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*Research Article*

# Weighted approximation by generalized Baskakov operators reproducing affine functions

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ABSTRACT. Some extension of the Baskakov operators are introduced. The new operators reproduce affine functions. The rate of convergence of the associated approximation process is obtained in polynomial type weighted spaces. A Voronovskaja type result is included.

**Keywords:** Baskakov type operators, polynomial weighted spaces, rate of convergence.

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## 1. INTRODUCTION

For $a \geq 0$, $\mid t \mid < 1$ and $n \in \mathbb{N}$, let us consider the expansion

$$\frac{e^{at}}{(1-t)^n} = \sum_{k=0}^{\infty} p_{k,a}(n)\frac{t^k}{k!}, \qquad p_{k,a}(n) = \sum_{i=0}^{k} \binom{k}{i} (n)_i a^{k-i}, \tag{1.1}$$

where $(n)_0 = 1$, and for $i \in \mathbb{N}$,

$$(n)_i = n(n+1)\cdots(n+i-1). \tag{1.2}$$

If we take $t = x/(x+1)$, then

$$e^{ax/(x+1)}(1+x)^n = \sum_{k=0}^{\infty} \frac{p_{k,a}(n)}{k!}\left(\frac{x}{1+x}\right)^k.$$

For $a \geq 0$, Miheşan [4] defined

$$B_n^a(f,x) = \sum_{k=0}^{\infty} W_{n,k}^a(x) f\left(\frac{k}{n}\right),$$

where

$$W_{n,k}^a(x) = e^{-ax/(x+1)} \frac{p_{k,a}(n)x^k}{k!(1+x)^{n+k}}.$$

These operators have been studied by several authors, for instance see [1, 2, 5, 6]. Since $p_{k,0} = (n)_k$ (case $a = 0$), the Miheşan operators $B_n^0$ agree with the classical Baskakov operators. The Miheşan operators preserve constant functions, but do not reproduce affine functions. That is the reason why in this work we introduce a new construction. In what follows, for $n > 1$ set

$$y_{n,0}^a = 0 \qquad \text{and} \qquad y_{n,k}^a = \frac{kp_{k-1,a}(n+1)}{p_{k,a}(n)} \tag{1.3}$$

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for $k \in \mathbb{N}$. For $n > 1$, we define the positive linear operators $C_n^a(f,x)$ by the equation

$$C_n^a(f,x) = \sum_{k=0}^{\infty} f(y_{n,k}^a) W_{n,k}^a(x),$$

whenever the series converges. Sometimes, we also use the notation $B_n^a(f(t),x)$ instead of $B_n^a(f,x)$. We prove in Proposition 4.2 that this operator reproduces affine functions. In this paper, we present some approximation properties of the operators $C_n^a$ in a weighted space $C_q[0,\infty)$ defined as follows:

For a fixed real $q \geq 0$,

$$C_{\varrho_q}[0,\infty) = \Big\{ f \in C[0,\infty) \, : \, \|f\|_{\varrho_q} < \infty \Big\},$$

where

$$\varrho_q(x) = 1/(1+x)^q$$

for $x \geq 0$ and

$$\|f\|_{\varrho_q} = \sup_{x \geq 0} |\varrho(x) f(x)|.$$

The work is organized as follows: In Section 2, we collect several known results. In Section 3, some properties of the nodes $y_{n,k}^a$ and the coefficients of $C_n^a$ are given. Section 4 is devoted to verify that the operators $C_n^a$ are well defined in the weighted space $C_{\varrho_q}[0,\infty)$ and to estimate the norm of the operators. The proofs of our main results depend on some properties of the moments of the operators proved in Section 5. An upper estimate for the rate of convergence to zero of $\|C_n^a(f) - f\|_{\varrho_q}$ is given in Section 6.

## 2. Known results

**Lemma 2.1** ([1]). *If $a \geq 0$ and $n, k \in \mathbb{N}$, then*

$$p_{k,a}(n) = a p_{k-1,a}(n) + n p_{k-1,a}(n+1).$$

**Lemma 2.2** ([4]). *If $a \geq 0$, $n > 1$ and $x \geq 0$, then*

$$B_n^a(1,x) = 1, \qquad B_n^a(t,x) = x + \frac{ax}{n(1+x)} \tag{2.4}$$

*and*

$$B_n^a((t-x)^2,x) = \frac{\varphi^2(x)}{n}\Big(1 + \frac{a}{n(1+x)^2} + \frac{a^2 x}{n(1+x)^3}\Big). \tag{2.5}$$

**Lemma 2.3** ([2]). *If $a \geq 0$, there exists a constant $C$ such that for each $n > 1$ and $x \geq 0$, one has*

$$B_n^a(|t-x|^j,x) \leq C\frac{\varphi^j(x)}{n^{j/2}}, \qquad j \in \{1,2,3,4\}. \tag{2.6}$$

*Moreover*

$$0 \leq B_n^a((t-x)^6,x) \leq C\frac{(1+x)\varphi^4(x)}{n^3}\Big(x + \frac{1}{n}\Big). \tag{2.7}$$

**Theorem 2.1** ([1]). *If $a \geq 0$ and $q \geq 0$ are real numbers, there exists a constant $M_q(a)$ such that*

$$M_a(q) := \sup_{n>1} \sup_{x \geq 0} \frac{B_n^a((1+t)^q,x)}{(1+x)^q} < \infty.$$

**Proposition 2.1** ([1]). *If $a > 0$, $r \in [0,1]$, there exists a constant $C$ such that for each integer $n > 1$ and each $x \geq 0$,*

$$B_n^a\Big(\frac{1}{(1+t)^r}, x\Big) \leq \Big(\frac{n}{n-1}\Big)^r \frac{1}{(1+x)^r}.$$

**Remark 2.1.** *It is known a similar result for $a = 0$. We can use the arguments in [1] to verify that, if $a \geq 0$ and $r \in [0,2]$ there exists a constant $C = C(a,r)$ such that for $n > 2$,*

$$B_n^a\Big(\frac{1}{(1+t)^r}, x\Big) \leq \frac{C}{(1+x)^r}. \tag{2.8}$$

**Lemma 2.4** ([3], Proposition 3.3). *Assume $r \geq 0$, $m, p \in \mathbb{R}$ and $m - r + 1 > 0$. Then $x > 0$ and $t \geq 0$, one has*

$$\Big|\int_x^t \frac{(t-s)^m}{s^r}(1+u)^p ds\Big| \leq \frac{|\, t - x\,|^{m+1}}{(m-r+1)x^r}\Big((1+x)^p + (1+t)^p\Big).$$

## 3. Properties of the nodes and the coefficients of $C_n^a$

Moreover, if $a = 0$, then

$$y_{n,k}^a = \frac{k}{n}$$

and we recover the Baskakov operators.

**Lemma 3.5.** *If $a \geq 0$ and $k \in \mathbb{N}_0$, then*

$$0 \leq \frac{k}{n} - y_{n,k}^a \leq \frac{k}{n} - \frac{k}{n+a} = \frac{ak}{n(n+a)}.$$

*Proof.* For $k = 0$ the inequalities are trivial, because $y_{n,0}^a = 0$.
**(A)** For $k \geq 1$, it follows from Lemma 2.1 that

$$0 < p_{k-1,a}(n+1) = \frac{1}{n}(p_{k,a}(n) - a p_{k-1,a}(n)) \leq \frac{1}{n} p_{k,a}(n).$$

Therefore

$$y_{n,k}^a = \frac{k p_{k-1,a}(n+1)}{p_{k,a}(n)} \leq \frac{k}{n}.$$

**(B)** It follows from (1.1) and (1.2) that for $k \in \mathbb{N}$ and $k > 1$,

$$\begin{aligned}
p_{k-1,a}(n) &= a^{k-1} + \sum_{i=1}^{k-1} \binom{k-1}{i} (n)_i a^{k-1-i} \\
&= a^{k-1} + n \sum_{i=1}^{k-1} \frac{1}{n+i} \binom{k-1}{i} (n+1)_i a^{k-1-i} \\
&\leq a^{k-1} + \frac{n}{n+1} \sum_{i=1}^{k-1} \binom{k-1}{i} (n+1)_i a^{k-1-i} \\
&\leq p_{k-1,a}(n+1).
\end{aligned}$$

Taking into account Lemma 2.1, for $k \in \mathbb{N}$ with $k > 1$, we obtain

$$\begin{aligned}
p_{k,a}(n) &= a p_{k-1,a}(n) + n p_{k-1,a}(n+1) \\
&\leq a p_{k-1,a}(n+1) + n p_{k-1,a}(n+1) \\
&= (n+a) p_{k-1,a}(n+1)
\end{aligned}$$

and

$$\frac{k}{n+a} \leq \frac{kp_{k-1,a}(n+1)}{p_{k,a}(n)} = y^a_{n,k}.$$

In the case $k = 1$ one has

$$y^a_{n,1} = \frac{p_{0,a}(n+1)}{p_{1,a}(n)} = \frac{1}{n+a}.$$

We have proved that

$$0 \leq \frac{k}{n} - y^a_{n,k} \leq \frac{k}{n} - \frac{k}{n+a} = \frac{ak}{n(n+a)}.$$

□

**Lemma 3.6.** *If $a \geq 0$ and $k \in \mathbb{N}$, then*

$$y^a_{n,k} W^a_{n,k}(x) = xW^a_{n+1,k-1}(x).$$

*Proof.* If $k \geq 1$,

$$\begin{aligned} y^a_{n,k} W^a_{n,k}(x) &= e^{-ax/(x+1)} x \frac{kp_{k-1,a}(n+1)}{p_{k,a}(n)} \frac{p_{k,a}(n)x^{k-1}}{k!(1+x)^{n+k}} \\ &= xe^{-ax/(x+1)} \frac{p_{k-1,a}(n+1)x^{k-1}}{(k-1)!(1+x)^{n+1+k-1}} \\ &= xW^a_{n+1,k-1}(x). \end{aligned}$$

□

## 4. The norm of the operators in weighted spaces

**Proposition 4.2.** *If $a \geq 0$, for all $n > 1$, one has*

$$C^a_n(1, x) = 1 \qquad \text{and} \qquad C^a_n(t, x) = x.$$

*Proof.* It is clear that $C^a_n(1, x) = B^a_n(1, x) = 1$ (see (2.4)). On the other hand, it follows from Lemma 3.6 that

$$\begin{aligned} C^a_n(e_1, x) &= \sum_{k=0}^{\infty} y^a_{n,k}\, W^a_{n,k}(x) = \sum_{k=1}^{\infty} y^a_{n,k}\, W^a_{n,k}(x) = x \sum_{k=1}^{\infty} W^a_{n+1,k-1}(x) \\ &= x \sum_{k=0}^{\infty} W^a_{n+1,k}(x) = x\, C^a_n(1, x) = x. \end{aligned}$$

□

The following result can be proven in different ways. For instance, it is a consequence of [1, Corollary 3.2].

**Lemma 4.7.** *If $a \geq 0$ and $m \in \mathbb{N}$, for all $n > 1$ and $x \geq 0$ the series*

$$B^a_n(t^m, x) = \sum_{k=0}^{\infty} \left(\frac{k}{n}\right)^m W^a_{n,k}(x) \tag{4.9}$$

*converges.*

**Corollary 4.1.** *If $a \geq 0$ and $m \in \mathbb{N}$, for all $n > 1$ and $x \geq 0$ the series*

$$C_n^a(t^m, x) = \sum_{k=0}^{\infty} (y_{n,k}^a)^m W_{n,k}^a(x) \tag{4.10}$$

*converges.*

*Proof.* It is a simple consequence of Lemma 3.5 and Lemma 4.7. □

Theorem 2.1 has an extension to the case of the operators $C_n^a$.

**Theorem 4.2.** *If $a \geq 0$ and $q \geq -2$ are real numbers, for each integer $n > 2$, $C_n^a((1+t)^q, x) \in C_\varrho[0,\infty)$. Moreover, there exists a constant $N_q(a)$ such that for each integer $n > 2$,*

$$\|C_n^a((1+t)^q, \cdot)\|_{\varrho_q} \leq N_q(a).$$

*Proof.* If $q \geq 0$, from Lemma 3.5, we know that

$$(1 + y_{n,k}^a)^q \leq \Big(1 + \frac{k}{n}\Big)^q.$$

Hence, the results follows as a consequence of Theorem 2.1. Notice that

$$\frac{k}{n} < \frac{k(1+a)}{n+a}.$$

Hence, (see Lemma 3.5)

$$1 + \frac{k}{n} \leq 1 + a + \frac{k(1+a)}{n+a} = (1+a)\Big(1 + \frac{k}{n+a}\Big) \leq (1+a)\Big(1 + y_{n,k}^a\Big).$$

Therefore, if $q \in [-2, 0)$,

$$\Big(1 + \frac{k}{n}\Big)^{-q} \leq \frac{1}{(1+a)^q}\Big(1 + y_{n,k}^a\Big)^{-q}.$$

It can be written as

$$(1 + y_{n,k}^a)^q \leq \frac{1}{(1+a)^q}\Big(1 + \frac{k}{n}\Big)^q$$

and the result follows from (2.8). □

**Remark 4.2.** *Notice that, in Theorem 4.2, the condition $n > 2$ is only needed in the case $q < 0$. But we need some negative parameters $q$ in the proof of the main result (Theorem 6.4).*

**Theorem 4.3.** *If $a \geq 0$, $q \geq 0$ are real numbers, $n > 1$, and $f \in C_{\varrho_q}[0,\infty)$, then $C_n^a(f) \in C_{\varrho_q}[0,\infty)$ and*

$$\|C_n^a(f)\|_{\varrho_q} \leq N_a(q)\|f\|_{\varrho_q},$$

*where $N_q(a)$ is the constant in Theorem 4.2.*

*Proof.* If $f \in C_{\varrho_q}[0,\infty)$ and $x \geq 0$, it follows from Theorem 4.2 that

$$\begin{aligned}\varrho_q(x)|C_n^a(f,x)| &\leq \varrho_q(x)\,\|f\|_{\varrho_q}\, C_n^a((1+t)^q, x)\\ &\leq N_q(a)\,\|f\|_{\varrho_q}.\end{aligned}$$

□

## 5. Estimates for some moments

**Proposition 5.3.** *If* $a \geq 0$, $n > 1$ *and* $x \geq 0$, *then*

$$C_n^a((t-x)^2, x) = x \sum_{k=0}^{\infty} y_{n,k+1}^a W_{n+1,k}^a(x) - x^2$$
$$\leq (1+a)\frac{\varphi^2(x)}{n}.$$

*Proof.* Taking into account Proposition 4.2 and Lemma 3.6, one has

$$\begin{aligned} C_n^a((t-x)^2, x) &= C_n^a((t-x)(t-x), x) \\ &= C_n^a(t(t-x), x) - xC_n^a((t-x), x) \\ &= C_n^a(t(t-x), x) \\ &= \sum_{k=1}^{\infty} y_{n,k}^a (y_{n,k}^a - x) W_{n,k}^a(x) \\ &= x \sum_{k=0}^{\infty} (y_{n,k+1}^a - x) W_{n+1,k}^a(x) \\ &= x \sum_{k=0}^{\infty} y_{n,k+1}^a W_{n+1,k}^a(x) - x^2. \end{aligned}$$

On the other hand, using Lemma 3.5 and (2.4), we obtain

$$\begin{aligned} 0 \leq C_n^a((t-x)^2, x) &= x \sum_{k=0}^{\infty} y_{n,k+1}^a W_{n+1,k}^a(x) - x^2 \\ &\leq x \sum_{k=1}^{\infty} \frac{k+1}{n} W_{n+1,k}^a(x) - x^2 \\ &\leq \frac{x}{n} + x \sum_{k=1}^{\infty} \frac{k}{n} W_{n+1,k}^a(x) - x^2 \\ &= \frac{x}{n} + x\frac{n+1}{n} B_{n+1}^a(t, x) - x^2 \\ &= \frac{x}{n} + \frac{(n+1)x}{n}\left(x + \frac{ax}{(n+1)(1+x)}\right) - x^2 \\ &= \frac{x}{n} + \frac{(n+1)}{n}x^2 + \frac{ax^2}{(n+1)(1+x)} - x^2 \\ &= \frac{x+x^2}{n} + \frac{ax^2}{n(1+x)} \\ &= \frac{\varphi^2(x)}{n}\left(1 + \frac{ax}{(1+x)^2}\right) \\ &\leq \frac{(1+a)\varphi^2(x)}{n}. \end{aligned}$$

□

**Proposition 5.4.** *If $a \geq 0$, there exists a constant $C$ such that for $n > 1$ and $x \geq 0$, one has*

$$C_n^a((t-x)^4, x) \leq C\frac{\varphi^4(x)}{n^2},$$
$$C_n^a(|t-x|^3, x) \leq C\,\frac{\varphi^3(x)}{n^{3/2}},$$
$$C_n^a((t-x)^6, x) \leq C\Big(\frac{\varphi^6(x)}{n^3} + \frac{(1+x)\varphi^4(x)}{n^3}\Big(x+\frac{1}{n}\Big)\Big).$$

*Proof.* If $k \in \mathbb{N}_0$ and $x \geq 0$, then

$$\begin{aligned}
|y_{n,k}^a - x| &= \Big|y_{n,k}^a - \frac{k}{n} + \frac{k}{n} - x\Big| \\
&\leq \frac{ak}{n^2} + \Big|\frac{k}{n} - x\Big| \\
&= \frac{a}{n}\Big(\frac{k}{n} - x\Big) + \frac{ax}{n} + \Big|\frac{k}{n} - x\Big| \\
&\leq \frac{ax}{n} + (1+a)\Big|\frac{k}{n} - x\Big| \\
&\leq 2(1+a)\max\Big\{\frac{x}{n}, \Big|\frac{k}{n} - x\Big|\Big\}.
\end{aligned}$$

Therefore, taking into account (2.6), one has

$$\begin{aligned}
\sum_{k=0}^{\infty}(y_{n,k}^a - x)^4 W_{n,k}^a(x) &\leq 2^4(1+a)^4\sum_{k=0}^{\infty}\Big(\frac{x^4}{n^4} + \Big(\frac{k}{n} - x\Big)^4\Big)W_{n,k}^a(x) \\
&= 2^4(1+a)^4\Big(\frac{x^4}{n^4} + B_n^a((t-x)^4, x)\Big) \\
&\leq C_1\frac{\varphi^4(x)}{n^2}.
\end{aligned}$$

With similar arguments, we obtain

$$\begin{aligned}
\sum_{k=0}^{\infty}(y_{n,k}^a - x)^6 W_{n,k}^a(x) &\leq 2^6(1+a)^6\sum_{k=0}^{\infty}\Big(\frac{x^6}{n^6} + \Big(\frac{k}{n} - x\Big)^6\Big)W_{n,k}^a(x) \\
&= 2^6(1+a)^6\Big(\frac{x^6}{n^6} + B_n^a((t-x)^6, x)\Big) \\
&\leq C_2\Big(\frac{\varphi^6(x)}{n^3} + \frac{(1+x)\varphi^4(x)}{n^3}\Big(x+\frac{1}{n}\Big)\Big),
\end{aligned}$$

where we use (2.7). Moreover

$$\begin{aligned}
C_n^a(|t-x|^3, x) &\leq \sqrt{C_n^a(|t-x|^2, x)}\sqrt{C_n^a(|t-x|^4, x)} \\
&\leq C\frac{\varphi^3(x)}{n^{3/2}}.
\end{aligned}$$

The proof is over. □

For the proof of the Voronovskaja type result we need other computations. Set

$$F_n^a(x) = \sum_{k=0}^{\infty} \left(\frac{k}{n} - y_{n,k}^a\right)^2 W_{n,k}^a(x).$$

**Proposition 5.5.** *If $a \geq 0$, $n > 1$ and $x \geq 0$, then*

$$C_n^a((t-x)^2, x) = F_n^a(x) - B_n^a((t-x)^2, x) + \frac{2\varphi^2(x)}{n}.$$

*Proof.* It is easy to prove that

$$\frac{\varphi^2(x)}{n}\left(\frac{d}{dx} W_{n,k}^a(x)\right) = \left(\frac{k}{n} - x - \frac{ax}{n(1+x)}\right) W_{n,k}^a(x).$$

Taking into account Proposition 4.2, (2.4), and (2.5), the assertion follows from the equations

$$\begin{aligned}
&C_n^a((t-x)^2, x) - F_n^a(x) - B_n^a((t-x)^2, x)\\
=&\sum_{k=0}^{\infty}\left(y_{n,k}^a - \frac{k}{n} + \frac{k}{n} - x\right)^2 W_{n,k}^a(x) - F_n^a(x) - B_n^a((t-x)^2, x)\\
=&2\sum_{k=0}^{\infty}\left(y_{n,k}^a - \frac{k}{n}\right)\left(\frac{k}{n} - x\right) W_{n,k}^a(x)\\
=&2\sum_{k=0}^{\infty}\left(y_{n,k}^a - \frac{k}{n}\right)\left(\frac{k}{n} - x - \frac{ax}{n(1+x)}\right) W_{n,k}^a(x) + \frac{2ax}{n(1+x)}\sum_{k=0}^{\infty}\left(y_{n,k}^a - x + x - \frac{k}{n}\right) W_{n,k}^a(x)\\
=&\frac{2\varphi^2(x)}{n}\frac{d}{dx}\sum_{k=0}^{\infty}\left(y_{n,k}^a - \frac{k}{n}\right) W_{n,k}^a(x) + \frac{2ax}{n(1+x)}(C_n^a(t-x, x) - B_n^a(t-x, x))\\
=&\frac{2\varphi^2(x)}{n}\frac{d}{dx}\left(C_n^a(t-x, x) - B_n^a(t-x, x)\right) - \frac{2axB_n^a(t-x, x)}{n(1+x)}\\
=&-\frac{2\varphi^2(x)}{n}\frac{d}{dx} B_n^a(t-x, x) - \frac{2axB_n^a(t-x, x)}{n(1+x)}\\
=&-\frac{2\varphi^2(x)}{n}\frac{d}{dx}\frac{ax}{n(1+x)} - \frac{2a^2x^2}{n^2(1+x)^2}\\
=&-\frac{2a\varphi^2(x)}{n^2(1+x)^2} - \frac{2a^2x\varphi^2(x)}{n^2(1+x)^3}\\
=&-\frac{2\varphi^2(x)}{n}\left(\frac{a}{n(1+x)^2} + \frac{a^2x}{n(1+x)^3}\right)\\
=&-\frac{2\varphi^2(x)}{n}\left(1 + \frac{a}{n(1+x)^2} + \frac{a^2x}{n(1+x)^3}\right) + \frac{2\varphi^2(x)}{n}\\
=&-2B_n^a((t-x)^2, x) + \frac{2\varphi^2(x)}{n}.
\end{aligned}$$

□

**Proposition 5.6.** *If $a \geq 0$, $n > 1$ and $x \geq 0$, then*

$$0 \leq F_n^a(x) \leq (2a^2 + 3a^3 + a^4)\frac{\varphi^2(x)}{n^2}.$$

*Proof.* It follows from Lemma 3.5, (2.5) and (2.4) that

$$\begin{aligned}\sum_{k=0}^{\infty}\Big(y_{n,k}^{a}-\frac{k}{n}\Big)^{2}W_{n,k}^{a}(x)&\leq\sum_{k=0}^{\infty}\frac{a^{2}k^{2}}{n^{4}}W_{n,k}^{a}(x)\\&=\frac{a^{2}}{n^{2}}\sum_{k=0}^{\infty}\Big(\frac{k}{n}-x+x\Big)^{2}W_{n,k}^{a}(x)\\&=\frac{a^{2}}{n^{2}}B_{n}^{a}((t-x)^{2},x)+\frac{2a^{2}x}{n^{2}}B_{n}^{a}((t-x),x)+\frac{a^{2}}{n^{2}}x^{2}\\&=\frac{a^{2}}{n^{2}}\frac{\varphi^{2}(x)}{n}\Big(1+\frac{a}{n(1+x)^{2}}+\frac{a^{2}x}{n(1+x)^{3}}\Big)+\frac{2a^{2}x}{n^{2}}\frac{ax}{n(1+x)}+\frac{a^{2}}{n^{2}}x^{2}\\&=\frac{a^{2}\varphi^{2}(x)}{n^{2}}\Big\{\frac{1}{n}+\frac{a}{n^{2}(1+x)^{2}}+\frac{a^{2}x}{n^{2}(1+x)^{3}}+\frac{2ax}{n(1+x)^{2}}+\frac{x}{(1+x)}\Big\}\\&\leq a^{2}(2+3a+a^{2})\frac{\varphi^{2}(x)}{n^{2}}.\end{aligned}$$

□

**Corollary 5.2.** *If $a\geq 0$, $n>1$ and $x\geq 0$, then*

$$\Big|\frac{\varphi^{2}(x)}{n}-C_{n}^{a}((t-x)^{2},x)\Big|\leq(a+3a^{2}+3a^{3}+a^{4})\frac{\varphi^{2}(x)}{n^{2}}.$$

*Proof.* Taking into account Proposition 5.5, (2.5) and Proposition 5.6, we obtain

$$\begin{aligned}\Big|\frac{\varphi^{2}(x)}{n}-C_{n}^{a}((t-x)^{2},x)\Big|&=\Big|\frac{\varphi^{2}(x)}{n}-F_{n}^{a}(x)+B_{n}^{a}((t-x)^{2},x)-\frac{2\varphi^{2}(x)}{n}\Big|\\&=\Big|B_{n}^{a}((t-x)^{2},x)-F_{n}^{a}(x)-\frac{\varphi^{2}(x)}{n}\Big|\\&=\Big|\frac{\varphi^{2}(x)}{n}\Big(1+\frac{a}{n(1+x)^{2}}+\frac{a^{2}x}{n(1+x)^{3}}\Big)-F_{n}^{a}(x)-\frac{\varphi^{2}(x)}{n}\Big|\\&=\Big|\frac{\varphi^{2}(x)}{n^{2}}\Big(\frac{a}{(1+x)^{2}}+\frac{a^{2}x}{(1+x)^{3}}\Big)-F_{n}^{a}(x)\Big|\\&\leq(a+3a^{2}+3a^{3}+a^{4})\frac{\varphi^{2}(x)}{n^{2}}.\end{aligned}$$

□

**Proposition 5.7.** *If $a,q\geq 0$, there exists a constant $C(a)$ (depending only on $a$), such that if $n>1$ and $f\in C_{\varrho_q}[0,\infty)$, then*

$$\Big\|f(\cdot)\Big(\frac{\varphi^{2}(\cdot)}{n}-C_{n}^{a}((t-\cdot)^{2},\cdot)\Big)\Big\|_{\varrho_q}\leq\frac{C(a)}{n^{2}}\|\varphi^{2}f\|_{\varrho_q}.$$

*Proof.* Taking into account Corollary 5.2, there exists a constant $C(a)$ (depending only on $a$) such that for any $x\geq 0$,

$$\begin{aligned}\Big|\varrho_{\varrho}(x)f(x)\Big(\frac{\varphi^{2}(x)}{n}-C_{n}^{a}((t-x)^{2},x)\Big)\Big|&\leq\frac{C(a)}{n^{2}}\Big|\varrho_{q}(x)\varphi^{2}(x)f(x)\Big|\\&\leq\frac{C(a)}{n^{2}}\|\varphi^{2}f\|_{\varrho_q}.\end{aligned}$$

This yields the result. □

## 6. THE RATE OF CONVERGENCE

For $f\in C_{\varrho_q}[0,\infty)$ and $t\geq 0$, set

$$K(f,t)_{\varrho_q}=\inf_{g\,\in\, D(\varrho_q)}\left(\|f-g\|_{\varrho_q}+t\left(\|\varphi^2 g''\|_{\varrho_q}\right)\right), \tag{6.11}$$

where

$$D(\varrho_q)=\Big\{g\in C_{\varrho_q}[0,\infty):g,g'\in AC_{loc}[0,\infty),\ \varphi^2 g''\in C_{\varrho_q}[0,\infty)\Big\}.$$

**Theorem 6.4.** *Assume $a,q\geq 0$ are real numbers. There exists a constant $C=C(a,q)$ such that for $n>2$ and $f\in C_{\varrho_q}[0,\infty)$, one has*

$$\|C_n^a(f)-f\|_{\varrho_q}\leq CK\Big(f,\frac{1}{n}\Big)_{\varrho_q},$$

*where $K(f,t)_{\varrho_q}$ is defined in* (6.11).

*Proof.* If $x,t>0$ and $g\in D(\varrho_q)$, we will use the representation

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t g''(s)(t-s)ds.$$

It follows from Lemma 2.4,

$$\begin{aligned}\Big|\int_x^t g''(s)(t-s)ds\Big|&\leq\|\varphi^2 g''\|_{\varrho_q}\Big|\int_x^t\frac{(t-s)}{\varphi^2(s)\varrho(s)}ds\Big|\\&=\|\varphi^2 g''\|_{\varrho_q}\Big|\int_x^t\frac{(t-s)}{s}(1+s)^{q-1}ds\Big|\\&\leq\|\varphi^2 g''\|_{\varrho_q}\frac{(t-x)^2}{x}((1+x)^{q-1}+(1+t)^{q-1}).\end{aligned}$$

Taking into account Proposition 5.3, Hölder inequality, Theorem 4.2 and Proposition 5.4, we obtain

$$\begin{aligned}&\Big|C_n^a\Big(\int_x^t g''(s)(t-s)ds,x\Big)\Big|\\&\leq\frac{\|\varphi^2 g''\|_{\varrho_q}}{x}\Big((1+x)^{q-1}C_n^a((t-x)^2,x)+C_n^a((t-x)^2(1+t)^{q-1},x)\Big)\\&\leq\frac{\|\varphi^2 g''\|_{\varrho_q}}{x}\Big(C_1(1+x)^{q-1}\frac{\varphi^2(x)}{n}+\sqrt{C_n^a((1+t)^{2q-2},x)}\sqrt{C_n^a(t-x)^4,x)}\Big)\\&\leq C_2\frac{\|\varphi^2 g''\|_{\varrho_q}}{x}\Big(\frac{x(1+x)^q}{n}+(1+x)^{q-1}\frac{\varphi^2(x)}{n}\Big)\\&\leq C_3\frac{\|\varphi^2 g''\|_{\varrho_q}}{n\varrho_q(x)}.\end{aligned}$$

Since

$$\begin{aligned}C_n^a(g,x)-g(x)&=g'(x)B_n^a(t-x,x)+C_n^a\Big(\int_x^t g''(s)(t-s)ds,x\Big)\\&=C_n^a\Big(\int_x^t g''(s)(t-s)ds,x\Big),\end{aligned}$$

from the previous estimate, one has

$$\begin{aligned}\varrho_q(x)\Big|C_n^a(g,x)-g(x)\Big| &\leq \varrho_q(x)C_n^a\Big(\Big|\int_x^t g''(s)(t-s)ds\Big|,x\Big)\\ &\leq C_3\frac{\|\varphi^2 g''\|_\varrho}{n}.\end{aligned}$$

Therefore, taking into account Theorem 4.3, for any $g\in D(\varrho)$,

$$\begin{aligned}\|C_n^a(f)-f\|_{\varrho_q} &\leq \|f-g\|_{\varrho_q}+\|C_n^a(f-g)\|_{\varrho_q}+\|C_n^a(g)-g\|_{\varrho_q}\\ &\leq (1+N_a(q))\|f-g\|_{\varrho_q}+\frac{C_3}{n}\|\varphi^2 g''\|_{\varrho_q}.\end{aligned}$$

Hence, the assertion follows from standard arguments. □

In order to simplify the arguments, in the next result we only consider the case $q\geq 1$.

**Theorem 6.5.** *Assume $a\geq 0$ and $q\geq 1$. There exists a constant $C$ such that if $n>1$, $g\in C^3[0,\infty)$ and $g,\varphi^2 g'',\varphi^3 g'''\in C_{\varrho_q}[0,\infty)$, then*

$$\Big\|C_n^a(g)-g-\frac{\varphi^2 g''}{2n}\Big\|_{\varrho_q}\leq C\Big(\frac{\|\varphi^2 g''\|_{\varrho_q}+\|\varphi^2 g'''\|_{\varrho_q}}{n^2}+\frac{\|\varphi^3 g'''\|_{\varrho_q}}{n^{3/2}}\Big).$$

*Proof.* For $g\in C^3[0,\infty)$, we use the Taylor expansion

$$g(t)=g(x)+g'(x)(t-x)+\frac{1}{2}g''(x)(t-x)^2+\frac{1}{2}\int_x^t g'''(s)(t-s)^2 ds$$

to obtain the representation

$$C_n^a(g,x)-Q_n^a(g,x)=\frac{1}{2}C_n^a\Big(\int_x^t g'''(s)(t-s)^2 ds,x\Big)$$

with

$$Q_n^a(g,x)=g(x)+\frac{1}{2}g''(x)C_n^a((t-x)^2,x).$$

We should estimate

$$C_n^a\Big(\Big|\int_x^t g'''(s)(t-s)^2 ds\Big|,x\Big)=\sum_{k=0}^{\infty}\Big|\int_x^{y_{n,k}^a} g'''(s)\Big(\frac{k}{n}-s\Big)^2 ds\Big|W_{n,k}^a(x).$$

**(A)** Suppose $0\leq (n+a)x<1$. In this case, we consider the inequality

$$\sum_{k=0}^{\infty}\Big|\int_x^{y_{n,k}^a} g'''(s)\Big(\frac{k}{n}-s\Big)^2 ds\Big|W_{n,k}^a(x)\leq \|\varphi^2 g'''\|_{\varrho_q}\sum_{k=0}^{\infty}\Big|\int_x^{y_{n,k}^a}\Big(\frac{k}{n}-s\Big)^2\frac{ds}{\varphi^2(s)\varrho_q(s)}\Big|W_{n,k}^a(x).$$

For $k=0$ one has

$$\begin{aligned}W_{n,0}^a(x)\int_0^x\frac{s^2}{\varphi^2(s)\varrho_q(s)}ds &= W_{n,0}^a(x)\int_0^x s(1+s)^{q-1}ds\\ &\leq \frac{x^2(1+x)^{q-1}}{(1+x)^n}\\ &\leq \frac{(1+x)^q}{n^2}.\end{aligned}$$

On the other hand, if $q \geq 1$, then

$$\sum_{k=1}^{\infty} W_{n,k}^a(x) \int_x^{y_{n,k}^a} \frac{(k/n-s)^2}{\varphi^2(s)\varrho_q(s)} ds \leq \sum_{k=1}^{\infty} W_{n,k}^a(x) \int_x^{k/n} \frac{(k/n-s)^2}{\varphi^2(s)\varrho_q(s)} ds \leq \frac{C_1(1+x)^q}{n^2},$$

where the second inequality was proved in [2]. Therefore, for $0 \leq (n+a)x < 1$

$$\varrho_q(x)|C_n^a(g,x) - Q_n^a(g,x)| \leq \frac{C_2}{n^2}\|\varphi^2 g'''\|_{\varrho_q}.$$

**(B)** Assume that $(n+a)x \geq 1$. In this case, we consider the inequality

$$\sum_{k=0}^{\infty} \left| \int_x^{y_{n,k}^a} g'''(s)\left(\frac{k}{n}-s\right)^2 ds \right| W_{n,k}^a(x) \leq \|\varphi^3 g'''\|_{\varrho_q} \sum_{k=0}^{\infty} \left| \int_x^{y_{n,k}^a} \left(\frac{k}{n}-s\right)^2 \frac{ds}{\varphi^3(s)\varrho_q(s)} \right| W_{n,k}^a(x).$$

Since $1/(n+a) \leq x$, it follows from Proposition 5.4,

$$\begin{aligned} C_n^a((t-x)^6,x) &\leq C_3 \frac{(1+x)\varphi^4(x)}{n^3}\left(x+\frac{1}{n}\right) \\ &\leq C_4 \frac{\varphi^6(x)}{n^3}. \end{aligned}$$

Thus, we can use Lemma 2.4 and Theorem 4.2 to obtain

$$\begin{aligned} &\varrho_q(x)\left|C_n^a(g,x) - Q_n^a(g,x)\right| \\ \leq &\frac{1}{2}\varrho_q(x)\|\varphi^3 g'''\|_{\varrho_q} C_n^a\left(\left|\int_x^t \frac{(t-s)^2}{\varphi^3(s)\varrho_q(s)} ds\right|, x\right) \\ = &\frac{1}{2}\varrho_q(x)\|\varphi^3 g'''\|_{\varrho_q} C_n^a\left(\left|\int_x^t \frac{(t-s)^2(1+s)^{q-3/2}}{s^{3/2}} ds\right|, x\right) \\ \leq &\frac{1}{3}\varrho_q(x)\|\varphi^3 g'''\|_{\varrho_q} C_n^a\left(|t-x|^3\left(\frac{(1+x)^{q-3/2}}{x^{3/2}} + \frac{(1+t)^{q-3/2}}{x^{3/2}}\right), x\right) \\ = &\frac{1}{3}\|\varphi^3 g'''\|_{\varrho_q}\left(\frac{B_n^a(|t-x|^3,x)}{\varphi^3(x)} + \frac{\varrho_q(x)}{x^{3/2}}\sqrt{C_n^a((t-x)^6,x)}\sqrt{C_n^a((1+t)^{2q-3},x)}\right. \\ \leq &C_5\|\varphi^3 g'''\|_{\varrho_q}\left(\frac{1}{n^{3/2}} + \frac{\varrho_q(x)}{x^{3/2}}\frac{\varphi^3(x)}{n^{3/2}}(1+x)^{q-3/2}\right) \\ \leq &C_6\frac{\|\varphi^3 g'''\|_{\varrho_q}}{n^{3/2}}. \end{aligned}$$

We have proved that

$$\|C_n^a(g,) - Q_n^a(g)\|_{\varrho_q} \leq \frac{C_2}{n^2}\|\varphi^2 g'''\|_{\varrho_q} + C_6\frac{\|\varphi^3 g'''\|_{\varrho_q}}{n^{3/2}}.$$

Taking into account Proposition 5.7, we obtain

$$\begin{aligned} \left\|C_n^a(g) - g - \frac{\varphi^2 g''}{2n}\right\|_{\varrho_q} &\leq \|C_n^a(g) - Q_n^a(g)\|_{\varrho_q} + \frac{1}{2}\left\|g''\left(\frac{\varphi^2}{n} - C_n^a((t-\cdot)^2,\cdot)\right)\right\|_{\varrho_q} \\ &\leq \frac{C_2}{n^2}\|\varphi^2 g'''\|_{\varrho_q} + C_6\frac{\|\varphi^3 g'''\|_{\varrho}}{n^{3/2}} + \frac{C_7}{n^2}\|\varphi^2 g''\|_{\varrho_q}. \end{aligned}$$

□

**Corollary 6.3.** *Assume $a \geq 0$ and $q \geq 1$. If $g \in C^3[0,\infty)$ and $g, \varphi^2 g'', \varphi^3 g''' \in C_{\varrho_q}[0,\infty)$, then*

$$\lim_{n\to\infty} \left\| n\Big(C_n^a(g) - g\Big) - \frac{\varphi^2 g''}{2} \right\|_{\varrho_q} = 0.$$

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