bounds: at least* L^2*-critical growth*, Calc. Var. Partial Differ. Equ., **61** (10) (2022), 1–31.

[24] N. Soave: *Normalized ground states for the NLS equation with combined nonlinearities*, J. Differ. Equ., **269** (9) (2020), 6941–6987.

[25] N. Soave: *Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case*, J. Funct. Anal., **279** (6) (2020), 1–43.

[26] A. Stefanov: *On the normalized ground states of second order PDEs with mixed power non-linearities*, Comm. Math. Phys., **369** (3) (2019), 929–971.

[27] C. A. Stuart: *Bifurcation in $L^p(\mathbb{R}^N)$ for a semilinear elliptic equation*, Proc. Lond. Math. Soc., **57** (1988), 511–541.
[28] T. Tao, M. Visan and X. Zhang: *The nonlinear Schrödinger equation with combined power-type nonlinearities*, Comm. Partial Differential Equations, **32** (7-9) (2007), 1281–1343.
[29] M. Weinstein: *Nonlinear Schrödinger equations and sharp interpolation estimates*, Commun. Math. Phys., **87** (1983), 567–576.

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