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Survey Article

A short review on (p, q)-equations with Carathéodory perturbation

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ABSTRACT. We review some recent works dealing with (p, q)-Laplacian equations in the setting of Sobolev spaces and Dirichlet boundary condition. We aim to underline the key role of growth conditions on the Carathéodory perturbation, in establishing both the existence and multiplicity of positive weak solutions. We focus on $(p-1)$-superlinear perturbations which do not satisfy the Ambrosetti-Rabinowitz condition, and a special attention is paid to those problems involving a singular term in the reaction. We refer both to variational tools and topological tools, and point out the dependence of the multiplicity result on a real parameter, when possible.

Keywords: (p, q)-Laplacian equation, Carathéodory perturbation, singular term, positive weak solution, convection.

2020 Mathematics Subject Classification: 35J66, 35J75, 35J92.

1. INTRODUCTION

We focus on some recent results concerning a class of (p, q)-Laplacian equations posed in a bounded domain $\Omega \subseteq \mathbb{R}^N$ with a C^2-boundary $\partial\Omega$. We know that boundary value problems driven by a combination of two, or more, operators of different nature, arise in many models of physical systems. We recall the mathematical model described in the work of Cahn & Hilliard (see [4]) to represent the process of separation of binary alloys, in the context of plasma physics we refer to the work of Zakharov (see [30]), in the framework of quantum physics we cite the work of Benci et al. (see [3]), for reaction-diffusion systems there is the work of Cherfils & Il'yasov (see [5]), finally for transonic flow problems we refer to the paper of Bahrouni et al. (see [2]). Now, for $r \in (1, +\infty)$, by Δ_r we denote the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u) \quad \text{for all } u \in W_0^{1,r}(\Omega),$$

where $W_0^{1,r}(\Omega)$ is the usual Dirichlet Sobolev space. Then, we consider the following operator:

$$L_{p,q}(u) = -\Delta_p u - \Delta_q u \quad \text{for all } u \in W_0^{1,p}(\Omega), \text{ with } 1 < q < p < +\infty.$$

Since we have the sum of two r-Laplacian operators, the differential operator $L_{p,q}(\cdot)$ is nonhomogeneous and this is a source of difficulties in the study of boundary value problems. We note that $L_{p,q}(\cdot)$ is a special case of a more general operator, known as the double-phase operator, corresponding to the energy functional given as

$$u \mapsto \int_\Omega [b(z)|\nabla u|^p + |\nabla u|^q]dz.$$

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The integrand in this functional, is the function

$$I(z,t)=b(z)t^p+t^q \quad \text{for all } z\in\Omega, \text{ all } t\geq 0, \text{ with } 1<q<p<+\infty,$$

where $b(\cdot)\in L^\infty(\Omega)\setminus\{0\}$, $b(z)\geq 0$ for a.a. $z\in\Omega$, acts like a modulating coefficient. This is because the behavior of the operator related to $I(z,\cdot)$ changes its ellipticity switching between two cases. Roughly speaking, on the set $\{z\in\Omega : b(z)>0\}$ the operator is controlled by the power of order p, and on the set $\{z\in\Omega : b(z)=0\}$ it is controlled by the power of order q. Consequently, a crucial hypothesis on $b(\cdot)$ is in imposing that it is bounded away from zero, that is

$$\operatorname*{ess\,inf}_{\Omega} b>0.$$

Under such hypothesis, the function $I(z,\cdot)$ has balanced growth, given as

$$c_1[t^p+t^q]\leq I(z,t)\leq c_2[t^p+t^q]$$

for some $c_1,c_2>0$, for a.a. $z\in\Omega$, all $t\geq 0$. Differently from this, when we do not impose the positivity condition on the essential infimum of modulating coefficient $b(\cdot)$, we have a more complicate unbalanced growth, namely

$$t^q\leq I(z,t)\leq c[t^p+t^q]$$

for some $c>0$, for a.a. $z\in\Omega$, all $t\geq 0$. Referring to these growth conditions, suitable integral functionals were first considered in the works of Marcellini (see [18, 19]) and of Zhikov (see [31, 32]), in the situation of problems of the calculus of variations and nonlinear elasticity theory. The unbalanced growth implies that for the study of such problems the right setting is given by Musielak-Orlicz (-Sobolev) spaces. Differently, in the case of balanced growth, we can pose the problem in the setting of classical Lebesgue and Sobolev spaces, with constant or variable exponents (respectively, for isotropic and anisotropic problems). Here, we assume $b\equiv 1$ and so, the function $I(z,t)$ reduces to

$$I(z,t)=t^p+t^q \quad \text{for all } z\in\Omega, \text{ all } t\geq 0.$$

Based on this, we consider Dirichlet problems of the form

$$\begin{cases} L_{p,q}(u)=\lambda_1 g(z,u)+\lambda_2 f(z,u) & \text{in } \Omega,\\ u\big|_{\partial\Omega}=0, \quad u>0,\ \lambda_1,\lambda_2>0,\ 1<q<p<+\infty. \end{cases} \tag{1.1}$$

In the reaction of (1.1), $\lambda_i>0$ $(i=1,2)$ is a parameter, and $f(z,t)$ is a Carathéodory perturbation, namely

- for all $t\in\mathbb{R}$, $z\to f(z,t)$ is measurable;
- for a.a. $z\in\Omega$, $t\to f(z,t)$ is continuous.

We concentrate on the case when for a.a. $z\in\Omega$, the function $t\to f(z,\cdot)$ is somehow $(p-1)$-superlinear near $+\infty$. However, the superlinear behavior of $f(z,\cdot)$ is not formulated using the common in the literature Ambrosetti-Rabinowitz condition (the AR-condition for short, see [1]). We know that the AR-condition is very useful in giving the compactness condition (i.e., the Palais-Smale condition, the PS-condition for short) for the energy functional associated to the problem. On the contrary, we employ a less restrictive condition to incorporate in our framework also superlinear nonlinearities with slower growth near $+\infty$, which may fail to satisfy the AR-condition. Further, the function $g(z,t)$ is built over a sigular term of the form $t^{-\eta}$, with $\eta>0$, for all $t\geq 0$, eventually combined with additional terms.

We look for positive solutions in integral form (i.e., weak solutions), in the situations when the existence and multiplicity results can be established by using variational tools from the critical point theory together with truncation and comparison techniques, as well as topological tools and tools from the operator theory (see, for example, the monography of Motreanu et al. [20]). If possible, we provide a bifurcation-type theorem, producing a critical parameter value $\lambda_i^* > 0$ $(i = 1, 2)$ such that

- for all $\lambda_i \in (0, \lambda_i^*)$ problem (1.1) admits at least two positive solutions (multiplicity);
- for $\lambda_i = \lambda_i^*$ problem (1.1) admits at least one positive solution (existence);
- for all $\lambda_i > \lambda_i^*$ problem (1.1) has no positive solutions (non-existence).

Precisely, we refer to some recent works of Papageorgiou et al., namely [23, 25] (isotropic problems) and [24] (anisotropic problem), and the references cited therein. In details the strategy of proofs is based on a judicious combinations of the tools provided in the works of Díaz & Saá [7], Gilbarg & Trudinger [11], Guedda & Veron [12], Ladyzhenskaya & Ural'tseva [14], Lieberman [16].

The manuscript is organized as follows. In Section 2, we introduce the notation and notions about the framework spaces and some operator properties involved in the analysis. In Section 3, we discuss a suitable set of hypotheses to describe the growth of the Carathéodory perturbation, in absence and in presence of a singular term in the reaction. In Section 4, we depict the strategy to establish a multiplicity result of weak solutions to a singular parametric Dirichlet problem with positive Carathéodory perturbation. Some variants are briefly discussed in Section 5 (negative perturbation), in Section 6 (sign-changing perturbation), and in Section 7 (locally defined perturbation with convection).

2. Mathematical background

The study of anisotropic (resp. isotropic) (p,q)-Laplacian equations uses Lebesgue and Sobolev spaces with variable (resp. constant) exponents. A comprehensive treatment of such spaces can be found in the monography of Diening et al. (see [6]). According to the finding in the works to review, we impose that $p, q \in C^1(\overline{\Omega})$ so that we can apply the anisotropic global regularity theory of Fan (see [9]). Also by $c, c_1, c_2 > 0$ we denote three constants that are not necessarily the same at each occurrence, further by $c_m > 0$ we mean that the constant depends on m. To shorten notation we write r for $r(z)$ $(r = p, q)$, such way we formally identify the notation of variable exponents with the one of constant exponents; some differences between the two settings will be noted in the sequel.

Let $p > 1$, we denote $p_- = \min_{\overline{\Omega}} p$ and $p_+ = \max_{\overline{\Omega}} p$, the similar notation is used for the exponent q. We now consider the set of measurable functions defined on Ω, namely $M(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is measurable}\}$. Clearly, we identify two functions which differ only on a Lebesgue-null set. Now, the (variable exponent) Lebesgue space $L^p(\Omega)$ is defined by

$$L^p(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u(z)|^p dz < +\infty \right\}.$$

This space is endowed with the so-called Luxemburg norm given as

$$\|u\|_p = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(z)}{\lambda} \right|^p dz \le 1 \right\}.$$

Remark 2.1. *For readers convenience, we recall that the passage from the constant exponents p, q to the variable exponents $p(\cdot), q(\cdot)$ discussed in this section, is natural but not trivial. Differently from the constant exponents setting, Kováčik & Rákosník note that the variable exponent Lebesgue space $L^p(\Omega)$ is not invariant with respect to translation (see [13]). This aspect implies difficulties directly linked to convolutions and continuity of functions in the mean.*

We know that $(L^p(\Omega), \|u\|_p)$ is a separable Banach space which is uniformly convex (hence, reflexive); further simple functions and continuous functions with compact support are dense in $L^p(\Omega)$. Suppose that $p, q \in C^1(\overline{\Omega})$ with $1 < q(z), p(z)$ for all $z \in \overline{\Omega}$. Then, we have the following classical embedding result

$$L^p(\Omega) \hookrightarrow L^q(\Omega) \text{ continuously if and only if } q(z) \le p(z) \text{ for all } z \in \overline{\Omega}.$$

Also let p' denote the Hölder conjugate exponent to p, that is $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$ for all $z \in \overline{\Omega}$. Thus, by $L^{p'}(\Omega) = L^p(\Omega)^*$ we denote the topological dual space of $L^p(\Omega)$, and we have the Hölder-type inequality

$$\int_\Omega |uv| dz \le \left[\frac{1}{p_-} + \frac{1}{p'_-}\right] \|u\|_p \|v\|_{p'} \quad \text{for all } u \in L^p(\Omega), \text{ all } v \in L^{p'}(\Omega).$$

Referring to the (variable exponent) Lebesgue space, we can introduce the corresponding (variable exponent) Sobolev space. Namely, we have

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : |\nabla u| \in L^p(\Omega)\}.$$

Let $\|\nabla u\|_p := \||\nabla u|\|_p$, we endow $W^{1,p}(\Omega)$ with the following norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p \quad \text{for all } u \in W^{1,p}(\Omega).$$

Then we define $W_0^{1,p}(\Omega)$ to be the completion of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$, that is

$$W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,p}}.$$

Recall that $C_c^\infty(\Omega) = \{u \in C^\infty(\Omega) : \text{ supp } u \text{ is compact}\}$. The spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are both separable and reflexive (in fact uniformly convex) Banach spaces. Moreover, on $W_0^{1,p}(\Omega)$ the Poincaré inequality holds, that is, there exists $c > 0$ such that

$$\|u\|_p \le c \|\nabla u\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Consequently, on $W_0^{1,p}(\Omega)$ we can use the norm $\|\cdot\|$ given as

$$\|u\| = \|\nabla u\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Further, we recall the Sobolev critical exponent of p defined by

$$p^*(z) = \begin{cases} \frac{Np(z)}{N-p(z)} & \text{if } p(z) < N, \\ +\infty & \text{if } p(z) \ge N, \end{cases} \quad z \in \overline{\Omega}.$$

Then if $q(z) \le p^*(z)$ for all $z \in \overline{\Omega}$ (resp. $q(z) < p^*(z)$ for all $z \in \overline{\Omega}$), we know about the following embedding result

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ continuously (resp. compactly)}.$$

By $W_0^{1,p}(\Omega)^* = W^{-1,p'}(\Omega)$ we denote the topological dual space of $W_0^{1,p}(\Omega)$. We refer the readers to the works of Edmunds & Rákosník (see [8]), and Kováčik & Rákosník (see [13]) to see more about Sobolev inequality and embedding theorems in the variable exponent setting, with respect to the constant exponent setting. For $r \in C^1(\overline{\Omega})$ (recall the assumption on the exponents at the beginning of this section), useful in the study of the spaces $L^r(\Omega)$ and $W^{1,r}(\Omega)$ is the following function

$$\rho_r(u) = \int_\Omega |u|^r dz \quad \text{for all } u \in L^r(\Omega),$$

which is known as modular function. Hence, we note a close relation between ρ_r and the Luxemburg norm $\|\cdot\|_r$. We can resume this by the following proposition. In the sequel by $\to$ we denote the strong convergence and by $\xrightarrow{w}$ we denote the weak convergence.

Proposition 2.1. *For any $u \in L^r(\Omega)$, $u \neq 0$, one has:*

- $\|u\|_r = \lambda \Leftrightarrow \rho_r\left(\frac{u}{\lambda}\right) = 1$;
- $\|u\|_r < 1$ (resp. $= 1, > 1$) $\Leftrightarrow \rho_r(u) < 1$ (resp. $= 1, > 1$);
- $\|u\|_r < 1 \Rightarrow \|u\|_r^{r+} \leq \rho_r(u) \leq \|u\|_r^{r-}$;
- $\|u\|_r > 1 \Rightarrow \|u\|_r^{r-} \leq \rho_r(u) \leq \|u\|_r^{r+}$;
- $\|u_n\|_r \to 0 \Leftrightarrow \rho_r(u_n) \to 0$;
- $\|u_n\|_r \to +\infty \Leftrightarrow \rho_r(u_n) \to +\infty$.

By $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ we denote the nonlinear map defined by

$$\langle A_r(u), h\rangle = \int_\Omega |\nabla u|^{r-2}(\nabla u, \nabla h)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,r}(\Omega),$$

where $\langle \cdot\,, \cdot\rangle$ denote the duality pairing of Banach spaces. The following useful properties of $A_r(\cdot)$ are known in the literature, for more details and information we refer to the monography of Rădulescu & Repovš [28].

Proposition 2.2. *The map $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$ (that is, $u_n \xrightarrow{w} u$ in $W_0^{1,r}(\Omega)$ and $\limsup\limits_{n\to+\infty}\langle A_r(u_n), u_n - u\rangle \leq 0$ imply $u_n \to u$ in $W_0^{1,r}(\Omega)$).*

Another space that we use in this work is the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega} : u|_{\partial\Omega} = 0)\}$. Namely, this is an ordered Banach space with positive cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given as

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

For $t \in \mathbb{R}$, we set $t^\pm = \max\{\pm t, 0\}$. Then for $u \in W^{1,r}(\Omega)$, we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We have

$$u^\pm \in W^{1,r}(\Omega),\ |u| = u^+ + u^-,\ u = u^+ - u^-.$$

Given $u, v \in W^{1,r}(\Omega)$ with $u \leq v$, we also consider the two sets

$$\begin{aligned}
[u, v] &= \left\{h \in W^{1,r}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\right\},\\
[u) &= \left\{h \in W^{1,r}(\Omega) : u(z) \leq h(z) \text{ for a.a. } z \in \Omega\right\},\\
\operatorname{int}_{C_0^1(\overline{\Omega})}[u, v] &= \text{ the interior in } C_0^1(\overline{\Omega}) \text{ of } [u, v] \cap C_0^1(\overline{\Omega}).
\end{aligned}$$

Given $h_1, h_2 \in L^\infty(\Omega)$, we write $h_1 \prec h_2$ if for all $K \subseteq \Omega$ compact we have

$$0 < c_K \leq h_2(z) - h_1(z) \quad \text{for a.a. } z \in K.$$

If $h_1, h_2 \in C(\Omega)$ and $h_1(z) < h_2(z)$ for all $z \in \Omega$, then $h_1 \prec h_2$. Let X be a Banach space and $\Phi \in C^1(X)$ be a functional, then we set

$$K_\Phi = \{u \in X : \Phi'(u) = 0\} \quad \text{(the critical set of } \Phi).$$

Also, we say that $\Phi(\cdot)$ satisfies the PS-condition, if the following property holds:

"Every sequence $\{u_n\}_{n\geq1} \subseteq X$ such that $\{\Phi(u_n)\}_{n\geq1} \subseteq \mathbb{R}$ is bounded and $\Phi'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence".

Further, we say that $\Phi(\cdot)$ satisfies the Cerami condition (C-condition for short), if the following property holds:

"Every sequence $\{u_n\}_{n\geq1} \subseteq X$ such that $\{\Phi(u_n)\}_{n\geq1} \subseteq \mathbb{R}$ is bounded and $(1+\|u_n\|_X)\Phi'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence".

These are compactness-type conditions on the functional $\Phi(\cdot)$. This is needed since in general the ambient space X is not locally compact (being infinite dimensional). Roughly speaking, the compactness-type conditions lead to a deformation theorem from which one can derive the minimax characterizations of the critical values of $\Phi(\cdot)$ (see, for example, Papageorgiou et al. [21]). A basic result in that theory is the well-known mountain pass theorem of Ambrosetti and Rabinowitz which we recall here (see the monography of Gasiński & Papageorgiou [10]).

Theorem 2.1. *If $\Phi \in C^1(X)$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > r > 0$,*

$$\max\{\Phi(u_0), \Phi(u_1)\} < \inf\{\Phi(u) : \|u - u_0\| = r\} = m_r,$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \Phi(\gamma(t)) \text{ where } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\},$$

then $c \geq m_r$ and c is a critical value of $\Phi(\cdot)$ (namely, there exists $u \in X$ such that $\Phi(u) = c$, $\Phi'(u) = 0$).

Evidently, the Cerami condition is weaker than the Palais-Smale condition however these conditions are equivalent provided that $\Phi \in C^1(X)$ is bounded below. Finally, we mention that in the proofs of forthcoming results we will use the following properties of operators.

Definition 2.1. *We say that the operator $A : X \to X^*$ is*

- *pseudomonotone if $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \le 0$ imply*
$$\liminf_{n\to+\infty}\langle A(u_n), u_n - v\rangle \ge \langle A(u), u - v\rangle \quad \text{for all } v \in X;$$
- *coercive if*
$$\lim_{\|u\|_X\to+\infty} \frac{\langle A(u), u\rangle}{\|u\|_X} = +\infty.$$

We remark that pseudomonotone operators have remarkable surjectivity properties.

3. Hypotheses on the data

Now, we are ready to introduce a suitable set of hypotheses on the data of problem (1.1), in the case when $g \equiv 0$. First we give the precise relation between the exponents p and q, namely we assume the following:

$$H_{p\&q} : p, q \in C^1(\overline{\Omega}) \text{ and } 1 < q_- \le q_+ < p_- \le p_+ < +\infty.$$

Clearly, this assumption reduces just to the usual condition $1 < q < p < +\infty$ in the case of constant exponents. Then, we depict the growth of the Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, namely we impose that

H_f: $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $0 \le f(z,t) \le a(z)[1 + t^{r-1}]$ for a.a. $z \in \Omega$, all $t \ge 0$, with $a \in L^\infty(\Omega)$, $p_+ < r < p^*$;

(ii) if $F(z,t) = \int_0^t f(z,s)ds$, then
$$\lim_{t\to+\infty} \frac{F(z,t)}{t^{p_+}} = +\infty \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii) if $\alpha(z,t) = f(z,t)t - p_+F(z,t)$, then there exists $\beta \in L^1(\Omega)$ such that
$$\alpha(z,t) \le \alpha(z,s) + \beta(z) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \le t \le s;$$

(iv) there exist $x > 0$ and $1 < \tau < q_-$ such that
$$c\,t^{\tau-1} \le f(z,t) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \le t \le x, \text{ with } c > 0$$
and for all $y > 0$, we have
$$c_y \le f(z,t) \quad \text{for a.a. } z \in \Omega, \text{ all } t \ge y, \text{ some } c_y > 0;$$

(v) for every $\rho > 0$, there exists $c_\rho > 0$ such that for a.a. $z \in \Omega$, the function $t \mapsto f(z,t) + c_\rho t^{p-1}$ is nondecreasing on $[0, \rho]$.

Remark 3.2. *The goal of this short survey is to discuss positive solutions for the prototype problem* (1.1), *and we note that all the above assumptions concern the positive semiaxis, that is the interval* $[0, +\infty)$. *Then, without any loss of generality, one can assume the following condition*

$$f(z,t) = 0 \quad \textit{for a.a. } z \in \Omega, \textit{ all } t \leq 0. \tag{3.2}$$

Hypotheses H_f (ii), (iii) *imply that*

$$\lim_{t\to+\infty} \frac{f(z,t)}{t^{p_+-1}} = +\infty \quad \textit{uniformly for a.a. } z \in \Omega,$$

namely, for a.a. $z \in \Omega$ *the perturbation* $f(z,\cdot)$ *is* (p_+-1)*-superlinear. Often in the literature superlinear problems are treated using the AR-condition (see again* [1], *and also Willem* [29]*). In our case, on account of* (3.2), *we will state a unilateral version of this condition. According to the AR-condition, there exist* $\mu > p$ *and* $M > 0$ *such that*

$$0 < \mu F(z,t) \leq f(z,t)t \quad \textit{for a.a. } z \in \Omega, \textit{ all } t \geq M, \tag{3.3}$$

$$0 < \operatorname*{essinf}_{\Omega} F(\cdot, M). \tag{3.4}$$

Integrating (3.3) *and using* (3.4), *we obtain the following weaker condition*

$$c\,t^{\mu} \leq F(z,t) \quad \textit{for a.a. } z \in \Omega, \textit{ all } t \geq M, \textit{ some } c > 0,$$

which means that

$$c\,t^{\mu-1} \leq f(z,t) \quad \textit{for a.a. } z \in \Omega, \textit{ all } t \geq M, \textit{ some } c > 0.$$

So, the AR-condition restricts $f(z,\cdot)$ *to have at least* $(\mu-1)$*-polynomial growth near* $+\infty$. *The quasimonotonicity condition in hypothesis* H_f (iii) *does not imply such a restriction on the growth of* $f(z,\cdot)$. *Consequently, it permits also the consideration of superlinear nonlinearities with slower growth near* $+\infty$. *We mention that hypothesis* H_f (iii) *is an assumption used by Li & Yang (see* [17]*) in dealing with a Dirichlet problem driven by a single* p*-Laplace differential operator, in the case of constant exponent* p. *As known in the literature, there are convenient ways to verify* H_f (iii). *So, this situation holds if we can find* $M > 0$ *such that for a.a.* $z \in \Omega$ *we get one of the following outcomes:*

$$t \to \frac{f(z,t)}{t^{p_+-1}} \textit{ is nondecreasing on } [M, +\infty),$$
$$t \to \alpha(z,t) \textit{ is nondecreasing on } [M, +\infty).$$

About the remaining hypotheses, we note that assumption H_f (iv) *leads to the presence of a concave term near zero, while the situation in* H_f (v) *gives us a one sided Hölder condition. We know that* H_f (iv) *is satisfied if for a.a.* $z \in \Omega$, $f(z,\cdot)$ *is differentiable and for every* $\rho > 0$ *one can find a positive constant* $c_\rho > 0$ *satisfying the following inequality*

$$-c_\rho t^{p-1} \leq f'_t(z,t)t \quad \textit{for a.a. } z \in \Omega, \textit{ all } 0 \leq t \leq \rho.$$

According to the finding in the work of Papageorgiou et al. (see [22]) and other studies, we present the following illustrative example depicting the role of assumptions H_f stated above. For the sake of clarity and simplicity, we drop the z-dependence of $f(z,t)$, hence we refer to a real valued function $f : \mathbb{R} \to \mathbb{R}$.

Example 3.1. *Let* $1 < \tau < q < p$ *and consider two functions* $f_i : \mathbb{R} \to \mathbb{R}$ *(*$i = 1, 2$*) defined by*

$$f_1(t) = \begin{cases} t^{\tau-1} & \textit{if } \ 0 \leq t \leq 1 \\ t^{r-1} & \textit{if } \ 1 < t \end{cases} \quad \textit{with } p < r < p^*,$$

$$f_2(t) = \begin{cases} t^{\tau-1} & \textit{if } \ 0 \leq t \leq 1 \\ t^{s-1} + t^{p-1}\ln t & \textit{if } \ 1 < t \end{cases} \quad \textit{with } 1 < s < p.$$

By routine calculations, one can show that both $f_1(\cdot)$ and $f_2(\cdot)$ satisfy hypotheses H_f, but only the first one fulfills the AR-condition, namely $f_2(\cdot)$ has a slower growth near $+\infty$ in respect to the corresponding growth of $f_1(\cdot)$.

From the above discussion, the crucial role of hypotheses H_f should be evident, however we remark this point by showing what need to be revised in the case we also deal with a non-null term g. As mentioned in Section 1 (on page 3), we first consider the basic situation where

$$g(z,t) = t^{-\eta} \quad \text{for some } \eta > 0, \text{ all } t \geq 0, \tag{3.5}$$

namely g is a singular term. To approach this situation is sufficient to change $H_f\,(iii)$ in the definition of function $\alpha(z,t)$ as follows

$$\alpha(z,t) = \left[1 - \frac{p_+}{1-\eta}\right] t^{1-\eta} + f(z,t)t - p_+F(z,t). \tag{3.6}$$

For more discussion on this condition, see Papageorgiou et al. [24]. In the next sections, we will consider some special cases of problem (1.1).

4. Singular problem - weak solutions

In this section, we consider the isotropic version of problem (1.1) assuming g is the purely singular term given in (3.5), and a unique parameter $\lambda_1 = \lambda_2 = \lambda$ (λ-problem (1.1)-(3.5) for short). Further, we make the following hypothesis about the singular term:

H_η : There exists $t^* \in C_+$ such that $(t^*)^{-\eta} \in L^r(\Omega)$ with $r > N$.

Remark 4.3. *In the isotropic context, let us denote by $\widehat{u}_1(p)$ the positive, L^p-normalized (that is, $\|\widehat{u}_1(p)\|_p = 1$) principal eigenfunction of the Dirichlet p-Laplacian problem. We know that $\widehat{u}_1(p) \in$ int C_+ (see [10]). So, we note that if $\eta < \frac{1}{N}$, then assumption H_η is satisfied with $t^* = \widehat{u}_1(p)$. The reader can see this, using the Lemma in Lazer & McKenna (see [15]). We remark that this lemma (originally proved for the Laplacian, case $p = 2$) is valid also in the case $p \neq 2$, this because its proof depends only on the fact that the principal positive eigenfunction is in* int C_+. *For more details and information on hypothesis H_η, we also refer to the work of Perera & Silva (see condition (H) in [26] and related discussion).*

As mentioned above, we are interested to weak solutions (namely, solutions in integral form). We know that a weak solution of λ-problem (1.1)-(3.5) is a function $u \in W_0^{1,p}(\Omega)$ such that $u^{-\eta}h \in L^1(\Omega)$ for all $h \in W_0^{1,p}(\Omega)$ and

$$\langle A_p(u), h\rangle + \langle A_q(u), h\rangle = \int_\Omega \lambda \left[u^{-\eta} + f(z,u)\right] hdz \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

We note that, due to the singular term, the energy functional of the problem, namely $\Psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Psi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \lambda \int_\Omega \left[\frac{1}{1-\eta}(u^+)^{1-\eta} + F(z,u^+)\right] dz$$

for all $u \in W_0^{1,p}(\Omega)$, is not C^1. This is a source of difficulties in the analysis of λ-problem (1.1)-(3.5), since we cannot use the minimax methods of critical point theory directly on $\Psi(\cdot)$ (see [21, Chapter 5]. In such situation, we need to bypass the singularity, so that we can deal with C^1-functionals. Briefly, we describe a consolidated strategy used to get this goal. Precisely, if $H_f\,(i), (iv)$ hold, then one can find $c_1 > 0$ satisfying

$$f(z,t) \geq c\,t^{\tau-1} - c_1 t^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } t \geq 0, \tag{4.7}$$

where $c > 0$ is the constant given in $H_f\ (iv)$. This growth bound for the Carathéodory function $f(z, \cdot)$ gives us the possibility to introduce the auxiliary problem in the form

$$\begin{cases} L_{p,q}(u) = \lambda \left[c\, u^{\tau-1} - c_1 u^{r-1} \right] & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad u > 0,\ \lambda > 0,\ 1 < \tau < q < p < r. \end{cases} \tag{4.8}$$

This way, we are in position to introduce the associated C^1-functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q + \lambda \frac{c_1}{r}\|u^+\|_r^r - \lambda \frac{c}{\tau}\|u^+\|_\tau^\tau \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Now, one can show the regularity of this functional, referring to classical tools and notions in the theory of energy functionals, hence linking to the general theory of operators. The restrictions on the involved exponents, namely $1 < \tau < q < p < r$, give us the coercivity of $\Phi(\cdot)$. Next, using Sobolev embedding result, one can conclude that it is sequentially weakly lower semicontinuous. By the Weierstrass-Tonelli theorem, there exists $\underline{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\Phi(\underline{u}_\lambda) = \min \left[\Phi(u) : u \in W_0^{1,p}(\Omega) \right].$$

By routine calculations, one can show that $\underline{u}_\lambda \in L^\infty(\Omega)$ (according to Theorem 7.1 of Ladyzhenskaya & Ural'tseva [14]), hence the nonlinear regularity theory of Lieberman (see [16]) permits to conclude that $\underline{u}_\lambda \in C_+ \setminus \{0\}$. Finally, the nonlinear maximum principle of Pucci & Serrin (see [27]) implies that $\underline{u}_\lambda \in \operatorname{int} C_+$. This way, one can establish the following existence result.

Proposition 4.3. *Problem (4.8) admits a positive solution $\underline{u}_\lambda \in \operatorname{int} C_+$, for every $\lambda > 0$. Further, $\underline{u}_\lambda \to 0$ in $C_0^1(\overline{\Omega})$ as $\lambda \to 0^+$.*

By contradiction, one can easily show that the solution $\underline{u}_\lambda \in \operatorname{int} C_+$ is unique (namely, one can adapt the similar arguments in the proof of Proposition 5 in [24]). Next we introduce the following auxiliary problem

$$\begin{cases} L_{p,q} = \lambda \underline{u}_\lambda^{-\eta} + 1 & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad u > 0,\ \lambda > 0,\ \eta > 0,\ 1 < q < p. \end{cases} \tag{4.9}$$

For this problem, we can establish an existence (and uniqueness) result of positive weak solution, that can be linked to the solution obtained for problem (4.8). Precisely, we mean the following result.

Proposition 4.4. *If hypothesis H_η holds then problem (4.9) admits a unique positive solution $\overline{u}_\lambda \in \operatorname{int} C_+$, for every $\lambda > 0$. Further, we can find $\lambda_0 > 0$ such that $\underline{u}_\lambda \le \overline{u}_\lambda$ for all $\lambda \in (0, \lambda_0]$.*

A salient point in introducing problem (4.9) is linked to the fact that hypothesis $H_f\ (i)$ ensures that one can find a value $\lambda_0 \in (0, 1]$ such that

$$\lambda f(z, \overline{u}_\lambda(z)) \le 1 \quad \text{for a.a. } z \in \Omega, \text{ all } \lambda \in (0, \lambda_0].$$

In details, the proof of this proposition is based on a discussion of the regularity properties of the operator $A_{p,q} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ defined by

$$A_{p,q}(u) = A_p(u) + A_q(u) \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{4.10}$$

We just remark that $A_{p,q}(\cdot)$ is continuous, strictly monotone (hence maximal monotone too), and coercive. Therefore $A_{p,q}(\cdot)$ is surjective (see the monography of Papageorgiou et al. [21]). Hence by the surjectivity and strict monotonicity of $A_{p,q}(\cdot)$, we can find a unique $\overline{u}_\lambda \in W_0^{1,p}(\Omega)$, $\overline{u}_\lambda \ge 0$, $\overline{u}_\lambda \ne 0$ to infer the properties stated in Proposition 4.4 for all $0 < \lambda \le \lambda_0$. The two unique positive solutions established in Propositions 4.3 and 4.4 can be successfully involved in defining truncation of the reaction of λ-problem (1.1)-(3.5). So, referring again to the classical approaches (see [7, 12, 14, 16]) and results mentioned above, one can define suitable energy functionals associated to this problem to obtain the following abstract result, involving a critical value of the parameter λ, namely λ^*.

Theorem 4.2. *If hypotheses $H_{p\&q}$, H_f, H_η hold, then there exists $\lambda^* > 0$ such that*

- (a) *for every $\lambda \in (0, \lambda^*)$, λ-problem (1.1)-(3.5) has at least two positive solutions $u_0, u_1 \in \operatorname{int} C_+$, $u_0 \leq u_1$, $u_0 \neq u_1$;*
- (b) *λ^*-problem (1.1)-(3.5) has at least one positive solution $u^* \in \operatorname{int} C_+$;*
- (c) *for every $\lambda > \lambda^*$, λ-problem (1.1)-(3.5) has no positive solutions.*

Remark 4.4. *Cleary in Theorem 4.2, hypothesis H_f (iii) is given for the function $\alpha(z,t)$ defined by* (3.6).

5. Singular problem - negative perturbation

In this section, we show how the finding described in previous section changes if we assume a negative Carathéodory perturbation term. According to Papageorgiou et al. (see [25]), we will remark that this time there is not need to involve a parameter λ in the problem. So, we consider the following isotropic model problem

$$\begin{cases} L_{p,q}(u) = u^{-\eta} - f(z,u) & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad u > 0,\ 0 < \eta < 1,\ 1 < q < p. \end{cases} \tag{5.11}$$

The salient point of this model is the fact that the negative perturbation, originates some difficulties in producing lower solutions for the problem. We note that in previous section, lower solutions (recall the solution to auxiliary problem) are used to bypass the singularity, so that we can retrieve C^1-energy functionals. Roughly speaking, this time we can not use solution of the associated auxiliary purely singular problem to act as a lower solution, differently from the above studied situation of problems with positive perturbation. However, Papageorgiou et al. (see [25]) devoloped a suitable strategy based on both upper solutions and regularizations of the singular term, of course they continue to use certain truncation and other tools of critical point theory. For readers convenience we recall the precise hypotheses about the Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, namely we consider the case when
H_f^-: $f(z,0) = 0$ for a.a. $z \in \Omega$ and

- (i) $0 \leq f(z,t) \leq a(z)[1 + t^{r-1}]$ for a.a. $z \in \Omega$, all $t \geq 0$, with $a \in L^\infty(\Omega)$, $p_+ < r < p^*$;
- (ii) there exists $1 < \tau \leq q$ such that $x \to \dfrac{f(z,t)}{t^{\tau-1}}$ is nondecreasing on the interval $(0, +\infty)$.

Here, we note that substantially we use the classical growth condition in hypothesis H_f (i) (here called $H_f^-(i)$), while we already encountered the hypothesis $H_f^-(ii)$ in our discussion about the $(p_+ - 1)$-superlinearity of f with respect to the AR-condition and the other assumptions in H_f. On this basis (recall the discussion on problems (4.8) and (4.9) too), the purely singular problem

$$\begin{cases} L_{p,q}(u) = u^{-\eta} & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad u > 0,\ 0 < \eta < 1,\ 1 < q < p \end{cases}$$

admits a unique positive solution $u_* \in \operatorname{int} C_+$ (see Proposition 3.1 of [25]). One can use this solution to define appropriate truncation of the reaction in the following regularized version of problem (5.11), namely

$$\begin{cases} L_{p,q}(u) = [u+\varepsilon]^{-\eta} - f(z,u) & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, & u>0,\ 0<\eta<1,\ 1<q<p,\ \varepsilon>0. \end{cases} \tag{5.12}$$

Working on the energy functional associated to problem (5.12) and adapting the similar arguments mentioned in the proof of previous results (namely, coercivity, convexity and monotonicity properties of energy functional associated to the involved Dirichlet problem), one can establish the existence and uniqueness of positive solution to (5.12) in the sense that, for every $\varepsilon > 0$ problem (5.12) has a unique positive solution $u_\varepsilon^* \in \operatorname{int} C_+$. Involving another appropriate truncation of the reaction, and developing some standard calulations, is also possible to prove the non-decreasing monotonicity property of the solution map $\varepsilon \mapsto u_\varepsilon^*$. Next, under the assumptions on the Carathéodory perturbation, and passing to the limit as $\varepsilon \to 0^+$, one can obtain a positive solution for problem (5.11). The proof is based on a precise analysis of the operator $A_{p,q}$ and usual convergence processes (for details see again [25]). Summarizing, the authors established the following result.

Theorem 5.3. *If hypotheses H_f^- hold, then problem* (5.11) *admits a unique positive solution $u^* \in \operatorname{int} C_+$.*

Remark 5.5. *Since no critical value of some parameter is required in establishing the proof of this result, we can conclude that there is no need to involve it in the model problem* (5.11)*, see Remark 3.5 of* [25].

6. Singular problem - sign-changing perturbation

In this section, we consider problem (1.1) assuming g is the purely singular term given in (3.5) plus a power term, and we have a unique parameter $\lambda_1 = \lambda$ as we pose $\lambda_2 = 1$. Precisely, we refer to the following anisotropic model problem

$$\begin{cases} L_{p,q}(u) = \lambda[u^{-\eta} + u^{\tau-1}] + f(z,u) & \text{in } \Omega, \\ u\Big|_{\partial\Omega} = 0, & u>0,\ \lambda>0,\ 0<\eta<1,\ 1<\tau<q<p. \end{cases} \tag{6.13}$$

To complicate the context, Papageorgiou et al. ([24]) assume an indefinite Carathéodory perturbation $f(z,t)$, that is, it may change sign. As in previous sections, the perturbation continues to exhibit a superlinear (convex) behavior without satisfying the AR-condition. Finally, we note that the additional power term $\lambda u^{\tau-1}$ is assumed to be concave. Precisely, the set of hypotheses on the exponents of the problem (recall $H_{p,q}$ and H_η) are given as follows

$$p, q \in C^1(\overline{\Omega}),\ \tau \in C(\overline{\Omega}),\ 1 < \tau_- \le \tau_+ < q_- \le q_+ < p_- \le p_+ < N,$$
$$\eta \in C(\overline{\Omega}),\ 0 < \eta_- \le \eta_+ < 1.$$

Summing up, (6.13) is a singular concave-convex problem. To design their strategy, the authors in [24] point their attention on a parametric auxiliary problem of the form (that is, without singular term and Carathéodory perturbation)

$$\begin{cases} L_{p,q}(u) = \lambda u^{\tau-1} & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, & u>0,\ \lambda>0,\ 1<\tau<q<p. \end{cases} \tag{6.14}$$

The interest in establishing the existence and uniqueness of positive solution to (6.14) reflects in bypassing the singular term in problem (6.13), this way authors identify suitable parameters for problem (6.13) too. Briefly, one can manipulate problem (6.14) following the similar arguments mentioned with respect to the problem in Section 4 (namely, using coercivity and monotonicity properties of suitable energy functionals, standard Sobolev embedding results, the Weierstrass-Tonelli theorem, and so on). Hence, one can start the study referring to the C^1-functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \lambda\frac{1}{\tau}\|u^+\|_\tau^\tau \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

and properly discuss its regularity. So, using variational tools together with truncation and comparison techniques, one can obtain the counterparts of Proposition 4.3 (uniqueness of positive solution to (6.14)) and Theorem 4.2 (bifurcation-type theorem for (6.13), depending on $\lambda > 0$). Since the perturbation here can be sign-changing, the set of hypotheses H_f need to be revisited as follows. Here, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$H_f^{\pm}$: $f(z, 0) = 0$ for a.a. $z \in \Omega$, and

- (i) $|f(z,t)| \le a(z)[1 + t^{r-1}]$ for a.a. $z \in \Omega$, all $t \ge 0$, with $a \in L^\infty(\Omega)$, $r \in C(\overline{\Omega})$, $p_+ < r_- \le r_+ < p_-^*$;
- (ii) if $F(z,t) = \int_0^x f(z,s)ds$, then $\lim\limits_{t\to+\infty} \frac{F(z,t)}{t^{p_+}} = +\infty$ uniformly for a.a. $z \in \Omega$;
- (iii) if $\alpha(z,t) = \lambda\left[t^{1-\eta} + t^\tau\right] + f(z,t)t - \lambda p_+\left[\frac{1}{1-\eta}t^{1-\eta} + \frac{1}{\tau}t^\tau\right] + p_+F(z,t)$, then there exists $\beta \in L^1(\Omega)$ such that
$$\alpha(z,t) \le \alpha(z,s) + \beta(z) \text{ for a.a. } z \in \Omega, \text{ all } 0 \le t \le s;$$
- (iv) $\lim\limits_{t\to 0^+} \frac{f(z,t)}{t^{q_+-1}} = 0$ uniformly for a.a. $z \in \Omega$ and there exists $\delta > 0$ such that
$$0 < c_\delta \le f(z,t) \text{ for a.a. } z \in \Omega, \text{ all } 0 < s \le t \le \delta;$$
- (v) for every $\rho > 0$ there exists $c_\rho > 0$ such that for a.a. $z \in \Omega$, the function $t \mapsto f(z,t) + c_\rho|t|^{p-1}$ is nondecreasing on $[0, \rho]$.

7. Singular problem - locally defined perturbation

In this section, we conclude our survey focusing on the situation where the reaction exhibits dependence on the gradient of the solution, and the Carathéodory perturbation is defined only locally. Referring to the work of Papageorgiou et al. (see [23]), we recall the following isotropic model problem

$$\begin{cases} L_{p,q}(u) = \lambda_1 u^{-\eta} + \lambda_2|\nabla u|^{p-1} + f(z,u) & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad u > 0, \ \lambda_i > 0\,(i = 1,2), \ 0 < \eta < 1, \ 1 < q < p. \end{cases} \tag{7.15}$$

Precisely, we deal with a parametric Dirichlet problem where the reaction is built over a parametric singular term, a gradient dependent term (convection) and $f(z,\cdot)$ is defined only near 0^+ (hence locally). However, the main feature of problem (7.15) is the presence of the gradient of u in the reaction, since this fact implies that the problem loses its variational structure. Consequently the right way to study the problem is in developing a topological approach based on truncation techniques and on the theory of nonlinear operators of monotone type. Papageorgiou et al. [23] in designing their hypotheses, make use of the first eigenvalue of the q-Laplace Dirichlet problem, we denote by $\widehat{\lambda}_1(q)$ this eigenvalue. According to the classical theory we know that $\widehat{\lambda}_1(q) > 0$, it is simple and isolated and all the eigenfunctions corresponding to it, have fixed sign. We note that $\widehat{\lambda}_1(q)$ is the only eigenvalue with eigenfunctions of fixed sign, and we have already mentioned the properties of the corresponding eigenfunction in Remark 4.3. On this basis, we get the precise assumptions used in this investigation, namely we consider a Carathéodory function $f : \Omega \times [0, d] \to \mathbb{R}$ such that

H_ℓ: $f(z,0) = 0$ for a.a. $z \in \Omega$, and

(i) $|f(z,t)| \le a_d(z)$ for a.a. $z \in \Omega$, all $0 \le t \le d$, with $a_d \in L^\infty(\Omega)$;

(ii) $f(z,d) \le -c < 0$ for a.a. $z \in \Omega$;

(iii) there exist $0 < x < d$ and $\eta \in L^\infty(\Omega) \setminus \{\widehat{\lambda}_1(q)\}$ such that

$$\widehat{\lambda}_1(q) \le \eta(z) \quad \text{for a.a. } z \in \Omega,$$
$$\eta(z)t^{q-1} \le f(z,t) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \le t \le x;$$

(iv) there exists $c_d > 0$ such that for a.a. $z \in \Omega$, the function $t \mapsto f(z,t) + c_d t^{p-1}$ is nondecreasing on $[0,d]$.

Hypotheses $H_\ell\,(ii), (iii)$ imply that we can find $c > 0$ such that

$$f(z,t) \ge \eta(z)t^{q-1} - c\,t^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \le t \le d. \tag{7.16}$$

This permits us to consider the following auxiliary problem

$$\begin{cases} L_{p,q}(u) = \begin{cases} \eta(z)(u^+)^{q-1} - c(u^+)^{p-1} & \text{if } \ u^+(z) \le d, \\ \eta(z)d^{q-1} - c\,d^{p-1} & \text{if } \ d < u^+(z), \end{cases} \\ u\big|_{\partial\Omega} = 0, \quad u > 0,\ 0 < \eta < 1,\ 1 < q < p, \end{cases} \tag{7.17}$$

and conclude by previous arguments in Section 4 that (7.17) admits a unique solution $u_* \in \mathrm{int}_{C_0^1(\overline{\Omega})}[0,d]$. We recall that a growth bound similar to (7.16) is given in (4.7). Instead of underlining the similarities between the new setting and the previous ones, we prefer to briefly pointing out the new features of problem (7.15). This means that to complement the arguments of proofs in previous sections, here we use the Lipschitz function $\omega : \mathbb{R} \to \mathbb{R}$ given as

$$\omega(t) = \begin{cases} t & \text{if } \ t \le d, \\ d & \text{if } \ d < t. \end{cases}$$

Referring to [21], one can use the chain rule for Sobolev functions to deduce the following implications

$$u \in W_0^{1,p}(\Omega) \quad \Longrightarrow \quad u^+ \in W_0^{1,p}(\Omega) \quad \Longrightarrow \quad \omega(u^+(\cdot)) \in W_0^{1,p}(\Omega).$$

Further, one has

$$\nabla\omega(u^+) = \omega'(u^+)\nabla u^+ = \begin{cases} \nabla u^+ & \text{if } \ u^+(z) \le d, \\ 0 & \text{if } \ d < u^+(z). \end{cases}$$

The above truncation and related remarks, give us a technical tool that we use to properly define a concept of Nemitsky operator as follows

$$G(u)(z) = \begin{cases} \lambda_1 \overline{u}(z)^{-\eta} + f(z, \overline{u}(z)) & \text{if } \ u(z) \le \overline{u}(z), \\ \lambda_1 u(z)^{-\eta} + f(z, \omega(u(z))) & \text{if } \ \overline{u}(z) < u(z). \end{cases}$$

This type operator is usually involved in the topological approach to various classes of boundary value problems with reaction exhiniting dependence on the gradient of solution. We refer to the monographies of Motreanu et al. [20], and Gasiński & Papageorgiou [10] for more details and information. Under the hypotheses H_ℓ, we remark that this operator is well-defined, in the sense that $G(u) \in W^{-1,p'}(\Omega)$. Further, we know the following embedding result

$$L^{p'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega) \text{ continuously and densely (see Lemma 2.2.27 of [10])},$$

then one can conclude that $|\nabla\omega(u^+)|^{p-1} \in L^{p'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$.

Now, hypotheses H_ℓ guarantee that Proposition 2.2 can be applied to the operator $A_G : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$A_G(u) = A_{p,q}(u) - \lambda_2 |\nabla \omega(u^+)|^{p-1} - G(u) \quad \text{(see also (4.10))}.$$

This says us that the operator is pseudomonotone, and hence one can conclude that it is surjective too. Using properly $G(\cdot)$ and proving suitable estimates (with respect to the growth bound of f), Papageorgiou et al. [23] established the existence of a positive solution in the following form.

Theorem 7.4. *If hypotheses H_ℓ hold, then problem (7.15) admits a positive solution $u^* \in \operatorname{int} C_+$ for every $\lambda_1 > 0$ small. Further, $u^*(z) < d$ for all $z \in \overline{\Omega}$.*

8. Conclusions

The qualitative study of weak solutions to anisotropic problems, as well as isotropic problems, is interesting to deal with framework structures useful in modeling materials' properties and diffusion processes. We have discussed the impact of different singular reaction terms on the solvability of a parametric Dirichlet (p,q)-problem. Special attention is paid to the role of Carathéodory perturbation, comparing the situations when this perturbation is positive, negative, sign-changing and only locally defined. We just referred to classical tools of variational and topological methods to cover the cases when the energy functional associated to the main problem are not C^1 (due to the singular term) and when the problem itself loses the variational structure (due to a convection term). Auxiliary results for certain purely singular problems, as well as some regularized problems, are designed to obtain the regularity properties of weak solutions. If possible the multiplicity of solutions is established, with respect to a positive parameter involved in the reaction.

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References

[1] A. Ambrosetti, P. H. Rabinowitz: *Dual variational methods in critical point theory and applications*, J. Funct. Anal., **14** (4) (1973), 349–381.

[2] A Bahrouni, V. D. Rădulescu and D. D. Repovš: *Double phase transonic flow problems with variable growth: nonlinear patterns and stationary waves*, Nonlinearity, **32** (7) (2019), 2481–2495.

[3] V. Benci, P. D'Avenia, D. Fortunato and L. Pisani: *Solitons in several space dimensions: Derrick's problem and infinitely many solutions*, Arch. Ration. Mech. Anal., **154** (4) (2000), 297–324.

[4] J. W. Cahn, J. E. Hilliard: *Free energy of a nonuniform system. I. Interfacial free energy*, J. Chem. Phys., **28** (2) (1958), 258–267.

[5] L. Cherfils, Y. Il'yasov: *On the stationary solutions of generalized reaction diffusion equations with p&q-Laplacian*, Commun. Pure Appl. Anal., **4** (1) (2005), 9–22.

[6] L. Diening, P. Harjulehto, P. Hästö and M. Růžicka: *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Math., vol. 2017, Springer-Verlag, Heidelberg (2011).

[7] J. I. Díaz, J. E. Saá: *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, C.R. Acad. Sci. Paris, serie I, Math., **305** (12) (1987), 521–524.

[8] D. E. Edmunds, J. Rákosník: *Sobolev embeddings with variable exponent*, Studia Math., **143** (3) (2000), 267–293.

[9] X. Fan: *Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form*, J. Differential Equations, **235** (2) (2007), 397–417.

[10] L. Gasiński, N. S. Papageorgiou: *Nonlinear analysis*, Chapman & Hall / CRC, Boca Raton, FL (2006).

[11] D. Gilbarg, N. S. Trudinger: *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin (2001).

[12] M. Guedda, L. Veron: *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal., **13** (8) (1989), 879–902.

[13] O. Kováčik, J. Rákosník: *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J., **41** (4) (1991), 592–618.

[14] O. A. Ladyzhenskaya, N. N. Ural′tseva: *Linear and quasilinear elliptic equations*, Academic Press, New York-London (1968).

[15] A. C. Lazer, P. J. McKenna: *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc., **111** (3) (1991), 721–730.

[16] G. M. Lieberman: *The natural generalization of the natural conditions of Ladyzhenskaya and Ural′tseva for elliptic equations*, Comm. Partial Diff. Equations, **16** (2-3) (1991), 311–361.

[17] G. Li, C. Yang: *The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti-Rabinowitz condition*, Nonlinear Anal., **72** (12) (2010), 4602–4613.

[18] P. Marcellini: *Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions*, Arch. Ration. Mech. Anal., **105** (1989), 267–284.

[19] P. Marcellini: *Regularity and existence of solutions of elliptic equations with p, q-growth conditions*, J. Differential Equations, **90** (1) (1991) 1–30.

[20] D. Motreanu, V. Motreanu and N. S. Papageorgiou: *Topological and variational methods with applications to nonlinear boundary value problems*, Springer, New York (2014).

[21] N. S. Papageorgiou, V. D. Rădulescu and D. D. Repovš: *Nonlinear analysis - theory and methods*, Springer, Cham (2019).

[22] N. S. Papageorgiou, D. D. Repovš and C. Vetro: *Anisotropic (p, q)-equations with convex and negative concave terms*, Recent Advances in Mathematical Analysis. Trends in Mathematics. Birkhäuser, Cham (2023).

[23] N. S. Papageorgiou, C. Vetro and F. Vetro: *A singular (p, q)-equation with convection and a locally defined perturbation*, Appl. Math. Lett., **118** (2021), 107175.

[24] N. S. Papageorgiou, C. Vetro and F. Vetro: *Singular anisotropic problems with competition phenomena*, J. Geom. Anal., **33** (2023), 173.

[25] N. S. Papageorgiou, C. Vetro and F. Vetro: *Positive solutions for singular (p, q)-Laplacian equations with negative perturbation*, Electron. J. Differential Equations, **2023** (2023), 25.

[26] K. Perera, E. A. B. Silva: *Existence and multiplicity of positive solutions for singular quasilinear problems*, J. Math. Anal. Appl., **323** (2) (2006) 1238–1252.

[27] P. Pucci, J. Serrin: *The maximum principle*, Birkhäuser Verlag, Basel (2007).

[28] V. D. Rădulescu, D. D. Repovš: *Partial differential equations with variable exponents: variational methods and qualitative analysis*, CRC Press, Boca Raton, FL (2015).

[29] M. Willem: *Minimax theorems*, Birkhäuser, Boston (1996).

[30] V. E. Zakharov: *Collapse of Langmuir waves*, Soviet Jour. Exper. Theor. Physics, **35** (5) (1972), 908–914.

[31] V. V. Zhikov: *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR Izv., **29** (1) (1987), 33–66.

[32] V. V. Zhikov: *On variational problems and nonlinear elliptic equations with nonstandard growth conditions*, J. Math. Sci. (N. Y.), **173** (5) (2011), 463–570.

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