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Research Article

Viscosity solutions to the ∞-Laplace equation in Grushin-type spaces

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ABSTRACT. In this paper, we prove the existence and uniqueness of viscosity solutions to the infinite Laplace equation in Grushin-type spaces whose tangent spaces consist of arbitrary triangular vector fields.

Keywords: p-Laplace equation, ∞-Laplace equation, viscosity solution, Grushin-type spaces, sub-Riemannian geometry.

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1. INTRODUCTION & MOTIVATION

In [2], the author proves existence-uniquness of ∞-harmonic functions in the viscosity sense for a large class of Grushin-type spaces. Specifically, for each point p in the class of Grushin-type spaces, the tangent space at $p = (x_1, x_2, \ldots, x_n)$ is defined by vector fields of the form

$$Q_i(p)\frac{\partial}{\partial x_i} = Q_i(x_1, \ldots, x_{i-1})\frac{\partial}{\partial x_i},$$

where $Q_1 \equiv 1$ and for each $2 \leq i \leq n$ the functions Q_i are polynomials determined only by the first $i-1$ coordinates of p. In [8], the authors obtain existence and uniqueness results for ∞-harmonic functions in the viscosity sense in spaces whose tangent space at each point p is defined by $\frac{\partial}{\partial x_i}$ for $1 \leq i \leq m < n$ and

$$\sigma(p)\frac{\partial}{\partial x_j} = \sigma(x_1, \ldots, x_m)\frac{\partial}{\partial x_j}$$

for $m+1 \leq j \leq n$, where σ is a C^2 function satisfying certain assumptions on its zeroes. In the current article, our objective is to expand upon and generalize both results to a broader class of Grushin-type spaces. In particular, we seek to show that the Dirichlet problem

$$\text{(DP)} \qquad \begin{cases} \Delta_\infty w = 0 & \text{in } \Omega \\ w = g & \text{on } \partial\Omega \end{cases}$$

will possess unique viscosity solutions when posed in bounded domains Ω in Grushin-type spaces whose tangent spaces are determined by

$$\rho_k(x_1, \ldots, x_{k-1})\frac{\partial}{\partial x_k},$$

where ρ_k is an arbitrary function subject to mild technical assumptions.

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The layout of the paper will be as follows. In Section 2, we will explore details of the Grushin-type spaces $\mathbb{G}$, pausing to mention previous examples of such spaces and their relation to the current situation, and then introduce notions of distance and calculus. In Section 3, we define notions of viscosity theory for elliptic equations and give results relating Euclidean elliptic jets to their Grushin counterparts (see Section 3 for definitions). We conclude with Section 4, in which we will prove that solutions to (DP) exist and are unique. The uniqueness of solutions requires us to produce useful estimates via a maximum principle; we then utilize these estimates to prove a comparison principle for sub- and supersolutions of (DP).

2. Grushin-type Spaces

To construct the Lie Algebras which are of interest to this paper, let $n \geq 2$ be given and, fixing any $p = (x_1, \ldots, x_n) \in \mathbb{R}^n$, consider the frame $\mathfrak{X} := \{X_1, X_2, \ldots, X_n\}$ consisting of the vector fields

$$X_1(p) := \frac{\partial}{\partial x_1} \tag{2.1}$$

(that is, we decree $\rho_1 \equiv 1$) and

$$X_j(p) := \rho_j(p)\frac{\partial}{\partial x_j} = \rho_j(x_1, x_2, \ldots, x_{j-1})\frac{\partial}{\partial x_j} \ (2 \leq j \leq n). \tag{2.2}$$

We will assume that for every $2 \leq j$:

(A) Each function ρ_j is Euclidean C^2 (denoted C^2_{eucl} for what follows).
(B) The set of zeroes for each ρ_j is given by $Z_j \times \mathbb{R}^{n-j+1}$, where Z_j is a discrete subset of $\mathbb{R}^{j-1}$.

The papers [2, 5] considered the stationary ∞-Laplace equation in these spaces under the additional assumption that each ρ_j is a polynomial; in [8], the ∞-Laplacian was studied in the case that $\rho_1, \ldots, \rho_m \equiv 1$ for some $m < n$ and that $\rho_j = \sigma$ for all $m \leq j$, where $\sigma \in C^2_{\text{eucl}}(\mathbb{R}^m)$. The fundamental solution to the p-Laplace equation was explored in [9] when the functions $\rho_j = \rho_k$ for all $2 \leq j, k$ are chosen to be monomials in x_1; a similar study was made in [3], with

$$\rho_j(p) = c\left(\sum_{i=1}^{m}(x_i - a_i)^2\right)^{\frac{k}{2}},$$

where $1 \leq m \leq j$ and for $c, k, a_i \in \mathbb{R}$ with $c \neq 0$. (Note we will require $k \geq 4$.)

The Lie Algebra $\mathfrak{g} := \operatorname{span}\mathfrak{X}$ may be endowed with a singular inner-product $\langle\cdot,\cdot\rangle$ that makes $\mathfrak{X}$ orthonormal. We then consider the space $\mathbb{G}$ which is the image of $\mathfrak{g}$ under the exponential map, and we denote points of $\mathbb{G}$ also by the n-tuples $p = (x_1, \ldots, x_n)$. One consequence of this definition is that these spaces are not groups: Indeed, $\dim\mathfrak{g}$ at p is n when every $\rho_j \neq 0$; otherwise, $\dim\mathfrak{g} < n$.

The natural metric to impose upon $\mathbb{G}$ is the *Carnot-Carathéodory* (or CC) metric

$$d_{CC}(p, q) := \inf_{\gamma\in\Gamma}\int_0^1 \|\gamma'(t)\|dt, \tag{2.3}$$

where Γ is the collection of all curves γ satisfying **(i)** $\gamma(0) = p, \gamma(1) = q$ and **(ii)** $\gamma' \in \mathfrak{g}$. Because some vector fields X_j and their derivatives may vanish at a point, Chow's Theorem (see, for example, [10]) may not apply. However, since X_1 is always nonzero, points of $\mathbb{G}$ can always be connected by concatenating curves and so $\Gamma \neq \emptyset$ and $d_{CC}(\cdot,\cdot)$ is an honest metric. We may therefore define balls in $\mathbb{G}$ by

$$B(p_0, r) := \{p \in \mathbb{G} : d_{CC}(p_0, p) < r\}.$$

Given a smooth function $w : \mathcal{O} \to \mathbb{R}$ where $\mathcal{O} \subseteq \mathbb{G}$ is open, the gradient of w in $\mathbb{G}$ is defined by

$$\nabla_{\mathbb{G}} w := (X_1 w, \dots, X_n w)$$

and the second derivative matrix $\left(D^2 u\right)^\star$ is the symmetric $n \times n$ matrix whose entries are given by

$$[(D^2 w)^\star]_{k\ell} := \frac{1}{2}\left(X_\ell X_k w + X_k X_\ell w\right).$$

We also have notions of regularity:

Definition 2.1. *A function $u : \mathcal{O} \to \mathbb{R}$ is said to be $C^1_{\mathbb{G}}(\mathcal{O})$ if $X_k u$ is continuous for each $1 \leq k \leq n$. The function u is $C^2_{\mathbb{G}}(\mathcal{O})$ if $X_\ell X_k u$ is continuous for each $1 \leq k, \ell \leq n$.*

The function spaces $L^{\mathrm{r}}, L^{\mathrm{r}}_{\mathrm{loc}}, W^{1,\mathrm{r}}, W^{1,\mathrm{r}}_0$, and $W^{1,\mathrm{r}}_{\mathrm{loc}}$ for $1 \leq \mathrm{r} \leq \infty$ over $\mathbb{G}$ mimic their Euclidean counterparts.

3. Viscosity Theory for Elliptic Equations

Throughout this article, we will have need of multiple operators; we will define them in the current section for the sake of convenience. First we have the p-Laplacian which, for smooth functions w, is given by

$$\begin{aligned}\Delta_{\mathrm{p}} w &:= -\operatorname{div}\left(\|\nabla_{\mathbb{G}} w\|^{\mathrm{p}-2}\nabla_{\mathbb{G}} w\right)\\ &= -\left(\|\nabla_{\mathbb{G}} w\|^{\mathrm{p}-2}\operatorname{tr}(D^2 w)^\star + (\mathrm{p}-2)\|\nabla_{\mathbb{G}} w\|^{\mathrm{p}-4}\left\langle (D^2 w)^\star \nabla_{\mathbb{G}} w, \nabla_{\mathbb{G}} w\right\rangle\right)\end{aligned}$$

for $1 < \mathrm{p} < \infty$. The formal limit of the p-Laplacian as $\mathrm{p} \to \infty$ is the ∞-Laplacian

$$\Delta_\infty w := -\left\langle (D^2 w)^\star \nabla_{\mathbb{G}} w, \nabla_{\mathbb{G}} w\right\rangle.$$

We also define Jensen's Auxiliary functions (see [13]) for $\mathbb{G}$: Given some $\varepsilon \in \mathbb{R}$, these are the operators

$$\mathcal{F}^\varepsilon\left(p, \nabla_{\mathbb{G}} w, \left(D^2 w\right)^\star\right) := \min\left\{\|\nabla_{\mathbb{G}} w(p)\|^2 - \varepsilon^2, \Delta_\infty w(p)\right\}$$

and

$$\mathcal{G}^\varepsilon\left(p, \nabla_{\mathbb{G}} w, \left(D^2 w\right)^\star\right) := \max\left\{\varepsilon^2 - \|\nabla_{\mathbb{G}} w(p)\|^2, \Delta_\infty w(p)\right\}.$$

These last two operators may also be thought of as functions mapping $\mathbb{G} \times \mathfrak{g} \times \mathcal{S}^n$ into $\mathbb{R}$, and so we represent any of the above generically by the function $\mathcal{H} : \mathbb{G} \times \mathfrak{g} \times \mathcal{S}^n \to \mathbb{R}$.

The elliptic equations which we wish to solve are of the form

$$\mathcal{H}w(p) = \mathcal{H}\left(p, \nabla_{\mathbb{G}} w, \left(D^2 w\right)^\star\right) = 0, \tag{3.4}$$

where it should be noted that $\mathcal{H}$ in each of the four cases above exhibits a property which [12] calls *proper*: Specifically, for each pair of matrices $X \leq Y$ in $\mathcal{S}^n$ and all $p \in \mathbb{G}, \eta \in \mathfrak{g}$, we will have

$$\mathcal{H}(p, \eta, Y) \leq \mathcal{H}(p, \eta, X).$$

Fixing any $p_0 \in \mathcal{O} \subseteq \mathbb{G}$ for an open set $\mathcal{O}$, and a function $u : \mathcal{O} \to \mathbb{R}$, we may now introduce two collections of functions necessary to viscosity theory: The "touching above functions" $\mathcal{TA}(u, p_0)$ consisting of all $\phi \in C^2_{\mathbb{G}}(\mathcal{O})$ satisfying

$$0 = \phi(p_0) - u(p_0) \leq \phi(p) - u(p) \text{ near } p_0;$$

and the "touching below functions" $\mathcal{TB}(u, p_0)$ consisting of all $\psi \in C^2_{\mathbb{G}}(\mathcal{O})$ such that

$$0 = u(p_0) - \psi(p_0) \leq u(p) - \psi(p) \text{ near } p_0.$$

As in [8], we then may define viscosity (sub-/super-)solutions.

Definition 3.2. *Let $\Omega \Subset \mathbb{G}$ be given. A function $u \in \mathrm{USC}(\Omega)$ is a viscosity subsolution of Equation (3.4) if for each $p_0 \in \Omega$ and every $\phi \in \mathcal{TA}(u, p_0)$ the inequality*

$$\mathcal{H}\phi(p_0) \le 0$$

is satisfied. Functions $v \in \mathrm{LSC}(\Omega)$ are said to be viscosity supersolutions of (3.4) if for each $p_0 \in \Omega$ and every $\psi \in \mathcal{TB}(v, p_0)$, we have

$$\mathcal{H}\psi(p_0) \ge 0.$$

Functions $w \in C(\Omega)$ are called viscosity solutions of (3.4) if they are both a viscosity sub- and supersolution.

Remark 3.1. *In the case that $\mathcal{H} = \Delta_\infty$, we shall use the term ∞-(sub-/super-)harmonic to refer to the viscosity (sub-/super-)solutions of (3.4).*

Remark 3.2. *In the case that $\mathcal{H} = \Delta_{\mathrm{p}}$, care needs to be taken in the $\mathrm{p} < 2$ case due to the singularity which occurs when $\| \nabla_{\mathbb{G}} w\| = 0$; however, since our aim is to use viscosity solutions of the p-Laplacian to produce an ∞-harmonic function, we only concern ourselves with the case $\mathrm{p} \ge 2$.*

The notion of viscosity (sub-/super-)solutions may be equivalently restated. For a given function $u : \mathcal{O} \to \mathbb{R}$, we may define the upper jet

$$J^{2,+} u(p_0) := \Big\{ \big(\nabla_{\mathbb{G}} \phi(p_0), (D^2\phi)^\star(p_0) \big) : \phi \in \mathcal{TA}\left(u(p_0)\right) \Big\}$$

and lower jet

$$J^{2,-} u(p_0) := \Big\{ \big(\nabla_{\mathbb{G}} \psi(p_0), (D^2\psi)^\star(p_0) \big) : \psi \in \mathcal{TB}\left(u(p_0)\right) \Big\} = -J^{2,+}(-u)(p_0).$$

By $\overline{J}^{2,+} u(p_0)$ we will denote the collection of all $(\eta, X) \in \mathbb{R}^n \times \mathcal{S}^n$ such that there exists $(p_k) \subset \mathbb{G}$ and $(\eta_k, X_k) \in J^{2,+} u(p_k)$ which satisfy $(p_k, u(p_k), \eta_k, X_k) \to (p_0, u(p_0), \eta, X)$ as $k \to \infty$; a similar definition is made for $\overline{J}^{2,-} u(p_0)$.

Definition 3.3. *Let $\Omega \Subset \mathbb{G}$ be given. A function $u \in \mathrm{USC}(\Omega)$ is a viscosity subsolution of Equation (3.4) if for every $p_0 \in \Omega$,*

$$\mathcal{H}\big(p_0, \eta, X\big) \le 0$$

for all $(\eta, X) \in \overline{J}^{2,+} u(p_0)$. A function $v \in \mathrm{LSC}(\Omega)$ is a viscosity supersolution of (3.4) if $-v$ is a viscosity subsolution of (3.4), and a viscosity solution of (3.4) is a function $w \in C(\Omega)$ which is both a viscosity sub- and supersolution of (3.4).

The advantage of this second definition is that we can easily state the following result which, for a given function $u : \mathcal{O} \to \mathbb{R}$ and $p_0 \in \mathcal{O}$, relates the Euclidean upper jet $J^{2,+}_{\text{eucl}} u(p_0)$ to $J^{2,+} u(p_0)$. The proof in [2] relies upon producing Grushin second-order Taylor Polynomials for $C^2_{\mathbb{G}}$ functions and then utilizing the twisting terms and factors in these polynomials to deduce the twisting necessary for jet entries (cf. [6, Corollary 3.2]). The proof is not significantly altered by replacing the vector fields considered in [2] with our frame $\mathfrak{X}$.

Lemma 3.1 (Elliptic $\mathbb{G}$ Twisting Lemma)**.** *Let $\mathcal{O} \subseteq \mathbb{G}$ be open, let $u : \mathcal{O} \to \mathbb{R}$, and let $p_0 \in \mathcal{O}$. Suppose that $(\eta, X) \in J^{2,+}_{\text{eucl}} u(p_0)$: Then*

$$\Big(\boldsymbol{A}(p_0) \cdot \eta, \boldsymbol{A}(p_0) \cdot X \cdot \boldsymbol{A}^{\mathrm{T}}(p_0) + \boldsymbol{M}(\eta, p_0) \Big) \in J^{2,+} u(p_0), \tag{3.5}$$

where

$$(\boldsymbol{A}(p_0))_{k\ell} = \begin{cases} 1, & k = 1 = \ell \\ \rho_k(p), & 2 \leq k = \ell \leq n \\ 0, & \textit{otherwise} \end{cases} \tag{3.6}$$

and

$$(\boldsymbol{M}(\eta, p_0))_{k\ell} = \begin{cases} \frac{1}{2} \cdot \frac{\partial \rho_k}{\partial x_\ell}(p)\rho_\ell(p)\eta_k, & \ell < k \\ \frac{1}{2} \cdot \frac{\partial \rho_\ell}{\partial x_k}(p)\rho_k(p)\eta_\ell, & k < \ell \\ 0, & \textit{otherwise}. \end{cases} \tag{3.7}$$

Given the properties of the collections $J^{2,+}\, u(p_0)$ and $J^{2,-}\, u(p_0)$, a similar relationship holds for $J^{2,-}_{\text{eucl}} u(p_0)$ and $J^{2,-}\, u(p_0)$.

4. Existence & Uniqueness of ∞-Harmonic Functions

Consider the Dirichlet problems

$$\begin{cases} \mathcal{F}^\varepsilon w = 0 \text{ in } \Omega \\ \quad w = g \text{ on } \partial\Omega \end{cases}, \tag{4.8}$$

$$\begin{cases} \mathcal{G}^\varepsilon w = 0 \text{ in } \Omega \\ \quad w = g \text{ on } \partial\Omega \end{cases}, \tag{4.9}$$

and

$$\begin{cases} \Delta_\infty\, w = 0 \text{ in } \Omega \\ \quad w = g \text{ on } \partial\Omega \end{cases}, \tag{DP}$$

where $\Omega \Subset \mathbb{G}$ is a domain and we always assume that $g \in C(\partial\Omega)$. We will use the Problems (4.8) and (4.9) to show that Problem (DP) possess a unique solution. Our first task is to extend our notion of viscosity solutions to Dirichlet problems.

Definition 4.4. *Let $\mathcal{H} : \mathbb{G} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ represent one of the operators, $\Delta_{\text{p}}, \Delta_\infty, \mathcal{F}^\varepsilon$, or $\mathcal{G}^\varepsilon$ ($\varepsilon \in \mathbb{R}$); suppose that Ω, g are as above and that we are given the Dirichlet problem*

$$\begin{cases} \mathcal{H}w = 0 \textit{ in } \Omega \\ \quad w = g \textit{ on } \partial\Omega \end{cases}.$$

A viscosity subsolution u to such a problem is a viscosity subsolution to the equation $\mathcal{H}u = 0$ which also satisfies $u \leq g$ on $\partial\Omega$. We define a viscosity supersolution v to the Dirichlet problem similarly. The function w is a viscosity solution to the Dirichlet problem if it is both a viscosity sub- and supersolution of the problem.

From here, we proceed as follows: We prove that each of the Problems (4.8), (4.9), and (DP) possesses a viscosity solution; with existence proven, we show that Problems (4.8) and (4.9) support comparison principles which verify that solutions to these two problems are unique; and finally, we exploit relationships between the Problems (4.8), (4.9), and (DP) to show that the solution to (DP) is unique.

Existence of solutions to the three Dirichlet problems above is standard and, following the approach of [7, Theorem 4.1], we condense the results supporting this finding into one theorem. As in [7], the proof follows the layout of [1, Section 4].

Theorem 4.1 (Existence of ∞-Harmonic Functions)**.** *The following are true:*

(A) *Let $\varepsilon \in \mathbb{R}$ and $\mathrm{p} \geq 2$. If $u_{\mathrm{p}} \in C(\Omega) \cap W^{1,\mathrm{p}}_{\mathrm{loc}}(\Omega)$ is a weak (sub-/super)solution to the* p*-Laplace problem*

$$\begin{cases} \Delta_{\mathrm{p}} w = 0 & \text{in } \Omega \\ w = g & \text{on } \partial\Omega \end{cases}, \tag{4.10}$$

then u_{p} is a viscosity (sub-/super)solution to (4.10).

(B) *Let u_{p} be as before. Then, possibly passing to a subsequence of $(u_{\mathrm{p}})_{\mathrm{p}\geq 2}$, there exists some $u_\infty \in C(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ so that*

$$u_{\mathrm{p}} \to u_\infty \text{ uniformly in } \Omega$$

as $\mathrm{p} \to \infty$.

(C) *The function u_∞ from the previous item is a viscosity solution of one of (4.8), (4.9), or (DP), the choice of problem depending only upon ε:*
   (i) *If $\varepsilon > 0$, then u_∞ is a viscosity solution to Problem (4.8).*
   (ii) *If $\varepsilon < 0$, then u_∞ is a viscosity solution to Problem (4.9).*
   (iii) *If $\varepsilon = 0$, then u_∞ is a viscosity solution to Problem (DP).*

It only remains to prove comparison principles and to employ the relationships between viscosity solutions of (4.8), (4.9), and (DP). To simplify our presentation, we divide our work between the Subsections 4.1 and 4.2.

Remark 4.3. *Theorem 4.1 was recently proved for general sub-Riemannian spaces in more generality in* [11].

4.1. **Estimates for ∞-Subharmonic & ∞-Superharmonic Functions.** To start, we require a penalty function which is suited to match the geometry of our family of Grushin-type spaces:

$$\varphi_{\tau_1,\ldots,\tau_n}(p,q) = \varphi_{\vec{\tau}}(p,q) := \frac{1}{2}\sum_{k=1}^{n} \tau_k (x_k - y_k)^2.$$

Utilizing the Iterated Maximum Principle of [2] and its corollaries, it was shown in [8] that, if $u \in \mathrm{USC}(\Omega)$ and $v \in \mathrm{LSC}(\Omega)$ possess the property

$$\sup_{p\in\Omega}(u - v) = u(p_0) - v(p_0) > 0$$

at some $p_0 = (x_1^0, \ldots, x_n^0) \in \Omega$, then denoting $\vec{\tau} := (\tau_1, \ldots, \tau_n)$ where $\tau_k > 0$ there exist $(p_{\vec{\tau}}, q_{\vec{\tau}}) \subset \Omega \times \Omega$ so that

$$\begin{cases} \lim\limits_{\tau_n\to\infty} \cdots \lim\limits_{\tau_1\to\infty} \big(u(p_{\vec{\tau}}) - v(q_{\vec{\tau}}) - \varphi_{\vec{\tau}}(p,q)\big) = u(p_0) - v(p_0) \\ \lim\limits_{\tau_n\to\infty} \cdots \lim\limits_{\tau_1\to\infty} \varphi_{\vec{\tau}}(p,q) = \lim\limits_{\tau_1,\ldots,\tau_n\to\infty} \varphi_{\vec{\tau}}(p,q) = 0 \end{cases} \tag{4.11}$$

and, writing $p_{\vec{\tau}} = (x_1^{\vec{\tau}}, \ldots, x_n^{\vec{\tau}})$ and $q_{\vec{\tau}} = (y_1^{\vec{\tau}}, \ldots, y_n^{\vec{\tau}})$,

$$\begin{cases} p_{\tau_1,\ldots,\tau_k} := \lim\limits_{\tau_k\to\infty} \cdots \lim\limits_{\tau_1\to\infty} p_{\vec{\tau}} = (x_1^0, \ldots, x_k^0, x_{k+1}^{\vec{\tau}}, \ldots, x_n^{\vec{\tau}}) \\ q_{\tau_1,\ldots,\tau_k} := \lim\limits_{\tau_k\to\infty} \cdots \lim\limits_{\tau_1\to\infty} q_{\vec{\tau}} = (x_1^0, \ldots, x_k^0, y_{k+1}^{\vec{\tau}}, \ldots, y_n^{\vec{\tau}}) \end{cases} \tag{4.12}$$

for each $1 \leq k \leq n$. Additionally, these limits will hold true even if the order of the iterated limits is changed (although the sequence $(p_{\vec{\tau}}, q_{\vec{\tau}})$ may change). The Iterated Maximum Principle and the results above are not dependent upon the frame $\mathfrak{X}$; hence we may utilize Equations (4.11) and (4.12) freely in the lemma below.

Lemma 4.2 (cf. [8, Lemma 4.4]). *Let $u, v, \varphi_{\vec{\tau}}$ and $(p_{\vec{\tau}}, q_{\vec{\tau}})$ be as above. Assume that at least one of the functions u, v is locally $\mathbb{G}$-Lipschitz. Then:*

(A) *There exist $(\eta^+_{\vec{\tau}}, \mathcal{X}_{\vec{\tau}}) \in \overline{J}^{2,+} u(p_{\vec{\tau}})$ and $(\eta^-_{\vec{\tau}}, \mathcal{Y}_{\vec{\tau}}) \in \overline{J}^{2,-} v(q_{\vec{\tau}})$.*

(B) *Define $(p\diamond q)_k$ to be the point whose k-th coordinate coincides with q and whose other coordinates coincide with p, in other words,*

$$(p \diamond q)_k = (x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n).$$

Then for each index $1 \leq k \leq n$,

$$\tau_k \left| x^{\vec{\tau}}_k - y^{\vec{\tau}}_k \right| \lesssim d_{CC}(p_{\vec{\tau}}, (p_{\vec{\tau}} \diamond q_{\vec{\tau}})_k) \text{ as } \tau_k \to \infty. \tag{4.13}$$

In particular, $\tau_k \left| x^{\vec{\tau}}_k - y^{\vec{\tau}}_k \right| = O(1)$ as $\tau_k \to \infty$.

(C) *The vector estimate*

$$\left| \|\eta^+_{\vec{\tau}}\|^2 - \|\eta^-_{\vec{\tau}}\|^2 \right| = o(1) \text{ as } \tau_k \to \infty \text{ for all } k \leq n \tag{4.14}$$

holds.

(D) *The matrix estimate*

$$\left\langle \mathcal{X}_{\vec{\tau}} \cdot \eta^+_{\vec{\tau}}, \eta^+_{\vec{\tau}} \right\rangle - \left\langle \mathcal{Y}_{\vec{\tau}} \cdot \eta^-_{\vec{\tau}}, \eta^-_{\vec{\tau}} \right\rangle = o(1) \text{ as } \tau_k \to \infty \text{ for all } k \leq n \tag{4.15}$$

holds.

Proof. The proof of the first two items proceeds precisely as in [8]. We will instead focus on the crucial differences in our proof of Items (C) and (D) arising from the frame $\mathfrak{X}$.

Item (C).

Owing to [12, Theorem 3.2] and Lemma 3.1 (The Elliptic $\mathbb{G}$ Twisting Lemma), we have that

$$\begin{cases} \eta^+_{\vec{\tau}} = \boldsymbol{A}(p_{\vec{\tau}}) \cdot D_{\text{eucl}(p)} \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}) \\ \eta^-_{\vec{\tau}} = \boldsymbol{A}(q_{\vec{\tau}}) \cdot -D_{\text{eucl}(q)} \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}) \end{cases}.$$

Direct calculation shows

$$\frac{\partial}{\partial x_k} \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}) = \tau_k (x^{\vec{\tau}}_k - y^{\vec{\tau}}_k) = -\frac{\partial}{\partial y_k} \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}),$$

so we conclude that

$$[\eta^+_{\vec{\tau}}]_k = \begin{cases} \tau_k (x^{\vec{\tau}}_k - y^{\vec{\tau}}_k), & k = 1 \\ \tau_k (x^{\vec{\tau}}_k - y^{\vec{\tau}}_k) \rho_k(p_{\vec{\tau}}), & 2 \leq k \end{cases}$$

and

$$[\eta^-_{\vec{\tau}}]_k = \begin{cases} \tau_k (x^{\vec{\tau}}_k - y^{\vec{\tau}}_k), & k = 1 \\ \tau_k (x^{\vec{\tau}}_k - y^{\vec{\tau}}_k) \rho_k(q_{\vec{\tau}}), & 2 \leq k \end{cases}.$$

This leads us to:

$$\left| \|\eta^+_{\vec{\tau}}\|^2 - \|\eta^-_{\vec{\tau}}\|^2 \right| = \sum_{k=2}^{n} \tau_k^2 (x^{\vec{\tau}}_k - y^{\vec{\tau}}_k)^2 \left| \rho_k^2(p_{\vec{\tau}}) - \rho_k^2(q_{\vec{\tau}}) \right|. \tag{4.16}$$

Fixing any $2 \le k \le n$, observe that by Equation (4.12) we must have

$$\begin{aligned}\lim_{\tau_{k-1}\to\infty}\cdots\lim_{\tau_1\to\infty}\tau_k^2(x_k^{\vec{\tau}}-y_k^{\vec{\tau}})^2\left|\rho_k^2(p_{\vec{\tau}})-\rho_k^2(q_{\vec{\tau}})\right| =&\left|\rho_k^2(x_1^0,\ldots,x_{k-1}^0)-\rho_k^2(x_1^0,\ldots,x_{k-1}^0)\right|\\ &\times\tau_k^2(x_k^{\vec{\tau}}-y_k^{\vec{\tau}})^2\\ =&0.\end{aligned}$$

Applying the above to Inequality (4.16) and utilizing the terminology of Equation (4.12),

$$\begin{aligned}\lim_{\tau_1\to\infty}\left|\|\eta_{\vec{\tau}}^+\|^2-\|\eta_{\vec{\tau}}^-\|^2\right| =&\sum_{k=3}^{n}\tau_k^2(x_k^{\vec{\tau}}-y_k^{\vec{\tau}})^2\left|\rho_k^2(p_{\tau_1})-\rho_k^2(q_{\tau_1})\right|\\ \lim_{\tau_2\to\infty}\lim_{\tau_1\to\infty}\left|\|\eta_{\vec{\tau}}^+\|^2-\|\eta_{\vec{\tau}}^-\|^2\right| =&\sum_{k=4}^{n}\tau_k^2(x_k^{\vec{\tau}}-y_k^{\vec{\tau}})^2\left|\rho_k^2(p_{\tau_1,\tau_2})-\rho_k^2(q_{\tau_1,\tau_2})\right|\\ &\vdots\\ \lim_{\tau_{n-2}\to\infty}\cdots\lim_{\tau_1\to\infty}\left|\|\eta_{\vec{\tau}}^+\|^2-\|\eta_{\vec{\tau}}^-\|^2\right| =&\tau_n^2(x_n^{\vec{\tau}}-y_n^{\vec{\tau}})^2\left|\rho_n^2(p_{\tau_1,\ldots,\tau_{n-2}})-\rho_n^2(q_{\tau_1,\ldots,\tau_{n-2}})\right|\\ \lim_{\tau_{n-1}\to\infty}\cdots\lim_{\tau_1\to\infty}\left|\|\eta_{\vec{\tau}}^+\|^2-\|\eta_{\vec{\tau}}^-\|^2\right| =&\tau_n^2(x_n^{\vec{\tau}}-y_n^{\vec{\tau}})^2\left|\rho_n^2(x_1^0,\ldots,x_{n-1}^0)-\rho_n^2(x_1^0,\ldots,x_{n-1}^0)\right|\\ =&0.\end{aligned}$$

From this, the limit

$$\lim_{\tau_n\to\infty}\cdots\lim_{\tau_1\to\infty}\left|\|\eta_{\vec{\tau}}^+\|^2-\|\eta_{\vec{\tau}}^-\|^2\right|=0$$

is clear.

Item (D).

We begin by decomposing the left-hand side of the Estimate (4.15) into two terms:

$$\left\langle \mathcal{X}_{\vec{\tau}}\cdot\eta_{\vec{\tau}}^+,\eta_{\vec{\tau}}^+\right\rangle-\left\langle \mathcal{Y}_{\vec{\tau}}\cdot\eta_{\vec{\tau}}^-,\eta_{\vec{\tau}}^-\right\rangle=I_1+I_2$$

where, invoking [12, Theorem 3.2] and the Elliptic $\mathbb{G}$ Twisting Lemma once again, we have defined

$$I_1:=\left\langle\left(\boldsymbol{A}(p_{\vec{\tau}})\cdot X_{\vec{\tau}}\cdot\boldsymbol{A}^{\mathrm{T}}(p_{\vec{\tau}})\right)\cdot\eta_{\vec{\tau}}^+,\eta_{\vec{\tau}}^+\right\rangle-\left\langle\left(\boldsymbol{A}(q_{\vec{\tau}})\cdot Y_{\vec{\tau}}\cdot\boldsymbol{A}^{\mathrm{T}}(q_{\vec{\tau}})\right)\cdot\eta_{\vec{\tau}}^-,\eta_{\vec{\tau}}^-\right\rangle$$

(recall that $X_{\vec{\tau}}, Y_{\vec{\tau}}$ are a result of [12, Theorem 3.2]), and

$$I_2:=\left\langle\boldsymbol{M}(D_{\mathrm{eucl}(p)}\varphi_{\vec{\tau}}(p_{\vec{\tau}},q_{\vec{\tau}}),p_{\vec{\tau}})\cdot\eta_{\vec{\tau}}^+,\eta_{\vec{\tau}}^+\right\rangle-\left\langle\boldsymbol{M}(D_{\mathrm{eucl}(q)}\varphi_{\vec{\tau}}(p_{\vec{\tau}},q_{\vec{\tau}}),q_{\vec{\tau}})\cdot\eta_{\vec{\tau}}^-,\eta_{\vec{\tau}}^-\right\rangle.$$

Writing $\tilde{\epsilon}:=\boldsymbol{A}(p_{\vec{\tau}})\cdot\epsilon$ and $\hat{\kappa}:=\boldsymbol{A}(q_{\vec{\tau}})\cdot\kappa$ to represent twisting according to Lemma 3.1,

$$\begin{aligned}I_1=&\left\langle X_{\vec{\tau}}\cdot\widetilde{\eta_{\vec{\tau}}^+},\widetilde{\eta_{\vec{\tau}}^+}\right\rangle-\left\langle Y_{\vec{\tau}}\cdot\widehat{\eta_{\vec{\tau}}^-},\widehat{\eta_{\vec{\tau}}^-}\right\rangle\\ \le&\left\langle\mathcal{C}\cdot\zeta,\zeta\right\rangle.\end{aligned}\tag{4.17}$$

Here, $\zeta:=\widetilde{\eta_{\vec{\tau}}^+}\oplus\widehat{\eta_{\vec{\tau}}^-}\in\mathbb{R}^{2n}$ and $\mathcal{C}$ is a $2n\times 2n$ matrix resulting from [12, Theorem 3.2] which can be represented in block form as

$$\begin{pmatrix}B & -B\\ -B & B\end{pmatrix},$$

where we define

$$[B]_{k\ell}:=\begin{cases}\tau_k+2\delta\tau_k^2, & k=\ell\\ 0, & k\neq\ell\end{cases}$$

and $\delta > 0$ is an arbitrary parameter resulting from the theorem of [12]. The definition of $\mathcal{C}$ and B and Inequality (4.17) together yield

$$\begin{aligned} I_1 &\leq \left\langle B \cdot (\widetilde{\eta^+_{\vec{\tau}}} - \widehat{\eta^-_{\vec{\tau}}}), (\widetilde{\eta^+_{\vec{\tau}}} - \widehat{\eta^-_{\vec{\tau}}}) \right\rangle \\ &= \sum_{k=2}^{n} (\tau_k + 2\delta\tau_k^2) \cdot \left(\rho_k^2(p_{\vec{\tau}}) - \rho_k^2(q_{\vec{\tau}})\right)^2 \cdot \tau_k^2 (x_k^{\vec{\tau}} - y_k^{\vec{\tau}})^2. \end{aligned} \tag{4.18}$$

Since the terms on the right-hand side of (4.18) contain no factors τ_ℓ for $\ell \leq k-1$,

$$\lim_{\tau_{k-1}\to\infty} \cdots \lim_{\tau_1\to\infty} (\tau_k + 2\delta\tau_k^2) \cdot \left(\rho_k^2(p_{\vec{\tau}}) - \rho_k^2(q_{\vec{\tau}})\right)^2 \cdot \tau_k^2 (x_k^{\vec{\tau}} - y_k^{\vec{\tau}})^2 = 0. \tag{4.19}$$

Equation (4.19), work similar to what was employed in Item (C), and Inequality (4.18) therefore show that

$$\lim_{\tau_n\to\infty} \cdots \lim_{\tau_1\to\infty} I_1 = 0. \tag{4.20}$$

It remains to show that I_2 tends to 0 as $\tau_k \to \infty$ for all $1 \leq k \leq n$. Recalling the definition of the matrix $\boldsymbol{M}(\cdot,\cdot)$ from Equation (3.7), we may calculate directly the first entry in both of the inner-products defining I_2. Writing $\boldsymbol{M}_p$ and $\boldsymbol{M}_q$ to refer to the matrices resulting from $\boldsymbol{M}(\cdot,\cdot)$ evaluated at $\left(D_{\mathrm{eucl}(p)}\varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), p_{\vec{\tau}}\right), \left(D_{\mathrm{eucl}(q)}\varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), q_{\vec{\tau}}\right)$ respectively:

$$[\boldsymbol{M}_p \cdot \eta^+_{\vec{\tau}}]_h = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{\ell=2}^{n} \left(\frac{\partial \rho_\ell}{\partial x_1}\rho_\ell\right)(p_{\vec{\tau}}) \cdot \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 & h = 1 \\ \dfrac{1}{2}\displaystyle\sum_{\ell=1}^{h-1} \left(\frac{\partial \rho_h}{\partial x_\ell}\rho_\ell^2\right)(p_{\vec{\tau}}) \cdot \tau_\ell (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}}) \cdot \tau_h (x_h^{\vec{\tau}} - y_h^{\vec{\tau}}) & \\ +\dfrac{1}{2}\displaystyle\sum_{\ell=h+1}^{n} \left(\frac{\partial \rho_\ell}{\partial x_h}\rho_\ell\rho_h\right)(p_{\vec{\tau}}) \cdot \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 & h \geq 2 \end{cases} \tag{4.21}$$

and

$$[\boldsymbol{M}_q \cdot \eta^-_{\vec{\tau}}]_h = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{\ell=2} \left(\frac{\partial \rho_\ell}{\partial x_1}\rho_\ell\right)(q_{\vec{\tau}}) \cdot \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 & h = 1 \\ \dfrac{1}{2}\displaystyle\sum_{\ell=1}^{h-1} \left(\frac{\partial \rho_h}{\partial x_\ell}\rho_\ell^2\right)(q_{\vec{\tau}}) \cdot \tau_\ell (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}}) \cdot \tau_h (x_h^{\vec{\tau}} - y_h^{\vec{\tau}}) & \\ +\dfrac{1}{2}\displaystyle\sum_{\ell=h+1}^{n} \left(\frac{\partial \rho_\ell}{\partial x_h}\rho_\ell\rho_h\right)(q_{\vec{\tau}}) \cdot \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 & h \geq 2. \end{cases} \tag{4.22}$$

Owing to Equations (4.21) and (4.22) and the observation that $\boldsymbol{M}(\cdot,\cdot)$ is symmetric, we may calculate I_2 as follows:

$$\begin{aligned}
I_2 =& \left\langle \boldsymbol{M}_p \cdot \eta^+_{\vec{\tau}}, \eta^+_{\vec{\tau}} \right\rangle - \left\langle \boldsymbol{M}_q \cdot \eta^-_{\vec{\tau}}, \eta^-_{\vec{\tau}} \right\rangle \\
=& \frac{1}{2}\sum_{\ell=2}^{n} \left(\frac{\partial \rho_\ell}{\partial x_1}\rho_\ell \right)(p_{\vec{\tau}}) \cdot \tau_\ell^2 (x^{\vec{\tau}}_\ell - y^{\vec{\tau}}_\ell)^2 \cdot \tau_1 (x^{\vec{\tau}}_1 - y^{\vec{\tau}}_1) \\
&+ \frac{1}{2}\sum_{h=2}^{n}\sum_{\ell=1}^{h-1} \left(\frac{\partial \rho_h}{\partial x_\ell}\rho_h \rho_\ell^2 \right)(p_{\vec{\tau}}) \cdot \tau_\ell (x^{\vec{\tau}}_\ell - y^{\vec{\tau}}_\ell) \cdot \tau_h^2 (x^{\vec{\tau}}_h - y^{\vec{\tau}}_h)^2 \\
&+ \frac{1}{2}\sum_{h=2}^{n-1}\sum_{\ell=h+1}^{n} \left(\frac{\partial \rho_\ell}{\partial x_h}\rho_\ell \rho_h^2 \right)(p_{\vec{\tau}}) \cdot \tau_h (x^{\vec{\tau}}_h - y^{\vec{\tau}}_h) \cdot \tau_\ell^2 (x^{\vec{\tau}}_\ell - y^{\vec{\tau}}_\ell)^2 \\
&- \frac{1}{2}\sum_{\ell=2}^{n} \left(\frac{\partial \rho_\ell}{\partial x_1}\rho_\ell \right)(q_{\vec{\tau}}) \cdot \tau_\ell^2 (x^{\vec{\tau}}_\ell - y^{\vec{\tau}}_\ell)^2 \cdot \tau_1 (x^{\vec{\tau}}_1 - y^{\vec{\tau}}_1) \\
&- \frac{1}{2}\sum_{h=2}^{n}\sum_{\ell=1}^{h-1} \left(\frac{\partial \rho_h}{\partial x_\ell}\rho_h \rho_\ell^2 \right)(q_{\vec{\tau}}) \cdot \tau_\ell (x^{\vec{\tau}}_\ell - y^{\vec{\tau}}_\ell) \cdot \tau_h^2 (x^{\vec{\tau}}_h - y^{\vec{\tau}}_h)^2 \\
&- \frac{1}{2}\sum_{h=2}^{n-1}\sum_{\ell=h+1}^{n} \left(\frac{\partial \rho_\ell}{\partial x_h}\rho_\ell \rho_h^2 \right)(q_{\vec{\tau}}) \cdot \tau_h (x^{\vec{\tau}}_h - y^{\vec{\tau}}_h) \cdot \tau_\ell^2 (x^{\vec{\tau}}_\ell - y^{\vec{\tau}}_\ell)^2.
\end{aligned}$$

The sums above may be combined as follows.

$$\begin{aligned}
2I_2 =& \sum_{\ell=2}^{n} \tau_\ell^2 (x^{\vec{\tau}}_\ell - y^{\vec{\tau}}_\ell)^2 \cdot \tau_1 (x^{\vec{\tau}}_1 - y^{\vec{\tau}}_1) \cdot \left(\left(\frac{\partial \rho_\ell}{\partial x_1}\rho_\ell \right)(p_{\vec{\tau}}) - \left(\frac{\partial \rho_\ell}{\partial x_1}\rho_\ell \right)(q_{\vec{\tau}}) \right) \\
&+ \sum_{h=2}^{n}\sum_{\ell=1}^{h-1} \tau_\ell (x^{\vec{\tau}}_\ell - y^{\vec{\tau}}_\ell) \cdot \tau_h^2 (x^{\vec{\tau}}_h - y^{\vec{\tau}}_h)^2 \cdot \left(\left(\frac{\partial \rho_h}{\partial x_\ell}\rho_h \rho_\ell^2 \right)(p_{\vec{\tau}}) - \left(\frac{\partial \rho_h}{\partial x_\ell}\rho_h \rho_\ell^2 \right)(q_{\vec{\tau}}) \right) \\
&+ \sum_{h=2}^{n-1}\sum_{\ell=h+1}^{n} \tau_h (x^{\vec{\tau}}_h - y^{\vec{\tau}}_h) \cdot \tau_\ell^2 (x^{\vec{\tau}}_\ell - y^{\vec{\tau}}_\ell)^2 \cdot \left(\left(\frac{\partial \rho_\ell}{\partial x_h}\rho_\ell \rho_h^2 \right)(p_{\vec{\tau}}) - \left(\frac{\partial \rho_\ell}{\partial x_h}\rho_\ell \rho_h^2 \right)(q_{\vec{\tau}}) \right).
\end{aligned} \tag{4.23}$$

Denote the terminal factor in each of the three sums of (4.23) evaluated at $(p,q) \in \Omega \times \Omega$ by

$$\begin{cases}
T_\ell(p,q) := \left(\frac{\partial \rho_\ell}{\partial x_1}\rho_\ell \right)(p) - \left(\frac{\partial \rho_\ell}{\partial x_1}\rho_\ell \right)(q) \\
S^1_{h\ell}(p,q) := \left(\frac{\partial \rho_\ell}{\partial x_\ell}\rho_h \rho_\ell^2 \right)(p) - \left(\frac{\partial \rho_\ell}{\partial x_\ell}\rho_h \rho_\ell^2 \right)(q) \\
S^2_{h\ell}(p,q) := \left(\frac{\partial \rho_\ell}{\partial x_h}\rho_\ell \rho_h^2 \right)(p) - \left(\frac{\partial \rho_\ell}{\partial x_h}\rho_\ell \rho_h^2 \right)(q)
\end{cases}$$

respectively, in order of their appearance in (4.23). Recalling $\tau_k(x^{\vec{\tau}}_k - y^{\vec{\tau}}_k) = O(1)$ as $\tau_k \to \infty$ for every $k \leq n$, we invoke Equation (4.12) to conclude:

- For $1 \leq k \leq \ell - 1$,

$$\lim_{\tau_k \to \infty} \cdots \lim_{\tau_1 \to \infty} \tau_\ell^2 (x^{\vec{\tau}}_\ell - y^{\vec{\tau}}_\ell)^2 \cdot \tau_1 (x^{\vec{\tau}}_1 - y^{\vec{\tau}}_1) \cdot T_\ell(p_{\vec{\tau}}, q_{\vec{\tau}}) \sim \tau_\ell^2 (x^{\vec{\tau}}_\ell - y^{\vec{\tau}}_\ell)^2 \cdot T_\ell(p_{\tau_1,\ldots,\tau_k}, q_{\tau_1,\ldots,\tau_k}). \tag{4.24}$$

- For $\ell \leq k \leq h-1$,

$$\lim_{\tau_k\to\infty}\cdots\lim_{\tau_1\to\infty}\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\cdot\tau_\ell(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})\cdot S^1_{h\ell}(p_{\vec{\tau}},q_{\vec{\tau}})\sim\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\cdot S^1_{h\ell}(p_{\tau_1,\ldots,\tau_k},q_{\tau_1,\ldots,\tau_k}). \tag{4.25}$$

- For $h \leq k \leq \ell-1$,

$$\lim_{\tau_k\to\infty}\cdots\lim_{\tau_1\to\infty}\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot\tau_h(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})\cdot S^2_{h\ell}(p_{\vec{\tau}},q_{\vec{\tau}})\sim\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot S^2_{h\ell}(p_{\tau_1,\ldots,\tau_k},q_{\tau_1,\ldots,\tau_k}). \tag{4.26}$$

Since ρ_k depends only upon $x_1,\ldots,x_{k-1}$,

$$\begin{cases} T_\ell(p_{\tau_1,\ldots,\tau_k},q_{\tau_1,\ldots,\tau_k}) & =0 \quad \text{when } k=\ell-1 \\ S^1_{h\ell}(p_{\tau_1,\ldots,\tau_k},q_{\tau_1,\ldots,\tau_k}) & =0 \quad \text{when } k=h-1 \\ S^2_{h\ell}(p_{\tau_1,\ldots,\tau_k},q_{\tau_1,\ldots,\tau_k}) & =0 \quad \text{when } k=\ell-1 \end{cases}.$$

Iterated limits of I_2 are now calculated from Equations (4.24), (4.25), and (4.26):

$$\begin{aligned}\lim_{\tau_1\to\infty}2I_2 \sim\; & \sum_{\ell=3}^{n}\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot T_\ell(p_{\tau_1},q_{\tau_1}) \\ & +\sum_{h=3}^{n}\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\cdot S^1_{h1}(p_{\tau_1},q_{\tau_1}) \\ & +\sum_{h=3}^{n}\sum_{\ell=2}^{h-1}\tau_\ell(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})\cdot\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\cdot S^1_{h\ell}(p_{\tau_1},q_{\tau_1}) \\ & +\sum_{h=2}^{n-1}\sum_{\ell=h+1}^{n}\tau_h(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot S^2_{h\ell}(p_{\tau_1},q_{\tau_1}),\end{aligned}$$

$$\begin{aligned}\lim_{\tau_2\to\infty}\lim_{\tau_1\to\infty}2I_2 \sim\; & \sum_{\ell=4}^{n}\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot T_\ell(p_{\tau_1,\tau_2},q_{\tau_1,\tau_2}) \\ & +\sum_{h=4}^{n}\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\cdot S^1_{h1}(p_{\tau_1,\tau_2},q_{\tau_1,\tau_2}) \\ & +\sum_{h=4}^{n}\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\cdot S^1_{h2}(p_{\tau_1,\tau_2},q_{\tau_1,\tau_2}) \\ & +\sum_{h=4}^{n}\sum_{\ell=3}^{h-1}\tau_\ell(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})\cdot\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\cdot S^1_{h\ell}(p_{\tau_1,\tau_2},q_{\tau_1,\tau_2}) \\ & +\sum_{\ell=4}^{n}\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot S^2_{2\ell}(p_{\tau_1,\tau_2},q_{\tau_1,\tau_2}) \\ & +\sum_{h=3}^{n-1}\sum_{\ell=h+1}^{n}\tau_h(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot S^2_{h\ell}(p_{\tau_1,\tau_2},q_{\tau_1,\tau_2}),\end{aligned}$$

$$\begin{aligned}
\lim_{\tau_3\to\infty}\lim_{\tau_2\to\infty}\lim_{\tau_1\to\infty} 2I_2 \sim & \sum_{\ell=5}^{n}\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot T_\ell(p_{\tau_1,\tau_2,\tau_3},q_{\tau_1,\tau_2,\tau_3}) \\
&+\sum_{h=5}^{n}\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\cdot S_{h1}^1(p_{\tau_1,\tau_2,\tau_3},q_{\tau_1,\tau_2,\tau_3}) \\
&+\sum_{h=5}^{n}\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\cdot S_{h2}^1(p_{\tau_1,\tau_2,\tau_3},q_{\tau_1,\tau_2,\tau_3}) \\
&+\sum_{h=5}^{n}\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\cdot S_{h3}^1(p_{\tau_1,\tau_2,\tau_3},q_{\tau_1,\tau_2,\tau_3}) \\
&+\sum_{h=5}^{n}\sum_{\ell=4}^{h-1}\tau_\ell(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})\cdot\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\cdot S_{h\ell}^1(p_{\tau_1,\tau_2,\tau_3},q_{\tau_1,\tau_2,\tau_3}) \\
&+\sum_{\ell=5}^{n}\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot S_{2\ell}^2(p_{\tau_1,\tau_2,\tau_3},q_{\tau_1,\tau_2,\tau_3}) \\
&+\sum_{\ell=5}^{n}\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot S_{3\ell}^2(p_{\tau_1,\tau_2,\tau_3},q_{\tau_1,\tau_2,\tau_3}) \\
&+\sum_{h=4}^{n-1}\sum_{\ell=h+1}^{n}\tau_h(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot S_{h\ell}^2(p_{\tau_1,\tau_2,\tau_3},q_{\tau_1,\tau_2,\tau_3}), \\
&\vdots \\
\lim_{\tau_{n-2}\to\infty}\cdots\lim_{\tau_1\to\infty} 2I_2 \sim & \tau_n^2(x_n^{\vec{\tau}}-y_n^{\vec{\tau}})^2\cdot T_n(p_{\tau_1,\dots,\tau_{n-2}},q_{\tau_1,\dots,\tau_{n-2}}) \\
&+\tau_n^2(x_n^{\vec{\tau}}-y_n^{\vec{\tau}})^2\sum_{r=1}^{n-1}S_{nr}^1(p_{\tau_1,\dots,\tau_{n-2}},q_{\tau_1,\dots,\tau_{n-2}}) \\
&+\tau_n^2(x_n^{\vec{\tau}}-y_n^{\vec{\tau}})^2\sum_{r=2}^{n-1}\cdot S_{rn}^2(p_{\tau_1,\dots,\tau_{n-2}},q_{\tau_1,\dots,\tau_{n-2}}).
\end{aligned}$$

The the iterated limits presented above, particularly the final limit, imply that

$$\lim_{\tau_{n-1}\to\infty}\lim_{\tau_{n-2}\to\infty}\cdots\lim_{\tau_1\to\infty} 2I_2 = 0,$$

hence

$$\lim_{\tau_n\to\infty}\cdots\lim_{\tau_1\to\infty} I_2 = 0.$$

This and the iterated limit (4.20) together prove Item (D). □

4.2. **Comparison Principles.** Lemma 4.2 can now be used to prove a comparison principle for the operators $\mathcal{F}^\varepsilon$ and $\mathcal{G}^\varepsilon$ defined in Section 3.

Theorem 4.2. *Assume that u_∞ is the viscosity solution to* (4.8) *proven to exist by Theorem* 4.1*; assume also that v is a viscosity subsolution to Problem* (4.8)*. Then $v \leq u_\infty$ on $\overline{\Omega}$.*

Proof. Suppose to the contrary and recall that, since u_∞ is both a viscosity sub- and supersolution to (4.8), we will have $v \leq g \leq u_\infty$ on $\partial\Omega$ by our definitions. It must be that

$$\sup_{\Omega}(v-u_\infty) = v(p_0)-u_\infty(p_0) > 0. \tag{4.27}$$

[1, Lemma 5.1] and [1, Theorem 5.3] permit us to assume that there exists $\mu(\cdot) > 0$ so that

$$\mathcal{F}^\varepsilon u_\infty(p) = \mu(p) > 0.$$

Taking the difference of $\mathcal{F}^\varepsilon u_\infty$ and $\mathcal{F}^\varepsilon v$ on the sequence $(p_{\vec{\tau}}, q_{\vec{\tau}}) \subset \Omega \times \Omega$,

$$
\begin{aligned}
0 < \mu(q_{\vec{\tau}}) <& \mathcal{F}^\varepsilon u_\infty(q_{\vec{\tau}}) - \mathcal{F}^\varepsilon v(p_{\vec{\tau}}) \\
=& \min\left\{\|\eta^-_{\vec{\tau}}\|^2 - \varepsilon^2, -\left\langle \mathcal{Y}_{\vec{\tau}} \cdot \eta^-_{\vec{\tau}}, \eta^-_{\vec{\tau}}\right\rangle\right\} - \min\left\{\|\eta^+_{\vec{\tau}}\|^2 - \varepsilon^2, -\left\langle \mathcal{X}_{\vec{\tau}} \cdot \eta^+_{\vec{\tau}}, \eta^+_{\vec{\tau}}\right\rangle\right\} \\
\le& \max\left\{\|\eta^-_{\vec{\tau}}\|^2 - \|\eta^+_{\vec{\tau}}\|^2, \left\langle \mathcal{X}_{\vec{\tau}} \cdot \eta^+_{\vec{\tau}}, \eta^+_{\vec{\tau}}\right\rangle - \left\langle \mathcal{Y}_{\vec{\tau}} \cdot \eta^-_{\vec{\tau}}, \eta^-_{\vec{\tau}}\right\rangle\right\}.
\end{aligned} \tag{4.28}
$$

Since $u_\infty \in C(\Omega) \cap W^{1,\infty}_{\text{loc}}(\Omega)$, the assumptions of Lemma 4.2 are satisfied – so we may apply it, [1, Lemma 5.1], and [1, Theorem 5.3] and notice

$$\mu(q_{\vec{\tau}}) \to \mu(p_0) > 0 \tag{4.29}$$

and

$$\max\left\{\|\eta^-_{\vec{\tau}}\|^2 - \|\eta^+_{\vec{\tau}}\|^2, \left\langle \mathcal{X}_{\vec{\tau}} \cdot \eta^+_{\vec{\tau}}, \eta^+_{\vec{\tau}}\right\rangle - \left\langle \mathcal{Y}_{\vec{\tau}} \cdot \eta^-_{\vec{\tau}}, \eta^-_{\vec{\tau}}\right\rangle\right\} \to 0 \tag{4.30}$$

as $\tau_1, \ldots, \tau_n \to \infty$. We arrive at a contradiction by applying (4.28), (4.29), and (4.30). □

In the same manner, we can prove a similar result for the operator $\mathcal{G}^\varepsilon$.

Corollary 4.1. *Assume that u_∞ is the viscosity solution to (4.9) proven to exist by Theorem 4.1; assume also that v is a viscosity supersolution to Problem (4.9). Then $u_\infty \le v$ on $\overline{\Omega}$.*

The following properties of of solutions to (4.8) and (4.9) are evident from the definition of the operators $\mathcal{F}^\varepsilon$ and $\mathcal{G}^\varepsilon$:

- If u is a viscosity solution to Problem (4.8), then it is a viscosity *supersolution* to Problem (DP) – that is, u is ∞-superharmonic.
- If u is a viscosity solution to Problem (4.9), then it is a viscosity *subsolution* to Problem (DP) – that is, u is ∞-subharmonic.

We now state a lemma which relates solutions of (4.8) and (4.9). In light of the comparisons above, the uniqueness of the ∞-harmonic function u_∞ follows as a corollary.

Lemma 4.3 (cf. [1, Lemma 5.6])**.** *Let u^ε and u_ε represent the solutions to Problems (4.8) and (4.9) respectively. Given $\delta > 0$, there exists $\varepsilon > 0$ so that*

$$u_\varepsilon \le u^\varepsilon \le u_\varepsilon + \delta.$$

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