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Research Article

On the solutions of a coupled system of proportional fractional differential inclusions of Hilfer type

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ABSTRACT. A multi-point and integro-multi-strip boundary value problem associated to a Hilfer type coupled system of fractional differential inclusions is studied. The existence of solutions is established in the case when the set-valued maps have non-convex values.

Keywords: Differential inclusion, fractional derivative, boundary value problem.

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1. INTRODUCTION

The late literature abounds in papers concerning the study of systems governed by fractional order derivatives. That is because this kind of systems proved to be more realistic models than the ones using classical derivatives (see [2, 9, 13, 15, 17] etc.).

In [11], Hilfer introduced a generalization of both Riemann-Liouville and Caputo fractional derivatives. In fact, this derivative is an interpolation between Riemann-Liouville and Caputo derivatives. Properties and applications of Hilfer fractional derivative may be found in [12]. Recently, in [14], this derivative was also extended; namely to ψ-Hilfer generalized proportional fractional derivative of a function with respect to another function. Some properties of this derivative were studied in [14].

In this paper, we are concerned the following boundary value problem

$$(1.1)\qquad \begin{cases} D_H^{\alpha_1,\beta_1,\sigma,\psi}x_1(t)\in F_1(t,x_1(t),x_2(t)), & a.e.\ t\in[a,b] \\ D_H^{\alpha_2,\beta_2,\sigma,\psi}x_2(t)\in F_2(t,x_1(t),x_2(t)), & a.e.\ t\in[a,b] \end{cases},$$

with multi-point and integro-multi-strip boundary conditions of the form

$$(1.2)\qquad \begin{cases} x_1(a)=0, & \int_a^b\psi'(s)x_1(s)ds=\sum_{i=1}^n k_i\int_{\xi_i}^{\eta_i}\psi'(s)x_2(s)ds+\sum_{j=1}^m\theta_jx_2(\zeta_j) \\ x_2(a)=0, & \int_a^b\psi'(s)x_2(s)ds=\sum_{i=1}^n \varphi_i\int_{\delta_i}^{\epsilon_i}\psi'(s)x_1(s)ds+\sum_{j=1}^m\nu_jx_1(z_j) \end{cases},$$

where $F_1(\cdot,\cdot,\cdot):[a,b]\times\mathbf{R}^2\to\mathcal{P}(\mathbf{R})$, $F_2(\cdot,\cdot,\cdot):[a,b]\times\mathbf{R}^2\to\mathcal{P}(\mathbf{R})$ are given set-valued maps, $D_H^{\alpha,\beta,\sigma,\psi}$ denotes the ψ-Hilfer generalized proportional fractional derivative operator of order $\alpha\in(1,2]$ and type $\beta\in[0,1]$, respectively, $\sigma\in(0,1]$, $a<\zeta_j<\xi_i<\eta_i<b$, $a<z_j<\delta_i<\epsilon_i<b$, $k_i,\theta_j,\varphi_i,\nu_j\in\mathbf{R}$, $j=\overline{1,m}$, $i=\overline{1,n}$ and $\psi(\cdot)\in C^1(I,\mathbf{R})$ is such that $\psi'(t)>0\ \forall\, t\in[a,b]$.

The starting point of our study is a very recent paper [16], where problem (1.1)-(1.2) is studied in the single-valued case; namely, the right-hand side in (1.1) is given by single-valued

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maps. Existence and uniqueness results are provided by using well-known fixed point theorems: Banach, Leray-Schauder and Krasnoselskii.

We also mention that in [8] is studied the problem

$$D_H^{\alpha,\beta,\sigma,\psi}x(t) \in F(t,x(t)) \quad a.e. \ ([a,b]), \tag{1.3}$$

$$x(a)=0, \quad \int_a^b \psi'(s)x(s)ds = \sum_{i=1}^n \varphi_i \int_{\xi_i}^{\eta_i} \psi'(s)x(s)ds + \sum_{j=1}^m \theta_j x(\zeta_j). \tag{1.4}$$

Also, in [7], differential inclusion (1.3) is studied with another boundary condition.

Our goal is to extend the study in [16] to the set-valued framework, and on the other hand, to generalize the study in [8] to the coupled case. The approach presented here avoids the applications of fixed point theorems and takes into account the case when the values of F_1 and F_2 are not convex; instead these set-valued maps are assumed to be Lipschitz in state variables. We establish an existence result for problem (1.1)-(1.2) by using Filippov's technique [10]; namely, the existence of solutions is obtained by starting from a pair of given "quasi" solutions. In addition, the result provides an estimate between the "quasi" solutions and the solutions obtained.

Even if the approach used here may be found in other classes of coupled systems of fractional differential inclusions [3]-[6], as far as we know, the present paper is the first in literature which contains an existence result of Filippov type for coupled systems of differential inclusions governed by ψ-Hilfer generalized proportional fractional derivatives.

The paper is organized as follows: in Section 2, we recall some preliminary results that we need in the sequel and in Section 3, we prove our main results.

2. Preliminaries

Let (X,d) be a metric space. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \ d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

Let $I=[a,b]$, we denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x(\cdot)||_C = \sup_{t\in I}|x(t)|$ and $L^1(I,\mathbf{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $||u(\cdot)||_1 = \int_a^b |u(t)|dt$. In what follows $\psi(\cdot) \in C^1(I,\mathbf{R})$ such that $\psi'(t) > 0 \ \forall \ t \in I$.

Definition 2.1. *Let $\sigma \in (0,1]$ and $\alpha \in \mathbf{R}_+$. The generalized proportional fractional integral of order α of $f(\cdot) \in L^1(I,\mathbf{R})$ with respect to $\psi(\cdot)$ is defined by*

$$I^{\alpha,\sigma,\psi}f(t) = \frac{1}{\sigma^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)f(s)ds,$$

where Γ is the (Euler's) Gamma function defined by $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt$.

Remark 2.1. *If $\sigma=1$, $\psi(t)=t$ the above definition yields the Riemann-Liouville fractional integral, if $\sigma=1$, $\psi(t)=\ln t$ the previous definition gives the Hadamard fractional integral and if $\sigma=1$, $\psi(t)=\frac{t^\rho}{\rho}$, $\rho>0$, Definition 2.1 covers the Katugampola fractional integral.*

Definition 2.2. *Let $\sigma \in (0,1]$ and $\alpha \in \mathbf{R}_+$. The generalized proportional fractional derivative of order α of $f(\cdot) \in C(I,\mathbf{R})$ with respect to $\psi(\cdot)$ is defined by*

$$D^{\alpha,\sigma,\psi}f(t) = \frac{1}{\sigma^{n-\alpha}\Gamma(n-\alpha)}D^n\left(\int_a^t e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{n-\alpha+1}\psi'(s)f(s)ds\right),$$

where $n = [\alpha] + 1$, $[\alpha]$ is the integer part of $\alpha \in \mathbf{R}$.

Definition 2.3. *Let $f(\cdot), \psi(\cdot) \in C^n(I, \mathbf{R})$ such that $\psi(t), \psi'(t) > 0 \; \forall \; t \in I$. The ψ-Hilfer generalized proportional fractional derivative operator of order α and type β, respectively, σ with respect to $\psi(\cdot)$ is defined by*

$$D_H^{\alpha,\beta,\sigma,\psi} f(t) = (I^{\beta(n-\alpha),\sigma,\psi}(D^{n,\sigma,\psi})I^{(1-\beta)(n-\alpha),\sigma,\psi} f)(t),$$

where $n - 1 < \alpha < n$, $\beta \in [0, 1]$, $\sigma \in (0, 1]$ and $n \in \mathbf{N}$.

In what follows $\alpha_i \in (1, 2]$ and $\gamma_i = \alpha_i + \beta_i(2 - \alpha_i)$, $i = 1, 2$, we use the notations:

$$A_i = \int_a^b \frac{e^{\frac{\sigma-1}{\sigma}(\psi(s)-\psi(a))}(\psi(s) - \psi(a))^{\gamma_i-1}\psi'(s)}{\sigma^{\gamma_i-1}\Gamma(\gamma_i)} ds \quad i = 1, 2,$$

$$B_1 = \sum_{i=1}^n k_i \int_{\xi_i}^{\eta_i} \frac{e^{\frac{\sigma-1}{\sigma}(\psi(s)-\psi(a))}(\psi(s) - \psi(a))^{\gamma_2-1}\psi'(s)}{\sigma^{\gamma_2-1}\Gamma(\gamma_2)} ds,$$

$$B_2 = \sum_{i=1}^n \varphi_i \int_{\delta_i}^{\epsilon_i} \frac{e^{\frac{\sigma-1}{\sigma}(\psi(s)-\psi(a))}(\psi(s) - \psi(a))^{\gamma_1-1}\psi'(s)}{\sigma^{\gamma_1-1}\Gamma(\gamma_1)} ds,$$

$$C_1 = \sum_{j=1}^m \theta_j \frac{e^{\frac{\sigma-1}{\sigma}(\psi(z_j)-\psi(a))}(\psi(z_j) - \psi(a))^{\gamma_2-1}}{\sigma^{\gamma_2-1}\Gamma(\gamma_2)},$$

$$C_2 = \sum_{j=1}^m \nu_j \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\zeta_j)-\psi(a))}(\psi(\zeta_j) - \psi(a))^{\gamma_1-1}}{\sigma^{\gamma_1-1}\Gamma(\gamma_1)}$$

and $L = A_1A_2 - (B_1 + C_1)(B_2 + C_2)$. The next result is proved in [16].

Lemma 2.1. *Let $p_1(\cdot) : [a, b] \to \mathbf{R}$, $p_2(\cdot) : [a, b] \to \mathbf{R}$ be continuous mappings and assume that $L \neq 0$. Then, the solution of the system*

$$\begin{cases} D_H^{\alpha_1,\beta_1,\sigma,\psi} x_1(t) = p_1(t), & t \in [a, b] \\ D_H^{\alpha_2,\beta_2,\sigma,\psi} x_2(t) = p_2(t), & t \in [a, b] \end{cases},$$

with boundary conditions (1.2) is given by

$$\begin{aligned} x_1(t) =& I^{\alpha_1,\sigma,\psi} p_1(t) + \frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(a))}(\psi(t) - \psi(a))^{\gamma_1-1}}{L\sigma^{\gamma_1-1}\Gamma(\gamma_1)} \\ &\times\{A_2(\sum_{i=1}^n k_i \int_{\xi_i}^{\eta_i} \psi'(s)I^{\alpha_2,\sigma,\psi}p_2(s)ds + \sum_{j=1}^m \theta_j I^{\alpha_2,\sigma,\psi}p_2(\zeta_j) - \int_a^b \psi'(s)I^{\alpha_1,\sigma,\psi}p_1(s)ds) \\ &+(B_1 + C_1)(\sum_{i=1}^n \varphi_i \int_{\delta_i}^{\epsilon_i} \psi'(s)I^{\alpha_1,\sigma,\psi}p_2(s)ds + \sum_{j=1}^m \nu_j I^{\alpha_1,\sigma,\psi}p_1(z_j) \\ &- \int_a^b \psi'(s)I^{\alpha_2,\sigma,\psi}p_2(s)ds)\}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} x_2(t) =& I^{\alpha_2,\sigma,\psi} p_2(t) + \frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(a))}(\psi(t) - \psi(a))^{\gamma_2-1}}{L\sigma^{\gamma_2-1}\Gamma(\gamma_2)} \\ &\times\{A_1(\sum_{i=1}^n \varphi_i \int_{\delta_i}^{\epsilon_i} \psi'(s) \cdot I^{\alpha_1,\sigma,\psi}p_1(s)ds + \sum_{j=1}^m \nu_j I^{\alpha_1,\sigma,\psi}p_1(z_j) - \int_a^b \psi'(s)I^{\alpha_2,\sigma,\psi}p_2(s)ds) \end{aligned}$$

$$+(B_2+C_2)(\sum_{i=1}^{n}k_i\int_{\xi_i}^{\eta_i}\psi'(s)I^{\alpha_2,\sigma,\psi}p_1(s)ds+\sum_{j=1}^{m}\theta_jI^{\alpha_2,\sigma,\psi}p_1(\zeta_j)$$

$$-\int_a^b\psi'(s)I^{\alpha_1,\sigma,\psi}p_1(s)ds)\}. \tag{2.6}$$

Definition 2.4. *The mappings $x_1(\cdot),x_2(\cdot)\in C(I,\mathbf{R})$ are said to be solutions of problem (1.1)-(1.2) if there exists $p_1(\cdot),p_2(\cdot)\in L^1(I,\mathbf{R})$ with $p_1(t)\in F_1(t,x_1(t),x_2(t))$ a.e. (I), $p_2(t)\in F_2(t,x_1(t),x_2(t))$ a.e. (I) and $x_1(\cdot)$ and $x_2(\cdot)$ are given by (2.5)-(2.6).*

In what follows, $\chi_A(\cdot)$ denotes the characteristic function of the set $A\subset\mathbf{R}$.

Remark 2.2. *We denote*

$$\begin{aligned}
\mathcal{A}_1(t,s)=&\frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\alpha_1-1}}{\sigma^{\alpha_1}\Gamma(\alpha_1)}\chi_{[a,t]}(s)\psi'(s),\\
\mathcal{B}_1(t,s)=&\frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\gamma_1-1}}{L\sigma^{\gamma_1-1}\Gamma(\gamma_1)}\\
&\times A_2\sum_{i=1}^{n}\frac{k_i}{\sigma^{\alpha_2}\Gamma(\alpha_2)}(\int_s^{\eta_i}e^{\frac{\sigma-1}{\sigma}(\psi(u)-\psi(s))}(\psi(u)-\psi(s))^{\alpha_2-1}\psi'(u)du)\chi_{[a,\eta_i]}(s)\psi'(s),\\
\mathcal{A}_2(t,s)=&-\frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\gamma_1-1}}{L\sigma^{\gamma_1-1}\Gamma(\gamma_1)}\\
&\times A_2\frac{1}{\sigma^{\alpha_1}\Gamma(\alpha_1)}(\int_s^{b}\psi'(s)e^{\frac{\sigma-1}{\sigma}(\psi(u)-\psi(s))}(\psi(u)-\psi(s))^{\alpha_1-1}\psi'(u)du)\psi'(s),\\
\mathcal{B}_2(t,s)=&\frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\gamma_1-1}}{L\sigma^{\gamma_1-1}\Gamma(\gamma_1)}\\
&\times A_2\sum_{j=1}^{m}\frac{\theta_j}{\sigma^{\alpha_2}\Gamma(\alpha_2)}e^{\frac{\sigma-1}{\sigma}(\psi(\zeta_j)-\psi(s))}(\psi(\zeta_j)-\psi(s))^{\alpha_2-1}\chi_{[a,\zeta_j]}(s)\psi'(s),\\
\mathcal{A}_3(t,s)=&\frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\gamma_1-1}}{L\sigma^{\gamma_1-1}\Gamma(\gamma_1)}\\
&\times(B_1+C_1)\sum_{j=1}^{m}\frac{\nu_j}{\sigma^{\alpha_1}\Gamma(\alpha_1)}e^{\frac{\sigma-1}{\sigma}(\psi(z_j)-\psi(s))}(\psi(z_j)-\psi(s))^{\alpha_1-1}\chi_{[a,z_j]}(s)\psi'(s),\\
\mathcal{B}_3(t,s)=&\frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\gamma_1-1}}{L\sigma^{\gamma_1-1}\Gamma(\gamma_1)}\\
&\times(B_1+C_1)\sum_{i=1}^{n}\frac{\varphi_i}{\sigma^{\alpha_1}\Gamma(\alpha_1)}(\int_s^{\epsilon_i}e^{\frac{\sigma-1}{\sigma}(\psi(u)-\psi(s))}(\psi(u)-\psi(s))^{\alpha_1-1}\psi'(u)du)\chi_{[a,\epsilon_i]}(s)\psi'(s),\\
\mathcal{B}_4(t,s)=&-\frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\gamma_1-1}}{L\sigma^{\gamma_1-1}\Gamma(\gamma_1)}\\
&\times(B_1+C_1)\frac{1}{\sigma^{\alpha_2}\Gamma(\alpha_2)}(\int_s^{b}\psi'(s)e^{\frac{\sigma-1}{\sigma}(\psi(u)-\psi(s))}(\psi(u)-\psi(s))^{\alpha_2-1}\psi'(u)du)\psi'(s),
\end{aligned}$$

$$\mathcal{D}_1(t,s) = \frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\alpha_2-1}}{\sigma^{\alpha_2}\Gamma(\alpha_2)}\chi_{[a,t]}(s)\psi'(s),$$

$$\mathcal{C}_1(t,s) = \frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\gamma_2-1}}{L\sigma^{\gamma_2-1}\Gamma(\gamma_2)}$$
$$\times A_1\sum_{i=1}^{n}\frac{\varphi_i}{\sigma^{\alpha_1}\Gamma(\alpha_1)}(\int_s^{\epsilon_i} e^{\frac{\sigma-1}{\sigma}(\psi(u)-\psi(s))}(\psi(u)-\psi(s))^{\alpha_1-1}\psi'(u)du)\chi_{[a,\epsilon_i]}(s)\psi'(s),$$

$$\mathcal{D}_2(t,s) = -\frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\gamma_2-1}}{L\sigma^{\gamma_2-1}\Gamma(\gamma_2)}$$
$$\times A_1\frac{1}{\sigma^{\alpha_2}\Gamma(\alpha_2)}(\int_s^b \psi'(s)e^{\frac{\sigma-1}{\sigma}(\psi(u)-\psi(s))}(\psi(u)-\psi(s))^{\alpha_2-1}\psi'(u)du)\psi'(s),$$

$$\mathcal{C}_2(t,s) = \frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\gamma_2-1}}{L\sigma^{\gamma_2-1}\Gamma(\gamma_2)}$$
$$\times A_1\sum_{j=1}^{m}\frac{\nu_j}{\sigma^{\alpha_1}\Gamma(\alpha_1)}e^{\frac{\sigma-1}{\sigma}(\psi(z_j)-\psi(s))}(\psi(z_j)-\psi(s))^{\alpha_1-1}\chi_{[a,z_j]}(s)\psi'(s),$$

$$\mathcal{D}_3(t,s) = \frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\gamma_2-1}}{L\sigma^{\gamma_2-1}\Gamma(\gamma_2)}$$
$$\times(B_2+C_2)\sum_{j=1}^{m}\frac{\theta_j}{\sigma^{\alpha_2}\Gamma(\alpha_2)}e^{\frac{\sigma-1}{\sigma}(\psi(\zeta_j)-\psi(s))}(\psi(\zeta_j)-\psi(s))^{\alpha_2-1}\chi_{[a,\zeta_j]}(s)\psi'(s),$$

$$\mathcal{C}_3(t,s) = \frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\gamma_2-1}}{L\sigma^{\gamma_2-1}\Gamma(\gamma_2)}$$
$$\times(B_2+C_2)\sum_{i=1}^{n}\frac{k_i}{\sigma^{\alpha_2}\Gamma(\alpha_2)}(\int_s^{\eta_i} e^{\frac{\sigma-1}{\sigma}(\psi(u)-\psi(s))}(\psi(u)-\psi(s))^{\alpha_2-1}\psi'(u)du)\chi_{[a,\eta_i]}(s)\psi'(s),$$

$$\mathcal{C}_4(t,s) = -\frac{e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\gamma_2-1}}{L\sigma^{\gamma_2-1}\Gamma(\gamma_2)}$$
$$\times\frac{B_2+C_2}{\sigma^{\alpha_1}\Gamma(\alpha_1)}(\int_s^b \psi'(s)e^{\frac{\sigma-1}{\sigma}(\psi(u)-\psi(s))}(\psi(u)-\psi(s))^{\alpha_2-1}\psi'(u)du)\psi'(s),$$

$\mathcal{R}_1(t,s) = \mathcal{A}_1(t,s)+\mathcal{A}_2(t,s)+\mathcal{A}_3(t,s)$, $\mathcal{R}_2(t,s) = \mathcal{B}_1(t,s)+\mathcal{B}_2(t,s)+\mathcal{B}_3(t,s)+\mathcal{B}_4(t,s)$, $\mathcal{R}_3(t,s) = \mathcal{C}_1(t,s)+\mathcal{C}_2(t,s)+\mathcal{C}_3(t,s)+\mathcal{C}_4(t,s)$ and $\mathcal{R}_4(t,s) = \mathcal{D}_1(t,s)+\mathcal{D}_2(t,s)+\mathcal{D}_3(t,s)$, then the solutions $(x_1(.), x_2(.))$ in Lemma 2.1 may be put as

$$\begin{array}{ll} x_1(t) = \int_a^b \mathcal{R}_1(t,s)p_1(s)ds + \int_a^b \mathcal{R}_2(t,s)p_2(s)ds, & t\in I \\ x_2(t) = \int_a^b \mathcal{R}_3(t,s)p_1(s)ds + \int_a^b \mathcal{R}_4(t,s)p_2(s)ds, & t\in I \end{array}.$$

Moreover, if we assume that there exists $M_0 > 0$ such that $0 < \psi'(t) \le M_0\ \forall\ t\in I$, it follows that $\forall\ t,s\in I$, we have

$$|\mathcal{A}_1(t,s)| \le \frac{1}{\sigma^{\alpha_1}\Gamma(\alpha_1)}(\psi(b)-\psi(a))^{\alpha_1-1}M_0 =: a_1,$$

$$|\mathcal{A}_2(t,s)| \le \frac{|A_2|M_0^2(b-a)(\psi(b)-\psi(a))^{\alpha_1+\gamma_1-2}}{|L|\sigma^{\alpha_1+\gamma_1-1}\Gamma(\gamma_1)\Gamma(\alpha_1)} =: a_2,$$

$$|\mathcal{A}_3(t,s)| \leq \frac{(|B_1|+|C_1|)M_0(\psi(b)-\psi(a))^{\gamma_1-1}}{|L|\sigma^{\alpha_1+\gamma_1-1}\Gamma(\gamma_1)\Gamma(\alpha_1)}\sum_{j=1}^{m}|\nu_j|(\psi(z_j)-\psi(a))^{\alpha_1-1} =: a_3,$$

$$|\mathcal{B}_1(t,s)| \leq \frac{|A_2|M_0^2(\psi(b)-\psi(a))^{\alpha_2+\gamma_1-2}}{|L|\sigma^{\alpha_2+\gamma_1-1}\Gamma(\gamma_1)\Gamma(\alpha_2)}\sum_{i=1}^{n}|k_i|(\eta_i-a) =: b_1,$$

$$|\mathcal{B}_2(t,s)| \leq \frac{|A_2|M_0(\psi(b)-\psi(a))^{\gamma_1-1}}{|L|\sigma^{\alpha_2+\gamma_1-1}\Gamma(\gamma_1)\Gamma(\alpha_2)}\sum_{j=1}^{m}|\theta_j|(\psi(\zeta_j)-\psi(a))^{\alpha_2-1} =: b_2,$$

$$|\mathcal{B}_3(t,s)| \leq \frac{(|B_1|+|C_1|)M_0^2(\psi(b)-\psi(a))^{\alpha_1+\gamma_1-2}}{|L|\sigma^{\alpha_1+\gamma_1-1}\Gamma(\gamma_1)\Gamma(\alpha_1)}\sum_{i=1}^{n}|\varphi_i|(\epsilon_i-a) =: b_3,$$

$$|\mathcal{B}_4(t,s)| \leq \frac{(|B_1|+|C_1|)M_0^2(\psi(b)-\psi(a))^{\alpha_2+\gamma_1-2}}{|L|\sigma^{\alpha_2+\gamma_1-1}\Gamma(\gamma_1)\Gamma(\alpha_2)}(b-a) =: b_4,$$

$$|\mathcal{D}_1(t,s)| \leq \frac{1}{\sigma^{\alpha_2}\Gamma(\alpha_2)}(\psi(b)-\psi(a))^{\alpha_2-1}M_0 =: d_1,$$

$$|\mathcal{D}_2(t,s)| \leq \frac{|A_1|M_0^2(b-a)(\psi(b)-\psi(a))^{\alpha_2+\gamma_2-2}}{|L|\sigma^{\alpha_2+\gamma_2-1}\Gamma(\gamma_2)\Gamma(\alpha_2)} =: d_2,$$

$$|\mathcal{D}_3(t,s)| \leq \frac{(|B_2|+|C_2|)M_0(\psi(b)-\psi(a))^{\gamma_2-1}}{|L|\sigma^{\alpha_2+\gamma_2-1}\Gamma(\gamma_2)\Gamma(\alpha_2)}\sum_{j=1}^{m}|\theta_j|(\psi(\zeta_j)-\psi(a))^{\alpha_2-1} =: d_3,$$

$$|\mathcal{C}_1(t,s)| \leq \frac{|A_1|M_0^2(\psi(b)-\psi(a))^{\alpha_1+\gamma_2-2}}{|L|\sigma^{\alpha_1+\gamma_2-1}\Gamma(\gamma_2)\Gamma(\alpha_1)}\sum_{i=1}^{n}|\varphi_i|(\epsilon_i-a) =: c_1,$$

$$|\mathcal{C}_2(t,s)| \leq \frac{|A_1|M_0(\psi(b)-\psi(a))^{\gamma_2-1}}{|L|\sigma^{\alpha_1+\gamma_2-1}\Gamma(\gamma_2)\Gamma(\alpha_1)}\sum_{j=1}^{m}|\nu_j|(\psi(z_j)-\psi(a))^{\alpha_1-1} =: c_2,$$

$$|\mathcal{C}_3(t,s)| \leq \frac{(|B_2|+|C_2|)M_0^2(\psi(b)-\psi(a))^{\alpha_2+\gamma_2-2}}{|L|\sigma^{\alpha_2+\gamma_2-1}\Gamma(\gamma_2)\Gamma(\alpha_2)}\sum_{i=1}^{n}|k_i|(\eta_i-a) =: c_3,$$

$$|\mathcal{C}_4(t,s)| \leq \frac{(|B_2|+|C_2|)M_0^2(\psi(b)-\psi(a))^{\alpha_1+\gamma_2-2}}{|L|\sigma^{\alpha_1+\gamma_2-1}\Gamma(\gamma_2)\Gamma(\alpha_1)}(b-a) =: c_4.$$

If we put $M_1 = a_1 + a_2 + a_3$, $M_2 = b_1 + b_2 + b_3 + b_4$, $M_3 = c_1 + c_2 + c_3 + c_4$ *and* $M_4 = d_1 + d_2 + d_3$, *one has* $|\mathcal{R}_i(t,s)| \leq M_i, \forall\, t, s \in I$ *and* $i = 1, 2, 3, 4$.

Finally, we recall a variant of Kuratowski and Ryll-Nardzewski selection theorem concerning measurable set-valued maps proved in [1].

Lemma 2.2. *Consider* X *a separable Banach space,* B *is the closed unit ball in* X, $H : I \to \mathcal{P}(X)$ *is a set-valued map with nonempty closed values and* $g : I \to X, L : I \to \mathbf{R}_+$ *are measurable functions. If*

$$H(t) \cap (g(t) + L(t)B) \neq \emptyset \quad a.e.\ (I),$$

then the set-valued map $t \to H(t) \cap (g(t) + L(t)B)$ *has a measurable selection.*

3. The Results

In what follows, we need the following hypotheses.

Hypothesis. (i) $F_1 : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ and $F_2 : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ have nonempty closed values and the set-valued maps $F_1(\cdot, y_1, y_2)$, $F_2(\cdot, y_1, y_2)$ are measurable for any $y_1, y_2 \in \mathbf{R}$.

(ii) There exist $l_1(\cdot), l_2(\cdot) \in L^1(I,(0,\infty))$ such that, for almost all $t \in I$, $F_1(t,\cdot,\cdot)$ is $l_1(t)$-Lipschitz and $F_2(t,\cdot,\cdot)$ is $l_2(t)$-Lipschitz, i.e.,

$$\begin{array}{ll} d_H(F_1(t,y_1,z_1),F_1(t,y_2,z_2)) \leq l_1(t)(|y_1-y_2|+|z_1-z_2|) & \forall\, y_1,y_2,z_1,z_2 \in \mathbf{R} \\ d_H(F_2(t,y_1,z_1),F_2(t,y_2,z_2)) \leq l_2(t)(|y_1-y_2|+|z_1-z_2|) & \forall\, y_1,y_2,z_1,z_2 \in \mathbf{R} \end{array}.$$

Next, we use the notation $l(t) = M_1 l_1(t) + M_2 l_2(t) + M_3 l_1(t) + M_4 l_2(t)$, $t \in I$.

Theorem 3.1. *Assume that $L \neq 0$, Hypothesis is satisfied, $|l(\cdot)|_1 < 1$ and the mappings $(y_1(\cdot), y_2(\cdot)) \in C(I,\mathbf{R})^2$ are such that there exist $L_1(\cdot), L_2(\cdot) \in L^1(I,\mathbf{R})$ that verify*

$$\begin{aligned} d(D_H^{\alpha_1,\beta_1,\sigma,\psi} y_1(t), F_1(t,y_1(t),y_2(t))) \leq & L_1(t) \;\; a.e.\ t \in I \\ d(D_H^{\alpha_2,\beta_2,\sigma,\psi} y_2(t), F_2(t,y_1(t),y_2(t))) \leq & L_2(t) \;\; a.e.\ t \in I \\ y_1(a) = & y_2(a) = 0 \\ \int_a^b \psi'(s) y_1(s) ds = & \sum_{i=1}^n k_i \int_{\xi_i}^{\eta_i} \psi'(s) \cdot y_2(s) ds + \sum_{j=1}^m \theta_j y_2(\zeta_j) \\ \int_a^b \psi'(s) y_2(s) ds = & \sum_{i=1}^n \varphi_i \int_{\delta_i}^{\epsilon_i} \psi'(s) y_1(s) ds + \sum_{j=1}^m \nu_j y_1(z_j). \end{aligned}$$

Then, there exists $(x_1(\cdot), x_2(\cdot)) \in C(I,\mathbf{R})^2$ a solution of problem (1.1)-(1.2) satisfying for all $t \in I$

$$|x_1(t)-y_1(t)| + |x_2(t)-y_2(t)| \leq \frac{(M_1+M_3)|L_1(\cdot)|_1 + (M_2+M_4)|L_2(\cdot)|_1}{1-|l(\cdot)|_1}. \tag{3.7}$$

Proof. The assumptions in the statement of the theorem may be rewritten as

$$\begin{array}{ll} F_1(t,y_1(t),y_2(t)) \cap \{D_H^{\alpha_1,\beta_1,\sigma,\psi} y_1(t) + L_1(t)[-1,1]\} \neq \emptyset & a.e.\ (I) \\ F_2(t,y_1(t),y_2(t)) \cap \{D_H^{\alpha_2,\beta_2,\sigma,\psi} y_2(t) + L_2(t)[-1,1]\} \neq \emptyset & a.e.\ (I) \end{array}.$$

We apply Lemma 2.2, in order to deduce the existence of mesurable selections $p_1^1(t) \in F_1(t,y_1(t),y_2(t))$, $p_2^1(t) \in F_2(t,y_1(t),y_2(t))$ $a.e.\ (I)$ such that

$$\begin{array}{ll} |p_1^1(t) - D_H^{\alpha_1,\beta_1,\sigma,\psi} y_1(t)| \leq L_1(t) & a.e.\ (I) \\ |p_2^1(t) - D_H^{\alpha_2,\beta_2,\sigma,\psi} y_2(t)| \leq L_2(t) & a.e.\ (I) \end{array}$$

and define

$$\begin{array}{ll} x_1^1(t) = \int_a^b \mathcal{R}_1(t,s) p_1^1(s) ds + \int_a^b \mathcal{R}_2(t,s) p_2^1(s) ds, & t \in I \\ x_2^1(t) = \int_a^b \mathcal{R}_3(t,s) p_1^1(s) ds + \int_a^b \mathcal{R}_4(t,s) p_2^1(s) ds, & t \in I \end{array}.$$

One has

$$\begin{array}{ll} |x_1^1(t) - y_1(t)| \leq M_1|L_1(\cdot)|_1 + M_2|L_2(\cdot)|_1 & \forall t \in I \\ |x_2^1(t) - y_2(t)| \leq M_3|L_1(\cdot)|_1 + M_4|L_2(\cdot)|_1 & \forall t \in I \end{array}$$

and so

$$|x_1^1(t)-y_1(t)| + |x_2^1(t)-y_2(t)| \leq (M_1+M_3)|L_1(\cdot)|_1 + (M_2+M_4)|L_2(\cdot)|_1 =: M.$$

Next, we construct, by induction, the sequences $x_n^1(\cdot), x_n^2(\cdot) \in C(I,\mathbf{R})$ and $p_n^1(\cdot)$, $p_n^2(\cdot) \in L^1(I,\mathbf{R})$, $n \geq 1$ such that

$$\begin{array}{ll} x_1^n(t) = \int_a^b \mathcal{R}_1(t,s) p_1^n(s) ds + \int_a^b \mathcal{R}_2(t,s) p_2^n(s) ds, & t \in I \\ x_2^n(t) = \int_a^b \mathcal{R}_3(t,s) p_1^n(s) ds + \int_a^b \mathcal{R}_4(t,s) p_2^n(s) ds, & t \in I \end{array}. \tag{3.8}$$

$$p_1^n(t) \in F_1(t, x_1^{n-1}(t), x_2^{n-1}(t)), \quad p_2^n(t) \in F_2(t, x_1^{n-1}(t), x_2^{n-1}(t)) \quad a.e.\,(I), \tag{3.9}$$

$$(3.10)\qquad \begin{array}{ll} |p_1^{n+1}(t)-p_1^n(t)|\leq l_1(t)(|x_1^n(t)-x_1^{n-1}(t)|+|x_2^n(t)-x_2^{n-1}(t)|) & a.e.\ (I) \\ |p_2^{n+1}(t)-p_2^n(t)|\leq l_2(t)(|x_1^n(t)-x_1^{n-1}(t)|+|x_2^n(t)-x_2^{n-1}(t)|) & a.e.\ (I) \end{array}.$$

Let us note that from (3.8)-(3.10), it follows

$$(3.11)\qquad |x_1^{n+1}(t)-x_1^n(t)|+|x_2^{n+1}(t)-x_2^n(t)|\leq k(|l(\cdot)|_1)^n \quad a.e.\ (I),\quad \forall n\in \mathbf{N}.$$

Indeed, since the case $n=0$ is proved, we assume (3.11) valid for $n-1$. For almost all $t\in I$, one may write

$$\begin{aligned} |x_1^{n+1}(t)-x_1^n(t)|\leq &\int_a^b |\mathcal{R}_1(t,s)|\cdot|p_1^{n+1}(s)-p_1^n(s)|ds+\int_a^b |\mathcal{R}_2(t,s)|\cdot|p_2^{n+1}(s)-p_2^n(s)|ds \\ \leq &M_1\int_a^b |p_1^{n+1}(s)-p_1^n(s)|ds+M_2\int_a^b |p_2^{n+1}(s)-p_2^n(s)|ds \\ \leq &M_1\int_a^b l_1(s)(|x_1^n(s)-x_1^{n-1}(s)|+|x_2^n(s)-x_2^{n-1}(s)|)ds \\ &+M_2\int_a^b l_2(s)(|x_1^n(s)-x_1^{n-1}(s)|+|x_2^n(s)-x_2^{n-1}(s)|)ds \\ \leq &M(|l(.)|_1)^{n-1}(M_1\int_a^b l_1(s)ds+M_2\int_a^b l_2(s)ds). \end{aligned}$$

Similarly, we get for almost all $t\in I$,

$$|x_2^{n+1}(t)-x_2^n(t)|\leq M(|l(\cdot)|_1)^{n-1}(M_3\int_a^b l_1(s)ds+M_4\int_a^b l_2(s)ds).$$

Thus, (3.11) is true for n. From (3.11), the sequences $\{x_1^n(\cdot)\}$, $\{x_2^n(\cdot)\}$ are Cauchy in the space $C(I,\mathbf{R})$. Let $x_1(\cdot)\in C(I,\mathbf{R})$ and $x_2(\cdot)\in C(I,\mathbf{R})$ be their limits in $C(I,\mathbf{R})$. Also from (3.10), we deduce that, for almost all $t\in I$, the sequences $\{p_1^n(t)\}$, $\{p_2^n(t)\}$ are Cauchy in $\mathbf{R}$. Let $p_1(\cdot)$, $p_2(\cdot)$ be their pointwise limit. From Hypothesis and inequality (3.11), we find

$$\begin{aligned} |x_1^n(t)-y_1(t)|+|x_2^n(t)-y_2(t)|\leq &|x_1^1(t)-y_1(t)|+|x_2^1(t)-y_2(t)| \\ &+\sum_{i=1}^{n-1}(|x_1^{i+1}(t)-x_1^i(t)|+|x_2^{i+1}(t)-x_2^i(t)|) \\ \leq &M+\sum_{i=1}^{n}M(|l(\cdot)|_1)^i \\ \leq &\frac{M}{1-|l(\cdot)|_1}. \end{aligned}\qquad (3.12)$$

and

$$\begin{aligned} &|p_1^n(t)-D_H^{\alpha_1,\beta_1,\sigma,\psi}y_1(t)|+|p_2^n(t)-D_H^{\alpha_2,\beta_2,\sigma,\psi}y_2(t)| \\ \leq &|p_1^1(t)-D_H^{\alpha_1,\beta_1,\sigma,\psi}y_1(t)| \\ &+|p_2^1(t)-D_H^{\alpha_2,\beta_2,\sigma,\psi}y_2(t)|+\sum_{i=1}^{n-1}(|p_1^{i+1}(t)-p_1^i(t)|+|p_2^{i+1}(t)-p_2^i(t)|) \end{aligned}$$

$$\leq |p_1^1(t) - D_H^{\alpha_1,\beta_1,\sigma,\psi} y_1(t)| + |p_2^1(t) - D_H^{\alpha_1,\beta_1,\sigma,\psi} y_2(t)|$$
$$+ \sum_{i=1}^{n-1} (l_1(t) + l_2(t))(|x_1^i(t) - x_1^{i-1}(t)| + |x_2^i(t) - x_2^{i-1}(t)|)$$
$$\leq L_1(t) + L_2(t) + (l_1(t) + l_2(t)) \frac{M}{1 - |l(\cdot)|_1} \tag{3.13}$$

for almost all $t \in I$. Therefore, the sequences $p_1^n(\cdot)$, $p_2^n(\cdot)$ are integrably bounded and $p_1(\cdot) \in L^1(I, \mathbf{R})$, $p_2(\cdot) \in L^1(I, \mathbf{R})$.

Now, we realize the construction in (3.8)-(3.10). We assume that for $K \geq 1$, already exist $x_1^k(\cdot), x_2^k(\cdot) \in C(I, \mathbf{R})$ and $p_1^k(\cdot), p_2^k(\cdot) \in L^1(I, \mathbf{R})$, $k = 1, 2, ..., K$ with (3.8) and (3.10) for $k = 1, 2, ..., K$ and (3.9) for $k = 1, 2, ..., K - 1$. Using the lipschitzianity of $F_1(t, \cdot, \cdot)$ and $F_2(t, \cdot, \cdot)$

$$F_1(t, x_1^K(t), x_2^K(t)) \cap \{p_1^K(t) + (l_1(t)|x_1^K(t) - x_1^{K-1}(t)| + l_1(t)|x_2^K(t) - x_2^{K-1}(t)|)[-1, 1]\} \neq \emptyset,$$

$$F_2(t, x_1^K(t), x_2^K(t)) \cap \{p_2^K(t) + (l_2(t)|x_1^K(t) - x_1^{K-1}(t)| + l_2(t)|x_2^K(t) - x_2^{K-1}(t)|)[-1, 1]\} \neq \emptyset$$

for almost all $t \in I$. Again, with Lemma 2.2, we find the existence of measurable selections $p_1^{K+1}(\cdot)$ of $F_1(\cdot, x_K^1(\cdot), x_K^2(\cdot))$ and $p_2^{K+1}(\cdot)$ of $F_2(\cdot, x_K^1(\cdot), x_K^2(\cdot))$ such that

$$\begin{array}{ll} |p_1^{K+1}(t) - p_1^K(t)| \leq l_1(t)(|x_1^K(t) - x_1^{K-1}(t)| + |x_2^K(t) - x_2^{K-1}(t)|) & a.e.\ (I) \\ |p_2^{K+1}(t) - p_2^K(t)| \leq l_2(t)(|x_1^K(t) - x_1^{K-1}(t)| + |x_2^K(t) - x_2^{K-1}(t)|) & a.e.\ (I) \end{array}.$$

We define $x_1^{K+1}(\cdot), x_2^{K+1}(\cdot)$ as in (3.8) with $n = K + 1$. Passing with $n \to \infty$ in (3.8) and (3.12), we finish the proof. □

In above theorem, if we take as "quasi" solutions $y_1(\cdot) = y_2(\cdot) = 0$, one may obtain a statement similar to a result that can be derived by using the set-valued contration principle.

Corollary 3.1. *Assume that $L \neq 0$, Hypothesis is satisfied, $d(0, F_1(t, 0, 0)) \leq l_1(t)$ a.e. $t \in I$, $d(0, F_2(t, 0, 0)) \leq l_2(t)$ a.e. $t \in I$ and $|l(\cdot)|_1 < 1$. Then, there exists $(x_1(\cdot), x_2(\cdot)) \in C(I, \mathbf{R})^2$ a solution of problem* (1.1)-(1.2) *satisfying for all $t \in I$*

$$|x_1(t)| + |x_2(t)| \leq \frac{(M_1 + M_3)|l_1(\cdot)|_1 + (M_2 + M_4)|l_2(\cdot)|_1}{1 - |l(\cdot)|_1}.$$

Proof. We apply Theorem 3.1 with $y_1(\cdot) = y_2(\cdot) = 0$, $L_1(\cdot) = l_1(\cdot)$ and $L_2(\cdot) = l_2(\cdot)$. □

Remark 3.3. *If in* (1.1), *F_1 and F_2 are single-valued maps, Corollary 3.1 provides a generalization to the set-valued framework of* [16, Theorem 1] *whose proof is done using Banach's contraction principle.*

Example 3.1. *As an example, we consider the problem*

$$\begin{cases} D_H^{\frac{4}{3},\frac{3}{4},\frac{2}{5},\psi} x_1(t) \in [-\frac{1}{5}\frac{|\cos(x_1(t))|}{1+|\cos(x_1(t))|}, 0] \cup [0, \frac{1}{5}\frac{|x_2(t)|}{1+|x_2(t)|}] & a.e.\ [\frac{1}{7}, \frac{20}{7}] \\ D_H^{\frac{6}{5},\frac{5}{6},\frac{2}{5},\psi} x_2(t) \in [-\frac{1}{5}\frac{|x_1(t)|}{1+|x_1(t)|}, 0] \cup [0, \frac{1}{5}\frac{|\sin(x_2(t))|}{1+|\sin(x_2(t))|}] & a.e.\ [\frac{1}{7}, \frac{20}{7}] \end{cases} \tag{3.14}$$

with $\psi(z) = \frac{z+2}{z+3}$ and nonlocal integral boundary conditions as in [16]

$$\begin{cases} x_1(\frac{1}{7}) = 0, \quad \int_{\frac{1}{7}}^{\frac{20}{7}} \frac{x_1(t)}{(t+3)^2} dt = \frac{1}{11} \int_{\frac{2}{7}}^{\frac{3}{7}} \frac{x_2(t)}{(t+3)^2} dt + \frac{2}{13} \int_{\frac{8}{7}}^{\frac{9}{7}} \frac{x_2(t)}{(t+3)^2} dt + \frac{3}{17} \int_{2}^{\frac{15}{7}} \frac{x_2(t)}{(t+3)^2} dt + \\ \frac{4}{17} x_2(\frac{4}{7}) + \frac{5}{21} x_2(\frac{10}{7}) + \frac{6}{23} x_2(\frac{16}{7}) \end{cases}, \tag{3.15}$$

$$\begin{cases} x_2(\frac{1}{7}) = 0, \quad \int_{\frac{1}{7}}^{\frac{20}{7}} \frac{x_2(t)}{(t+3)^2} dt = \frac{7}{24} \int_{\frac{5}{7}}^{\frac{6}{7}} \frac{x_1(t)}{(t+3)^2} dt + \frac{8}{27} \int_{\frac{11}{7}}^{\frac{12}{7}} \frac{x_1(t)}{(t+3)^2} dt + \frac{9}{29} \int_{\frac{17}{7}}^{\frac{18}{7}} \frac{x_1(t)}{(t+3)^2} dt + \\ \frac{10}{31} x_1(1) + \frac{11}{34} x_1(\frac{13}{7}) + \frac{12}{37} x_1(\frac{19}{7}). \end{cases} \tag{3.16}$$

Therefore, in this case $F_1(t,(x_1,x_2)) = [-\frac{1}{5}\frac{|\cos x_1|}{1+|\cos x_1|}, 0] \cup [0, \frac{1}{5}\frac{|x_2|}{1+|x_2|}]$, $F_2(t,(x_1,x_2))$ $= [-\frac{1}{5}\frac{|x_1|}{1+|x_1|}, 0] \cup [0, \frac{1}{5}\frac{|\sin x_2|}{1+|\sin x_2|}]$, $\alpha_1 = \frac{4}{2}$, $\alpha_2 = \frac{6}{5}$, $\beta_1 = \frac{3}{4}$, $\beta_2 = \frac{5}{6}$, $\sigma = \frac{2}{5}$, $a = \frac{1}{7}$, $b = \frac{20}{7}$, $n = 3$, $m = 3$, $k_1 = \frac{1}{11}$, $k_2 = \frac{2}{13}$, $k_3 = \frac{3}{17}$, $\eta_1 = \frac{3}{7}$, $\eta_2 = \frac{9}{7}$, $\eta_3 = \frac{15}{7}$, $\xi_1 = \frac{2}{7}$, $\xi_2 = \frac{8}{7}$, $\xi_3 = 2$, $\theta_1 = \frac{4}{19}$, $\theta_2 = \frac{5}{21}$, $\theta_3 = \frac{6}{23}$, $\zeta_1 = \frac{4}{7}$, $\zeta_2 = \frac{10}{7}$, $\zeta_3 = \frac{16}{7}$, $\varphi_1 = \frac{7}{24}$, $\varphi_2 = \frac{8}{27}$, $\varphi_3 = \frac{9}{29}$, $\epsilon_1 = \frac{6}{7}$, $\epsilon_2 = \frac{12}{7}$, $\epsilon_3 = \frac{18}{7}$, $\delta_1 = \frac{5}{7}$, $\delta_2 = \frac{11}{7}$, $\delta_3 = \frac{17}{7}$, $\nu_1 = \frac{10}{11}$, $\nu_2 = \frac{11}{34}$, $\nu_3 = \frac{12}{37}$, $z_1 = 1$, $z_2 = \frac{13}{7}$, $z_3 = \frac{19}{7}$, $\gamma_1 = \frac{11}{6}$, $\gamma_2 = \frac{28}{15}$. *For all* $t \in [\frac{1}{7}, \frac{20}{7}]$ *and all* $x_1, x_2, y_1, y_2 \in \mathbf{R}$, *we have*

$$\sup\{|z|;\ z \in F_i(t,(x_1,x_2))\} \leq \frac{1}{5}, \quad i = 1,2,$$

$$d_H(F_1(t,(x_1,x_2)), F_2(t,(y_1,y_2)) \leq \frac{1}{5}|x_1 - y_1| + \frac{1}{5}|x_2 - y_2| \quad i = 1,2.$$

By standard computations (e.g., [16]*),* $A_1 \approx 0.0323$, $B_1 \approx 0.0006$, $C_1 \approx 0.1733$, $A_2 \approx 0.0303$, $B_2 \approx 0.0001$, $C_2 \approx 0.2898$, $|L| \approx 0.04937$, $M_1 \approx 0.4615$, $M_2 \approx 0.0607$, $M_3 \approx 0.6458$, $M_4 \approx 0.5776$ *and* $(M_1 + M_3)\frac{5}{8}\frac{19}{17} + (M_2 + M_4)\frac{5}{8}\frac{19}{17} \approx 0.952 < 1$. *Hence, we apply Corollary 3.1 in order to deduce the existence of a solution for problem* (3.14)-(3.16).

References

[1] J. P. Aubin, H. Frankowska: *Set-valued Analysis*, Birkhăuser, Basel (1990).

[2] D. Băleanu, K. Diethelm, E. Scalas and J. J. Trujillo: *Fractional Calculus Models and Numerical Methods*, World Scientific, Singapore (2012).

[3] A. Cernea: *Existence of solutions for some coupled systems of fractional differential inclusions*, Mathematics, **8** (700) (2020), 1–10.

[4] A. Cernea: *On some coupled systems of fractional differential inclusions*, Fract. Differ. Calc., **11** (2021), 133–145.

[5] A. Cernea: *A note on a coupled system of Caputo-Fabrizio fractional differential inclusions*, Ann. Commun. Math., **4** (2021), 190–195.

[6] A. Cernea: *A note on a coupled system of Hilfer fractional differential inclusions*, Foundations, **2** (2022), 290–297.

[7] A. Cernea: *On a Hilfer type fractional integro-differential inclusion*, Bull. Transilv. Univ. Braşov Ser. III. Math. Comput. Sci., **16** (2023), 71–80.

[8] A. Cernea: *On a Hilfer generalized proportional fractional integro-differential inclusion*, Proceedings of International Conference of Fractional Differentiation and its Applications ICFDA 2024, Bordeaux (France), to appear.

[9] K. Diethelm: *The Analysis of Fractional Differential Equations*, Springer, Berlin (2010).

[10] A. F. Filippov: *Classical solutions of differential equations with multivalued right hand side*, SIAM J. Control, **5** (1967), 609–621

[11] R. Hilfer: *Applications of Fractional Calculus in Physics*, World Scientific, Singapore (2000).

[12] R. Hilfer, Y. Luchko and Z. Tomovski: *Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivative*, Fract. Calc. Appl. Anal., **12** (2009), 299–318.

[13] A. Kilbas, H. M. Srivastava and J. J. Trujillo: *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam (2006).

[14] I. Mallah, I. Ahmed, A. Akgul, F. Jarad and S. Alha: *On ψ-Hilfer generalized proportional fractional operators*, AIMS Mathematics, **7** (2021), 82–103.

[15] K. Miller, B. Ross: *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York (1993).

[16] S. K. Ntouyas, B. Ahmad and J. Tariboon: *Coupled systems of nonlinear proportional fractional differential equations of the Hilfer-type with multi-point and integro-multi-strip boundary conditions*, Foundations, **2** (3) (2023), 241–259.

[17] I. Podlubny: *Fractional Differential Equations*, Academic Press, San Diego (1999).

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