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Research Article

Approximation of bounded functions by positive linear operators

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ABSTRACT. A general family of positive linear operators associated with a power expansion is studied. An upper estimate of the rate of convergence is obtained for bounded continuous functions in $[0,\infty)$ that has limit when $x \to \infty$. Applications are included.

Keywords: Positive linear operators, power series, rate of convergence.

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1. INTRODUCTION

In order to simplify notations, we set $I = [0,\infty)$. Let $C_b(I)$ is the space of all bounded continuous functions $f : [0,\infty) \to \mathbb{R}$. Moreover, we set $C_{b,\infty}(I)$ for the functions $f \in C_b[0,\infty)$ such that the limit

$$\lim_{x\to\infty} f(x)$$

exists. Moreover, for $f \in C_{b,\infty}(I)$ we consider the norm

$$\|f\| = \sup_{x\in I} |f(x)|.$$

As usual, we denote $e_k(x) = x^k$, for $k \in \mathbb{N}_0$. For fixed sequences $\{a_{n,k}\}_{n,k=0}^{\infty}$ of positive real numbers and $x \geq 0$, set

$$g_n(x) = \sum_{k=0}^{\infty} \frac{a_{n,k}}{k!} x^k, \tag{1.1}$$

where we assume that the series converges for all $x \geq 0$. For $f \in C_b(I)$ and a fixed increasing sequence $\{\beta(n)\}$ such that $\beta(n) \geq 1$ and $\lim_{n\to\infty} \beta(n) = \infty$, we consider the positive linear operators

$$L_n(f,x) = \frac{1}{g_n(x)} \sum_{k=0}^{\infty} \frac{a_{n,k}}{k!} f(y_{n,k}) x^k, \qquad \text{where} \quad y_{n,k} = \frac{k}{\beta(n)}. \tag{1.2}$$

Throughout the work, we assume that L_n is defined by (1.2). We say that the sequence of operators $\{L_n\}$ is an approximation process in $C_{b,\infty}(I)$ if $L_n : C_{b,\infty}(I) \to C_{b,\infty}(I)$ and

$$\lim_{n\to\infty} \|L_n(f) - f\| = 0$$

for every $f \in C_{b,\infty}(I)$.

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In this work, we study the operators L_n is the space $C_{b,\infty}(I)$. There are essential differences between the spaces $C_{b,\infty}(I)$ and $C_b(I)$. There are sequences $\{L_n\}$ such that $L_n : C_{b,\infty}(I) \to C_{b,\infty}(I)$ is an approximation process, while there exists $f \in C_b(I)$ such that $L_n(f)$ does not converges to f in norm. Let us state some questions related with the operators L_n in (1.2).

Problem 1.1. *Is it true that $L_n(C_{b,\infty}(I)) \subset C_{b,\infty}(I)$ for each $n \in \mathbb{N}$?*

Problem 1.2. *Find conditions on $\{g_n\}$ so that $\{L_n\}$ is an approximation process in $C_{b,\infty}(I)$.*

We use the notations

$$I_{n,i}(x) = \frac{g_n^{(i)}(x)}{\beta^i(n)g_n(x)} \qquad \text{and} \qquad J_{n,i}(x) = x^i I_{n,i}(x), \tag{1.3}$$

and consider the following two conditions related with the functions g_n:

(i) for $i = 1$ and $i = 2$, there exists a constant K_i such that for every $x > 0$, $n \in \mathbb{N}$, one has

$$\Big| I_{n,i}(x) - 1 \Big| \leq \frac{K_i}{1 + \beta(n)x}, \tag{1.4}$$

(ii) there exists a constant C such that for each $x \geq 0$ and $n \in \mathbb{N}$,

$$| \, I_{n,2}(x) - 2I_{n,1}(x) + 1 \, | \leq \frac{C}{\beta^2(n)}. \tag{1.5}$$

In this work, we obtain upper estimates for the rate of convergence of the operators L_n in the case when condition (1.4) or condition (1.5) holds. In Section 2, we included a few known results. Section 3 is devoted to verify that the operators L_n are an endomorphisms in the space $C_{b,\infty}(I)$. In Section 4, we prove some Korovkin-type theorems. In Section 5, we show that the conditions presented above are sufficient to proof that the family $\{L_n\}$ is an approximation process in $C_{b,\infty}(I)$. Section 6 contains the main results, we obtain upper estimates for the rate of convergence associated of the family $\{L_n\}$. In the last section, we present several examples.

For $x \geq 0$, $n \in \mathbb{N}$, and a function $f : I \to \mathbb{R}$ Szász [17] defined

$$S_n(f, x) = e^{-nx} \sum_{k=0}^{\infty} \frac{n^k}{k!} f\Big(\frac{k}{n}\Big) x^k, \tag{1.6}$$

whenever the series converges. These operators were also studied by Mirakyan [13], that is the reason why they are usually called Szász-Mirakyan operators. There is a large collection of works devoted to study the operators S_n and some modifications. Here, we only recall the following recent works: [1, 5, 6, 7, 12, 14].

2. The First Moments of the Operators

Theorem 2.1. *If $j \in \mathbb{N}_0$, $x \geq 0$ and*

$$P_{j+1}(x) = x\Big(x - \frac{1}{\beta(n)}\Big) \cdots \Big(x - \frac{j}{\beta(n)}\Big), \tag{2.7}$$

then

$$L_n(P_{j+1}, x) = x^{j+1} \frac{g_n^{(j+1)}(x)}{\beta^{j+1}(n) g_n(x)}. \tag{2.8}$$

In particular, for each $j \in \mathbb{N}_0$, $\mathbb{P}_j \subset \mathcal{D}(L)$.

Proof. Notice that
$$\beta^{j+1}(n)P_{j+1}\Big(\frac{k}{\beta(n)}\Big) = k(k-1)\cdots(k-j).$$
Therefore, for each fixed $x > 0$,
$$\begin{aligned}\beta^{j+1}(n)g_n(x)L_n(P_{j+1},x) &= \sum_{k=j+1}^{\infty}\frac{a_{n,k}x^k}{(k-j-1)!}\\ &= x^{j+1}\sum_{k=0}^{\infty}\frac{a_{n,k+j+1}}{k!}x^k = x^{j+1}g_n^{(j+1)}(x).\end{aligned}$$
Since L_n is a linear operator in $\mathcal{D}(L)$, for each $j \in \mathbb{N}_0$, $\mathbb{P}_j \subset \mathcal{D}(L)$. □

Proposition 2.1. *If L_n is given by (1.2), for each $n \in \mathbb{N}$ and every $x \in I$ one has*
$$L_n(e_1,x) = J_{n,1}(x) \qquad \textit{and} \qquad L_n(e_2,x) = J_{n,2}(x) + \frac{J_{n,1}(x)}{\beta(n)},$$
where we use the notations (1.3).

Proof. The first assertion follows from Theorem 2.1 with $j = 0$. On the other hand, since
$$P_2(x) = x\Big(x - \frac{1}{\beta(n)}\Big)$$
one has
$$L_n(e_2,x) = L_n(P_2,x) + \frac{1}{\beta(n)}L_n(e_1,x) = x^2\frac{g_n''(x)}{\beta^2(n)g_n(x)} + x\frac{g_n'(x)}{\beta^2(n)g_n(x)}.$$
□

Corollary 2.1. *If $I_{n,i}(x) = 1$ for $i = 1$ and $i = 2$ and every $x \in I$, then*
$$L_n((e_1 - xe_0)^2, x) = \frac{x}{\beta(n)}.$$

Proposition 2.2. *If condition (1.4) holds, there exists a constant K such that, for each $n \in \mathbb{N}$ and $x \in I$, then*
$$\mid L_n(e_1,x) - x \mid \le \frac{K}{\beta(n)} \qquad \textit{and} \qquad L_n((e_1 - xe_0)^2, x) \le K\frac{x}{\beta(n)}.$$

Proof. From (1.4) and Proposition 2.1, we know that
$$|L_n(e_1,x) - x| = x\left|\frac{g_n'(x)}{\beta(n)g_n(x)} - 1\right| \le \frac{K_1 x}{1+\beta(n)x} \le \frac{K_1}{\beta(n)}.$$
Moreover
$$\begin{aligned}L_n((e_1 - xe_0)^2, x) &= L_n(e_2,x) - 2xL_n(e_1,x) + x^2\\ &= x^2\frac{g_n''(x)}{\beta^2(n)g_n(x)} - 2x^2\frac{g_n'(x)}{\beta(n)g_n(x)} + x^2 + x\frac{g_n'(x)}{\beta^2(n)g_n(x)}\\ &= x^2\Big\{\Big(\frac{g_n''(x)}{\beta^2(n)g_n(x)} - 1\Big) + 2\Big(1 - \frac{g_n'(x)}{\beta(n)g_n(x)}\Big)\Big\} + x\frac{g_n'(x)}{\beta^2(n)g_n(x)}\\ &\le C_1\Big(\frac{1}{\beta(n)}\frac{\beta(n)x^2}{(1+\beta(n)x)} + \frac{x}{\beta(n)}\Big) \le C_2\frac{x}{\beta(n)}.\end{aligned}$$
□

Proposition 2.3. *Suppose there exists a constant C_1 such that, for each $x \in I$ and every $n \in \mathbb{N}$,*

$$I_{n,1}(x) \le C_1. \tag{2.9}$$

If condition (1.5) holds, there exists a constant C_2 such that, for each $x \in I$ and every $n \in \mathbb{N}$, one has

$$L_n((e_1 - xe_0)^2, x) \le C_2 \left(\frac{x^2}{\beta^2(n)} + \frac{x}{\beta(n)} \right).$$

3. The Operator L_n As an Endomorphism

It is easy to see that $\{L_n\}$ is uniformly bounded sequence of linear operators from the space $C_{b,\infty}(I)$ to $C_b(I)$, but we need to verify that

$$L_n : C_{b,\infty}(I) \to C_{b,\infty}(I).$$

Theorem 3.2. *If $n \in \mathbb{N}$ and $f \in C_{b,\infty}(I)$, then $L_n(f) \in C_{b,\infty}(I)$. In particular*

$$\lim_{x\to\infty} L_n(f,x) = \lim_{x\to\infty} f(x).$$

Proof. Set $y_{n,k} = k/\beta(n)$. If $f \in C_{b,\infty}(I)$, there exists a real A such that $f(x) \to A$, as $x \to \infty$. We set $B = |A| + \|f\|$. Fix $\varepsilon > 0$. There exists $N_1 > 0$ such that, for $x > N_1$,

$$|f(x) - A| < \frac{\varepsilon}{2}.$$

Since $y_{n,k} \to \infty$ as $k \to \infty$, there exists $m \in \mathbb{N}$, $m > N_1$, such that $y_{n,k} > N_1$, for all $k > m$. Taking into account L'Hôpital's rule

$$\lim_{x\to\infty} \frac{1}{g_n(x)} \sum_{k=0}^{m} \frac{a_{n,k}}{k!} x^k = 0.$$

Hence, there exists $N_2 > N_1$ such that, for $x > N_2$,

$$\frac{1}{g_n(x)} \sum_{k=0}^{m} \frac{a_{n,k}}{k!} x^k \le \frac{\varepsilon}{2B}.$$

Therefore, if $x > N_2$, then

$$\begin{aligned} |L_n(f,x) - A| &= \left| \frac{1}{g_n(x)} \sum_{k=0}^{\infty} \frac{a_{n,k}}{k!} (f(y_k) - A) x^k \right| \\ &\le \sum_{k=0}^{m} \left| \frac{a_{n,k}}{k!} (f(y_k) - A) \frac{x^k}{g_n(x)} \right| + \left| \sum_{k=m+1}^{\infty} \frac{a_{n,k}}{k!} (f(y_k) - A) \frac{x^k}{g_n(x)} \right| \\ &\le B \frac{1}{g_n(x)} \sum_{k=0}^{m} \frac{a_{n,k}}{k!} x^k + \frac{\varepsilon}{2} \frac{1}{g_n(x)} \sum_{k=m+1}^{\infty} \frac{a_{n,k}}{k!} x^k \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This is sufficient to prove that

$$\lim_{x\to\infty} L_n(f,x) = A.$$

In particular, $L_n(f) \in C_{b,\infty}(I)$. □

4. A Korovkin Type Theorem

Let us denote

$$\psi(y) = \frac{y}{1-y}, \qquad y \in [0,1). \tag{4.10}$$

It is clear that $\psi : [0,1) \to [0,\infty)$ is a homeomorphism with inverse function

$$\psi^{-1}(x) = \frac{x}{1+x}, \qquad x \in [0,\infty).$$

For $g \in C[0,1]$, we consider the uniform norm $\|g\|_\infty = \sup_{y\in[0,1]} \mid g(y) \mid$.

Theorem 4.3 ([3]). *If the operator $\Phi : C_{b,\infty}(I) \to C[0,1]$ is defined by*

$$\Phi(f,y) = \begin{cases} f(\psi(y)) & \text{if} \quad y \in [0,1) \\ \lim\limits_{x\to\infty} f(x) & \text{if} \quad y = 1 \end{cases}, \tag{4.11}$$

$f \in C_{b,\infty}(I)$, then Φ is a positive linear isomorphism, with positive linear inverse $\Phi^{-1} : C[0,1] \to C_{b,\infty}(I)$ given by

$$\Phi^{-1}(g,x) = g\Big(\frac{x}{1+x}\Big), \qquad g \in C[0,1], \qquad x \in [0,\infty).$$

Moreover, for each $f \in C_{b,\infty}(I)$, $\|f\|_\infty = \|\Phi(f)\|_\infty$.

Now, we will study convergence in the spaces $C_{b,\infty}[0,\infty)$. The following result is known.

Theorem 4.4 ([3] and [4]). *A sequence $\{M_n\}$ of positive linear operators, $M_n : C_{b,\infty}[0,\infty) \to C_{b,\infty}[0,\infty)$, is an approximation process if and only if $\|f_i - M_n(f_i)\| \to 0$, for $i = 0,1,2$, where*

$$f_0(x) = 1, \qquad f_1(x) = \frac{x}{(1+x)} \quad \textit{and} \quad f_2(x) = \frac{x^2}{(1+x)^2}. \tag{4.12}$$

We will follows the ideas given in [3] and [4], but we need other text functions. Let us remember known facts.

Recall that three functions $h_0, h_1, h_2 \in C[0,1]$ are a Chebyshev system of order three in $[0,1]$, if any linear combination $\lambda_0 h_0 + \lambda_1 h_1 + \lambda_2 h_2$, with $|\lambda_0| + |\lambda_1| + |\lambda_2| > 0$, has at most two different zeros (see [2, p. 100]).

Lemma 4.1. *The functions $f_0(x) = 1$, $f_1(x) = \sqrt{x}$ and $f_2(x) = x$ are a Chebyshev system of order three in $[0,1]$.*

Proof. Assume that the function $\theta(x) = a + b\sqrt{x} + cx$ (where at least one coefficient is different from zero) has at least three different zeros in $[0,1]$, say x_0, x_1 and x_2. Then, the polynomial $P(x) = a + bx + cx^2$ satisfies $P(\sqrt{x_0}) = P(\sqrt{x_1}) = P(\sqrt{x_2}) = 0$, but this is not possible. □

Theorem 4.5 ([9], p. 49). *Let $h_0, h_1, h_2 \in C[0,1]$ be a Chebyshev system of order three in $[0,1]$. If $\{M_n\}$ is a sequence of linear positive operators, $M_n : C[0,1] \to C[0,1]$ and*

$$\lim_{n\to\infty} \|h_i - M_n(h_i)\|_\infty = 0, \qquad i \in \{0,1,2\},$$

then

$$\lim_{n\to\infty} \|g - M_n(g)\|_\infty = 0$$

for every $g \in C[0,1]$.

Theorem 4.6. *If the sequence $\{L_n\}$, $L_n : C_{b,\infty}(I) \to C_{b,\infty}(I)$, is given by (1.2), then the following assertions are equivalent:*

(i) $\{L_n\}$ *is an approximation process.*
(ii) *For* $i = 0, 1, 2$, $\|f_i - L_n(f_i)\| \to 0$, *where*

$$f_0(x) = 1, \qquad f_1(x) = \frac{\sqrt{x}}{\sqrt{1+x}} \quad \textit{and} \quad f_2(x) = \frac{x}{1+x}. \tag{4.13}$$

(iii) *For* $i = 0, 1, 2$, $\|h_i - L_n(h_i)\| \to 0$, *where*

$$h_0(x) = 1, \qquad h_1(x) = \frac{\sqrt{x}}{1+\sqrt{x}} \quad \textit{and} \quad h_2(x) = h_1^2(x). \tag{4.14}$$

Proof. The assertions (i) ⇒ (ii) and (i) ⇒ (iii) are simple because each the functions f_i and h_i are continuous bounded functions with finite limits as $x \to \infty$.

(ii) ⇒ (i). For each $g \in C[0,1]$, we define a function $G(g) \in C_{b,\infty}(I)$, by setting

$$G(g, x) = g(x/(1+x)).$$

Notice $\lim_{x\to\infty} G(g, \infty) = g(1)$. If L_n is given by (1.2) and $g \in C[0,1]$, define $M_n(g, 1) = g(1)$ and, for $y \in [0,1)$,

$$M_n(g, y) = L_n\Big(G(g), \frac{y}{1-y}\Big).$$

From Theorem 3.2, we know that

$$\lim_{y\to 1} M_n(g, y) = \lim_{x\to\infty} L_n\Big(G(g), x\Big) = \lim_{x\to\infty} G(g)(x) = g(1) = M_n(g, 1).$$

Therefore $M_n : C[0,1] \to C[0,1]$ and it is a positive linear operator. For $y \in [0,1]$, set $g_0(y) = 1$, $g_1(y) = \sqrt{y}$ and $g_2(y) = y$. Since $\{g_0, g_1, g_2\}$ is a Chebyshev system in $[0,1]$ (see Lemma 4.1), in order to use Theorem 4.5, we will verify that

$$\lim_{n\to\infty} \|M_n(g_i) - g_i\|_\infty = 0, \qquad i \in \{0, 1, 2\}.$$

To prove this, we consider (ii). If $g_0 = e_0$, then $G(g_0) = f_0$, but $\|L_n(e_0) - e_0\| = 0$. If $y \in [0,1)$ and $x = y/(1-y)$, then

$$G(g_1, y_{n,k}) = \frac{\sqrt{y_{n,k}}}{\sqrt{1+y_{n,k}}} = f_1(y_{n,k})$$

and

$$f_1(x) = \frac{\sqrt{x}}{\sqrt{1+x}} = \frac{\sqrt{y/(1-y)}}{\sqrt{1+y/(1-y)}} = \sqrt{y} = g_1(y).$$

With analogous arguments, we verify that

$$G(g_2, y_{n,k}) = f_2(y_{n,k}) \qquad \text{and} \qquad f_2(x) = g_2(y).$$

Therefore, for $i = 1$ and $i = 2$, if $y \in [0,1)$ and $x = y/(1-y)$, then

$$M_n(g_i, y) - g_i(y) = L_n\Big(G(g_i), \frac{y}{1-y}\Big) - g_i(y) = L_n(f_i, x) - f_i(x).$$

Moreover

$$M_n(g_i, 1) - g_i(1) = 0.$$

If (ii) holds, we have proved that $\|M_n(g_i) - g_i\|_\infty = \|L_n(f_i) - f_i\| \to 0$ as $n \to \infty$.

From Theorem 4.5, we know that $\{M_n\}$ is a approximation process in $C[0,1]$ and it is sufficient to verify that $\{L_n\}$ is a approximation process in $C_{b,\infty}(I)$. If fact, if $f \in C_{b,\infty}(I)$ we set $F(f, 1) = \lim_{y\to 1} f(y/(1-y))$ and, for $y \in [0,1)$,

$$F(f, y) = f(y/1-y),$$

then $F(f) \in C[0,1]$ and $\|M_n(F(f)) - F(f)\|_\infty \to 0$, as $n \to \infty$. But, for $y \in [0,1)$,

$$M_n(F(f), y) = L_n\Big(G(F(f)), \frac{y}{1-y}\Big),$$

and

$$G(F(f))(y_{n,k}) = F(f)\Big(\frac{y_{n,k}}{1+y_{n,k}}\Big) = f\Big(\frac{\frac{y_{n,k}}{1-y_{n,k}}}{1+\frac{y_{n,k}}{1-y_{n,k}}}\Big) = f(y_{n,k}),$$

and, if $x = y/(1-y)$, $F(f,y) = f(x)$. Hence

$$\|M_n(F(f)) - F(f)\|_\infty = \|L_n(f) - f\|.$$

This proves the result.

(iii) $\Rightarrow$ (i). The proof is similar to the case (ii) $\Rightarrow$ (i), but we use another change of variables. For each $g \in C[0,1]$, we define a function $H(g) \in C_{b,\infty}(I)$, by setting

$$H(g,x) = g(\sqrt{x}/(1+\sqrt{x})).$$

Notice $\lim_{x\to\infty} G(g,\infty) = g(1)$. If L_n is given by (1.2) and $g \in C[0,1]$, define $M_n^*(g,1) = g(1)$ and, for $y \in [0,1)$,

$$M_n^*(g,y) = L_n\Big(H(g), \frac{y}{1-y}\Big).$$

From Theorem 3.2, we know that

$$\lim_{y\to 1} M_n^*(g,y) = \lim_{x\to\infty} L_n\Big(H(g), x\Big) = \lim_{x\to\infty} H(g)(x) = g(1) = M_n^*(g,1).$$

Therefore $M_n^* : C[0,1] \to C[0,1]$ and it is a positive linear operator. For $y \in [0,1]$, set $g_0(y) = 1$, $g_1(y) = y$ and $g_2(y) = y^2$. Since $\{g_0, g_1, g_2\}$ is a Chebyshev system in $[0,1]$, in order to use Theorem 4.5, we will verify that

$$\lim_{n\to\infty} \|M_n(g_i) - g_i\|_\infty = 0, \qquad i \in \{0,1,2\}.$$

To prove this, we consider (iii). If $g_0 = e_0$, then $G(g_0) = f_0$, but $\|L_n(e_0) - e_0\| = 0$. If $y \in [0,1)$ and $x = (y/(1-y))^2$, then

$$H(g_1, y_{n,k}) = \frac{\sqrt{y_{n,k}}}{1+\sqrt{y_{n,k}}} = f_1(y_{n,k})$$

and

$$f_1(x) = \frac{\sqrt{x}}{1+\sqrt{x}} = \frac{y/(1-y)}{1+y/(1-y)} = y = g_1(y).$$

With analogous arguments, we verify that

$$H(g_2, y_{n,k}) = f_2(y_{n,k}) \qquad \text{and} \qquad f_2(x) = g_2(y).$$

Therefore, for $i = 1$ and $i = 2$, if $y \in [0,1)$ and $x = (y/(1-y))^2$, then

$$M_n^*(g_i, y) - g_i(y) = L_n\Big(H(g_i), \frac{y}{1-y}\Big) - g_i(y) = L_n(f_i, x) - f_i(x).$$

Moreover

$$M_n^*(g_i, 1) - g_i(1) = 0.$$

If (iii) holds, we have proved that $\|M_n^*(g_i) - g_i\|_\infty = \|L_n(f_i) - f_i\| \to 0$ as $n \to \infty$.

From Theorem 4.5, we know that $\{M_n^*\}$ is a approximation process in $C[0,1]$ and it is sufficient to verify that $\{L_n\}$ is a approximation process in $C_{b,\infty}(I)$. In fact, if $f \in C_{b,\infty}(I)$, we set $F(f,1) = \lim_{y\to 1} f(y^2/(1-y)^2)$ and for $y \in [0,1)$,

$$F(f,y) = f\Big(\frac{y^2}{(1-y)^2}\Big),$$

then $F(f) \in C[0,1]$ and $\|M_n^*(F(f)) - F(f)\|_\infty \to 0$, as $n \to \infty$. But, for $y \in [0,1)$,

$$M_n^*(F(f),y) = L_n\Big(H(F(f)), \frac{y}{1-y}\Big),$$

and

$$H(F(f))(y_{n,k}) = F(f)\Big(\frac{\sqrt{y_{n,k}}}{1+\sqrt{y_{n,k}}}\Big) = f\Bigg(\frac{\Big(\frac{\sqrt{y_{n,k}}}{1+\sqrt{y_{n,k}}}\Big)^2}{(1-\frac{\sqrt{y_{n,k}}}{1+\sqrt{y_{n,k}}})^2}\Bigg) = f(y_{n,k}),$$

and if $x = y^2/(1-y)^2$, $F(f,y) = f(x)$. Hence

$$\|M_n(F(f)) - F(f)\|_\infty = \|L_n(f) - f\|.$$

This proves the result. □

5. Approximation Process

In this section, we present sufficient conditions in order that $\{L_n\}$ be an approximation process in $C_{b,\infty}(I)$. It is sufficient to verify (ii) or (iii) in Theorem 4.6.

Proposition 5.4. *Assume that condition (1.4) holds. If $f_1(x)$ and $f_2(x)$ are given as in (4.13), then*

$$L_n(|\, f_1(e_1) - f_1(x)\,|, x) \le \frac{K}{\sqrt{\beta(n)}} \quad \textit{and} \quad L_n(|\, f_2(e_1) - f_2(x)\,|, x) \le \frac{K}{\sqrt{\beta(n)}},$$

where K is the constant in Proposition 2.2.

Proof. From Proposition 2.2, we obtain

$$\begin{aligned} L_n(|f_1(e_1) - f_1(x)|, x) &= L_n\left(\frac{|\sqrt{e_1(1+x)} - \sqrt{x(1+e_1)}|}{\sqrt{1+x}\sqrt{1+e_1}}, x\right) \\ &\le \frac{1}{\sqrt{1+x}} L_n\left(\frac{|x-e_1|}{(\sqrt{e_1(1+x)} + \sqrt{x(1+e_1)})}, x\right) \\ &\le \frac{1}{\sqrt{x}\sqrt{1+x}}\sqrt{L_n((e_1-x)^2, x)} \\ &\le \frac{1}{\sqrt{x}\sqrt{1+x}}\frac{K\,x}{\beta(n)} \le \frac{K}{\sqrt{\beta(n)}}. \end{aligned}$$

On the other hand

$$\begin{aligned} L_n(|f_2(e_1) - f_2(x)|, x) &= L_n\Big(\frac{|e_1 - x|}{(1+x)(1+e_1)}, x\Big) \\ &\le \frac{1}{(1+x)}\sqrt{L_n((e_1-x)^2, x)} \le \frac{K}{\sqrt{\beta(n)}}. \end{aligned}$$

□

Proposition 5.5. *Assume that conditions (1.5) and (2.9) hold. If $h_0(x)$, $h_1(x)$ and $h_2(x)$ are given as in (4.14), there exists a constant C such that, for each $n \in \mathbb{N}$ and $x \in I$ one has*

$$|L_n(h_i(e_1), x) - h_i(x)| \leq \frac{C}{\sqrt{\beta(n)}}.$$

Proof. It is clear that $L_n(f_0(e_1) - f_0(x), x) = 0$. From Proposition 2.3, we obtain

$$\begin{aligned} L_n(|h_1(e_1) - h_1(x)|, x) &= L_n\Big(\frac{|\sqrt{e_1(1+x)} - \sqrt{x(1+e_1)}|}{(1+\sqrt{x})(1+\sqrt{e_1})}, x\Big) \\ &\leq \frac{1}{1+\sqrt{x}} L_n\Big(\frac{|x-e_1|}{\sqrt{e_1(1+x)} + \sqrt{x(1+e_1)}}, x\Big) \\ &\leq \frac{\sqrt{L_n((e_1-x)^2, x)}}{\sqrt{x}(1+\sqrt{x})} \\ &\leq \frac{2C}{\sqrt{x}(1+\sqrt{x})}\Big(\frac{x}{\beta(n)} + \frac{\sqrt{x}}{\sqrt{\beta(n)}}\Big) \leq C_1 \frac{1}{\sqrt{\beta(n)}}. \end{aligned}$$

On the other hand, taking into account that, for $x, y \in I$, one has

$$\sqrt{xy} \leq (1+\sqrt{x})(1+\sqrt{y}),$$

for $x > 0$, we obtain

$$\begin{aligned} L_n(|f_2(e_1) - f_2(x)|, x) &= L_n\Big(\Big|\Big(\frac{\sqrt{x}}{1+\sqrt{x}}\Big)^2 - \Big(\frac{\sqrt{e_1}}{1+\sqrt{e_1}}\Big)^2\Big|, x\Big) \\ &= L_n\Big(\frac{|x(1+2\sqrt{e_1}+e_1) - e_1(1+2\sqrt{x}+x)|}{(1+\sqrt{x})^2(1+\sqrt{e_1})^2}, x\Big) \\ &= L_n\Big(\frac{|x - e_1 + 2\sqrt{xe_1}(\sqrt{x}-\sqrt{e_1}))|}{(1+\sqrt{x})^2(1+\sqrt{e_1})^2}, x\Big) \\ &\leq \frac{1}{(1+\sqrt{x})^2} L_n(|x-e_1|, x) + 2L_n\Big(\frac{|\sqrt{x}-\sqrt{e_1})|}{(1+\sqrt{x})(1+\sqrt{e_1})}, x\Big) \\ &\leq \frac{1}{\sqrt{x}(1+\sqrt{x})} L_n(|x-e_1|, x) + \frac{2}{(1+\sqrt{x})} L_n\Big(\frac{|x-e_1|}{\sqrt{x}+\sqrt{e_1}}, x\Big) \\ &\leq \frac{3}{\sqrt{x}(1+\sqrt{x})}\sqrt{L_n((x-e_1)^2, x)} \\ &\leq \frac{C_1}{\sqrt{x}(1+\sqrt{x})}\Big(\frac{x}{\beta(n)} + \frac{\sqrt{x}}{\sqrt{\beta(n)}}\Big) \leq \frac{C_2}{\sqrt{\beta(n)}}. \end{aligned}$$

□

Theorem 5.7. *(i) If condition (1.4) holds, then the sequence of operators $\{L_n\}$ is an approximation process in $C_{b,\infty}(I)$.*

(ii) If conditions (1.5) and (2.9) hold, then the sequence of operators $\{L_n\}$ is an approximation process in $C_{b,\infty}(I)$.

Proof. (i) From Theorem 3.2, we know that $L_n : C_{b,\infty}(I) \to C_{b,\infty}(I)$. Taking into account Theorem 4.6, we will verify that conditions (4.12) hold.

Since $L_n(e_0) = e_0$, it is sufficient to prove the assertion for each e_i, $i \in \{1, 2\}$. But it was done in Proposition 5.4.

(ii) The proof follows analogously, but we use Proposition 5.5. □

6. Main Results

We need some properties of functions in $C_{b,\infty}(I)$.

Proposition 6.6. *(i) If $f \in C_{b,\infty}(I)$, $\phi_1 : [0,\infty) \to [0,1)$ is given by $\phi(x) = x/(1+x)$ and ϕ^{-1} is the inverse function, then $f \circ \phi^{-1}$ is uniformly continuous in $[0,1)$.*

(ii) If $f \in C_{b,\infty}(I)$, $\phi_1 : [0,\infty) \to [0,1)$ is given by $\phi(x) = \sqrt{x}/(1+\sqrt{x})$ and ϕ^{-1} is the inverse function, then $f \circ \phi^{-1}$ is uniformly continuous in $[0,1)$.

Proof. Let $A = \lim_{x\to\infty} f(x)$. Notice that $\phi^{-1}(y) = y/(1-y)$, $y \in [0,1)$. If we define $g(1) = A$ and $g(y) = (f \circ \phi^{-1})(y)$ for $y \in [0,1)$, the $g \in C[0,1]$ and it is a uniformly continuous function.

The other assertion can be proved analogously. □

If $\phi(x) = x/(1+x)$ or $\phi(x) = \sqrt{x}/(1+\sqrt{x})$, for $f \in C_{b,\infty}(I)$, following Holhoş [8], define

$$\omega^{\phi}(f,t) = \sup_{x,y\in[0,\infty),|\phi(x)-\phi(t)|\le t} |f(x) - f(t)|. \tag{6.15}$$

Proposition 6.7. *If $f \in C_{b,\infty}(I)$ and $\phi(x) = x/(1+x)$ or $\phi(x) = \sqrt{x}/(1+\sqrt{x})$, then*

$$\lim_{n\to\infty} \omega^{\phi}(f,\delta_n) = 0$$

for any sequence $\{\delta_n\}$ of positive numbers satisfying $\lim_{n\to\infty} a_n = 0$.

Proof. If was proved in [8] that the assertion is true if $f \circ \phi^{-1}$ is uniformly continuous, but this property was verified in Proposition 6.6. □

The next result is due to Holhoş, but we present in a convenient form for our purpose.

Proposition 6.8 ([8])**.** *Assume that $\phi(x) = x/(1+x)$ or $\phi(x) = \sqrt{x}/(1+\sqrt{x})$. Let $A_n : C_{b,\infty}(I) \to C_{b,\infty}(I)$ be a sequence of positive linear operators preserving constant functions. If the sequence $\{a_n\}$,*

$$a_n = \sup_{x\ge 0} A_n(|\phi(e_1) - \phi(x)|, x) \tag{6.16}$$

is bounded, $\lim_{n\to\infty} a_n = 0$, and $f \circ \phi^{-1}$ is uniformly continuous, then

$$\lim_{n\to\infty} \|A_n(f) - f\| = 0 \qquad \text{and} \qquad \|A_n(f) - f\| \le 2\omega^{\phi}(f,a_n).$$

Theorem 6.8. *Assume that condition (1.4) holds. If $\phi(x) = x/(1+x)$, there exists a constant C such that, for each $n \in \mathbb{N}$ and every $f \in C_{b,\infty}(I)$, one has*

$$\|L_n(f) - f\| \le C\omega^{\phi}\Big(f, \frac{1}{\sqrt{\beta(n)}}\Big),$$

where $\omega^{\phi}(f,t)$ is given by (6.15). In particular, $\|L_n(f) - f\| \to 0$, as $n \to \infty$.

Proof. If $\{a_n\}$ is defined as in (6.16) and we prove that $a_n \to 0$, as $n \to \infty$, we can derive the result from Proposition 6.8, because we verified in Proposition 6.6 that $f \circ \phi^{-1}$ is uniformly continuous. Since $\phi(x) = f_2(x)$, where f_2 is the function in Theorem 4.6, it follows from Proposition 5.4 that, if condition (1.4) holds, then

$$a_n \le \frac{K}{\sqrt{\beta(n)}}.$$

Therefore

$$\|L_n(f) - f\| \le 2\omega^{\phi}(f,a_n).$$

It was proved in [8] that, if $\delta, \lambda > 0$, then

$$\omega^{\phi}(f,\lambda\delta) \le (1+\lambda)\omega^{\phi}(f,\delta).$$

Hence, we can replace a_n by its estimate and extract the constant K as in the statement of the Theorem. □

Theorem 6.9 can be proved as Theorem 6.8. In fact, the function $\phi(x) = \sqrt{x}/(1+\sqrt{x})$ agree with h_1 in equation (4.14) and instead of Proposition 5.4, we can use Proposition 5.5, if conditions (1.5) and (2.9) hold.

Theorem 6.9. *Assume that conditions* (1.5) *and* (2.9) *hold. If* $\phi(x) = \sqrt{x}/(1+\sqrt{x})$, *there exists a constant* C *such that, for each* $n \in \mathbb{N}$ *and every* $f \in C_{b,\infty}(I)$, *one has*

$$\|L_n(f) - f\| \leq C\omega^{\phi}\Big(f, \frac{1}{\sqrt{\beta(n)}}\Big),$$

where $\omega^{\phi}(f,t)$ *is given by* (6.15).

The next result shows how to construct some families of operators for which our approach can be applied.

Theorem 6.10. *Let* $\{b_k\}$ *be a decreasing sequence of positive real numbers, and assume there exists a constant* Λ *such that, for* $i \in \{1,2\}$ *and every* $k \in \mathbb{N}$, *one has*

$$b_{k-1} - b_{k-1+i} \leq \frac{\Lambda}{k} b_k. \tag{6.17}$$

Define

$$C_n(f,x) = \frac{1}{g_n(x)} \sum_{k=0}^{\infty} \frac{b_k n^k}{k!} f\Big(\frac{k}{n}\Big) x^k, \qquad \text{with} \quad g_n(x) = \sum_{k=0}^{\infty} \frac{b_k n^k}{k!} x^k.$$

If $\phi(x) = x/(1+x)$, *then there exists a constant* C *such that, for each* $f \in C_{b,\infty}(I)$ *and every* $n \in \mathbb{N}$, *one has*

$$\|C_n(f) - f\| \leq C\omega^{\phi}\Big(f, \frac{1}{\sqrt{n}}\Big),$$

where $\omega^{\phi}(f,t)$ *is given by* (6.15).

Proof. Notice

$$g_n'(x) = \sum_{k=1}^{\infty} \frac{n^k b_k}{(k-1)!} x^{k-1} = n \sum_{k=0}^{\infty} \frac{n^k b_{k+1}}{k!} x^k = n \sum_{k=0}^{\infty} \frac{n^k b_{k+1} - b_k}{k!} x^k + n g_n(x),$$

and

$$g_n^{(2)}(x) = n^2 \sum_{k=0}^{\infty} \frac{b_{k+2}}{k!} (nx)^k = n^i \sum_{k=0}^{\infty} \frac{b_{k+2} - b_k}{k!} (nx)^k + n^2 g_n(x).$$

Hence, for $i \in \{1,2\}$,

$$\left| \frac{g_n^{(i)}(x)}{n^i g_n(x)} - 1 \right| = \frac{1}{g_n(x)} \sum_{k=0}^{\infty} \frac{b_k - b_{k+i}}{k!} (nx)^k,$$

because $\{b_k\}$ decreases. Taking into account (6.17), we obtain

$$\begin{aligned}(1+nx)\sum_{k=0}^{\infty}\frac{b_k-b_{k+i}}{k!}(nx)^k &= \sum_{k=0}^{\infty}\frac{b_k-b_{k+i}}{k!}(nx)^k+\sum_{k=0}^{\infty}\frac{b_k-b_{k+i}}{k!}(nx)^{k+1}\\ &= \sum_{k=0}^{\infty}\frac{b_k-b_{k+i}}{k!}(nx)^k+\sum_{k=1}^{\infty}\frac{b_{k-1}-b_{k+i-1}}{(k-1)!}(nx)^k\\ &\leq \sum_{k=0}^{\infty}\frac{b_k}{k!}(nx)^k+\Lambda\sum_{k=1}^{\infty}\frac{b_k}{k!}(nx)^k\leq(1+\Lambda)g_n(x).\end{aligned}$$

Therefore

$$\left|\frac{g_n^{(i)}(x)}{n^i g_n(x)}-1\right|\leq\frac{1+\Lambda}{1+nx}.$$

Thus, conditions (1.4) holds for $i=1$ and $i=2$ and the announced result follows from Theorem 6.8. □

7. Examples

First Example. In this example, we apply the results of the previous section to Szász-Schurer operators.

For a fixed $p\geq 0$, Schurer introduced the operators

$$L^*_{n,p}(f,x)=e^{-(n+p)x}\sum_{k=0}^{\infty}\frac{(n+p)^k}{k!}f\Big(\frac{k}{n}\Big)x^k. \tag{7.18}$$

This operator have been studied by several authors (see [10, 11, 15, 16]). The case $p=0$ gives place to Szász-Mirakyan operators.

In this work, we study the more general version

$$L_{n,p}(f,x)=e^{-(\beta(n)+p)x}\sum_{k=0}^{\infty}\frac{(\beta(n)+p)^k}{k!}f\Big(\frac{k}{\beta(n)}\Big)x^k, \tag{7.19}$$

where $\beta(n)\geq 1$ and $\beta(n)\to\infty$, as $n\to\infty$. The operator $L_{n,p}$ has the form (1.2) with

$$a_{n,k}=(\beta(n)+p)^k \qquad \text{and} \qquad g_n(x)=e^{(\beta(n)+p)x}.$$

Notice that

$$a_{n,k+1}=a_{n,k}(\beta(n)+p) \quad \text{and} \quad I_{n,i}(x)=\frac{g_n^{(i)}(x)}{\beta^i(n)g_n(x)}=\frac{(\beta(n)+p)^i}{\beta^i(n)}. \tag{7.20}$$

In particular, the functions $I_{n,1}(x)$ is uniformly bounded (condition (2.9)).

If $p=0$ (Szász-Mirakyan operators) then $I_{n,i}(x)=1$, and we can apply Proposition 2.1. Taking into Proposition 2.1, we know that

$$L_{n,p}(e_1,x)-x=x\Big(\frac{g_n'(x)}{\beta(n)g_n(x)}-1\Big)=x\Big(\frac{\beta(n)+p}{\beta(n)}-1\Big).$$

If $p>0$, since the expression in brackets depends not on x and it is different from zero, the behaviour of the operators $L_{n,p}$ is different from the Szász-Mirakyan operators. On the other hand

$$L_{n,p}((e_1-xe_0)^2,x)=J_{n,2}(x)+\frac{J_{n,1}(x)}{\beta(n)}-2xJ_{n,1}(x)+x^2$$

$$= x^2\big(I_{n,2}(x) - 2I_{n,1} + 1\big) + \frac{J_{n,1}(x)}{\beta(n)}$$
$$= x^2\Big(\frac{p}{\beta(n)}\Big)^2 + x\frac{\beta(n)+p}{\beta^2(n)}.$$

Hence condition (1.5) is satisfied and Theorem 6.9 can be applied. We think that Theorem 7.11 is the first result where uniform estimates for the Schurer operators in the space $C_{b,\infty}(I)$ are given.

Theorem 7.11. *Assume $p > 0$ and $\phi(x) = \sqrt{x}/(1+\sqrt{x})$. If $L_{n,p}$ is given by (7.19), there exists a constant C such that for each $n \in \mathbb{N}$ and every $f \in C_{b,\infty}(I)$, one has*

$$\|L_{n,p}(f) - f\| \le C\omega^{\phi}\Big(f, \frac{1}{\sqrt{\beta(n)}}\Big),$$

where $\omega^{\phi}(f,t)$ is given by (6.15).

Second Example. For $0 < \gamma < 1$ and $p > 0$, set

$$g_n(x) = g_{n,\gamma,p}(x) = \sum_{k=0}^{\infty} \frac{(n+p)^{\gamma k}}{k!} x^k = e^{(n+p)^{\gamma} x},$$

and define

$$B_{n,\gamma}(f,x) = \frac{1}{g_{n,\gamma}(x)} \sum_{k=0}^{\infty} \frac{(n+p)^{\gamma k}}{k!} f\Big(\frac{k}{n^{\gamma}}\Big) x^k. \tag{7.21}$$

In this case $a_{n,k} = (n+p)^{\gamma k}$ and $\beta(n) = n^{\gamma}$. Moreover

$$g_n'(x) = (n+p)^{\gamma} g_n(x) \qquad \text{and} \qquad g_n''(x) = (n+p)^{2\gamma} g_n(x).$$

Hence

$$I_{n,1}(x) = \frac{g_n'(x)}{\beta(n) g_n(x)} = \frac{(n+p)^{\gamma}}{n^{\gamma}} \qquad \text{and} \qquad I_{n,2}(x) = \frac{(n+p)^{2\gamma}}{n^{2\gamma}} \tag{7.22}$$

Let us verify that condition (1.5) holds.

Lemma 7.2. *If $0 < \gamma < 1$ and $p > 0$, for each $n \in \mathbb{N}$, one has*

$$|I_{n,2}(x) - 2I_{n,1}(x) + 1| \le \gamma^2 \frac{p^2}{n^{2\gamma}},$$

where $I_{n,1}(x)$ and $I_{n,2}(x)$ are given as in (7.22).

Proof. By the mean value theorem, if $0 < \gamma < 1$ and $y > 1$, there exists $\theta \in (1, y)$ such that

$$0 < y^{\gamma} - 1 = \gamma \frac{(y-1)}{\theta^{1-\gamma}} < \gamma(y-1),$$

Taking into the previous inequality, we obtain

$$|I_{n,2}(x) - 2I_{n,1}(x) + 1| = \left| \frac{(n+p)^{2\gamma}}{n^{2\gamma}} - 2\frac{(n+p)^{\gamma}}{n^{\gamma}} + 1 \right|$$
$$= \Big(\frac{(n+p)^{\gamma}}{n^{\gamma}} - 1\Big)^2 \le \gamma^2\Big(\frac{(n+p)}{n} - 1\Big)^2$$
$$= \gamma^2 \frac{p^2}{n^2} \le \gamma^2 \frac{p^2}{n^{2\gamma}}.$$

□

Since $I_{n,1}(x) \leq (1+p)^{\gamma}$, we can apply Theorem 6.9 (with $\beta(n) = n^{\gamma}$).

Theorem 7.12. *Assume $0 < \gamma < 1$, $p > 0$, and $\{B_{n,\gamma}\}$ is given by (7.21). If $\phi(x) = \sqrt{x}/(1+\sqrt{x})$, there exists a constant C such that, for each $n \in \mathbb{N}$ and every $f \in C_{b,\infty}(I)$, one has*

$$\|B_{n,\gamma}(f) - f\| \leq 2\omega^{\phi}\Big(f, \frac{C}{n^{\gamma/2}}\Big),$$

where $\omega^{\phi}(f,t)$ is given by (6.15).

Third Example. For a fixed $j \in \mathbb{N}$, each $n \in \mathbb{N}$, and every $x \geq 0$ set

$$c_{n,j}(x) = \sum_{k=0}^{\infty} \frac{n^k}{(k+j)!} x^k.$$

For $f \in C_{b,\infty}(I)$, define

$$C_{n,j}(f,x) = \frac{1}{c_{n,j}(x)} \sum_{k=0}^{\infty} \frac{n^k}{(k+j)!} f\Big(\frac{k}{n}\Big) x^k. \tag{7.23}$$

We will apply Theorem 6.10 by considering the decreasing sequence

$$\nu_{k,j} = \frac{1}{(k+1)\cdots(k+j)}, \qquad k \in \mathbb{N}_0. \tag{7.24}$$

In Lemma 7.3, we verify that condition (6.17) holds.

Lemma 7.3. *For each fixed $i \in \{1,2\}$ and every $k \in \mathbb{N}_0$, one has*

$$\nu_{k,j} - \nu_{k+i,j} \leq \frac{ij}{(k+1)} \nu_{k+1,j},$$

where $\nu_{k,j}$ is given by (7.24).

Proof. If $i = 1$ and $k \in \mathbb{N}_0$,

$$\begin{aligned} \nu_{k,j} - \nu_{k+1,j} &= \frac{1}{(k+1)\cdots(k+j)} - \frac{1}{(k+2)\cdots(k+j+1)} \\ &= \frac{j}{(k+1)\cdots(k+j+1)} = \frac{j}{(k+1)} \nu_{k+1,j}. \end{aligned} \tag{7.25}$$

If $i = 2$,

$$\begin{aligned} \nu_{k,j} - \nu_{k+2,j} &= \nu_{k,j} - \nu_{k+1,j} + \nu_{k+1,j} - \nu_{k+2,j} \\ &\leq \frac{j}{(k+1)} \nu_{k+1,j} + \frac{j}{(k+2)} \nu_{k+2,j} \leq \frac{2j}{(k+1)} \nu_{k+1,j}, \end{aligned}$$

because the sequence decreases. □

Theorem 7.13. *Fix $j \in \mathbb{N}$ and let $C_{n,j}$ be defined by (7.23). If $\phi(x) = x/(1+x)$, then there exists a constant C such that, for each $f \in C_{b,\infty}(I)$ and every $n \in \mathbb{N}$, one has*

$$\|C_{n,j}(f) - f\| \leq C\omega^{\phi}\Big(f, \frac{1}{\sqrt{n}}\Big),$$

where $\omega^{\phi}(f,t)$ is given by (6.15).

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