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Survey Article

Weighted approximation: Korovkin and quantitative type theorems

ALI ARAL*

ABSTRACT. In the present paper, we consider Korovkin and quantitative theorems, which have been treated by various authors to date, under weighted approximation. After giving the basic definitions and some of well-known spaces, we mention the main theorems and their applications to linear positive operators, which have been specially treated by the authors. Therefore, this study which can be considered as a survey study will direct the readers to literature information. Furthermore, we give a general operator including well-known operators as an application of some theorems given at the end of the paper.

Keywords: Quantitative type theorems, Korovkin type theorems, weighted approximations, weighted modulus of continuity.

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1. INTRODUCTION

Let $J=[a,b]\subset\mathbb{R}$, $P_n(J)=Span\{1,x,x^2,\ldots,x^n\}$ and $C(J)$ be the space of all contiunous functions defined on J. Bernstein polynomials [8] which map $C(J)$ into $P_n(J)$ are an algebraic elegant method for proof of Weierstrass approximation theorem. Bernstein polynomials are recognised as a pioneer of the linear positive operators in approximation theory. After construction of the Bernstein polynomials, many sequences of linear positive operators were introduced (for example, see [6, 30]). Since Weierstrass approximation theorem was presented for the functions $f\in C(J)$, the studies were restricted on closed and bounded, that is compact intervals of $\mathbb{R}$. In order to overcome this problem, A. D. Gadjiev [18, 19] introduced weighted spaces of functions. Since the similar problem occurs in Bohman-Korovkin theorem, Bohman-Korovkin type theorems were re-presented in the weighted spaces by Gadjiev.

The Banach and Steinhaus theorem yields an initial outcome regarding sets of test functions for the norm convergence of $L_n(f;x)$ to $f(x)$, where $(L_n)_{n\in\mathbb{N}}$ is a sequence of bounded linear operators. Korovkin showed that, in the case where the operators under consideration are positive, the set of test functions can be effectively reduced to a finite set. Curtis Jr. [12] and Dzjadyk [16] presented Korovkin-type theorems for the functions $f\in L_p(-\pi,\pi)$. Afterwards, in [25, 23] weighted Korovkin type theorems were given in the space of locally integrable functions on $\mathbb{R}$. For further studies, see [1, 5, 20, 24].

In the present paper, with necessary references, Korovkin-type and quantitative theorems obtained in previous works on certain weighted spaces are presented. Korovkin type theorems in weighted spaces are mentioned and then quantitative theorems are discussed. Furthermore,

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*Corresponding author: Ali Aral; aliaral73@yahoo.com

we give a general operator including well-known operators. Since the work is survey, we suggest the curious readers to consult the references mentioned for the proofs of the theorems.

2. Korovkin Type Theorems in Weighted Space of Continuous Functions

In this section, we mention that uniform convergence and approximation properties of a sequence of positive linear operators in weighted space of continuous functions.

2.1. Korovkin type theorems in $C_\rho(\mathbb{R})$.

Let $B[a,b]$ denote the space of all bounded functions on $[a,b]$. Furthermore, let $\|\cdot\|$ stands for the usual sup-norm in $C[a,b]$. If the sequence of positive linear operators $A_n : C[a,b] \to B[a,b]$ satisfy the following three conditions:

$$\lim_{n\to\infty} \|A_n(e_0,\cdot) - e_0\| = 0$$
$$\lim_{n\to\infty} \|A_n(e_1,\cdot) - e_1\| = 0$$
$$\lim_{n\to\infty} \|A_n(e_2,\cdot) - e_2\| = 0$$

where $e_i(x) = x^i, i = 0,1,2$, then, we have

$$\lim_{n\to\infty} \|A_n(f,x) - f\| = 0$$

for all function $f \in C[a,b]$ for which $|f(x)| \le M_f(1+x^2)$ hold on $\mathbb{R}$. This theorem is known as Korovkin theorem (see, [3, 31]) and it is important in approximation theory. The theorem shows that convergence on three functions may be extended to all functions which are continuous on $[a,b]$ and bounded on $\mathbb{R}$. Baskakov [7] generalized this result to unbounded functions on $\mathbb{R}$. In [18] and [19], by considering the general weight function ρ, the author defined the weighted spaces B_ρ and C_ρ given by

$$B_\rho(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : |f(x)| < M_f \rho(x),\ \text{for every } x \in \mathbb{R}\},$$
$$C_\rho(\mathbb{R}) := B_\rho(\mathbb{R}) \cap C(\mathbb{R}),$$

where $\rho(x) = 1 + \varphi^2(x)$, φ is an increasing function belonging to $C(\mathbb{R})$.

Now, let $C_\rho^0(\mathbb{R})$ be the subspace of all function $f \in C_\rho(\mathbb{R})$ for which $\lim_{|x|\to\infty} \frac{f(x)}{\rho(x)}$ exists finitely. $C_\rho(\mathbb{R})$ and $B_\rho(\mathbb{R})$ are normed linear space with the ρ-norm given by

$$\|f\|_\rho = \sup_{x\in\mathbb{R}} \frac{|f(x)|}{\rho(x)}.$$

In [19], the author proved following Lemma and Theorems:

Lemma 2.1. *If a sequence of positive linear operators $A_n : C_\rho(\mathbb{R}) \to B_\rho(\mathbb{R})$ satisfies the conditions*

$$\lim_{n\to\infty} \|A_n(\varphi^v) - \varphi^v\|_\rho = 0,\ \ v = 0,1,2,$$

then, for every $f \in C_\rho(\mathbb{R})$ and any finite interval $[a,b]$, we get

$$\lim_{n\to\infty} \sup_{x\in[a,b]} |A_n(f;x) - f(x)| = 0.$$

Theorem 2.1. *If a sequence of positive linear operators $A_n : C_\rho(\mathbb{R}) \to B_\rho(\mathbb{R})$ satisfies the conditions*

$$\lim_{n\to\infty} \|A_n(\varphi^v) - \varphi^v\|_\rho = 0,\ \ v = 0,1,2,$$

then for every $f \in C_\rho^0(\mathbb{R})$, we have

$$\lim_{n\to\infty} \|A_n(f) - f\|_\rho = 0,$$

and there exists a function $f^* \in C_\rho(\mathbb{R}) \setminus C_\rho^0(\mathbb{R})$ *such that*

$$\lim_{n\to\infty} \|A_n(f^*) - f^*\|_\rho \geq 1.$$

If we choose $\rho(x) = 1 + x^2$, we obtain the what follow:

Theorem 2.2. *If a sequence of positive linear operators* $A_n : C_\rho(\mathbb{R}) \to B_\rho(\mathbb{R})$ *satisfies the conditions*

$$\lim_{n\to\infty} \|A_n(\eta^v) - \eta^v\|_\rho = 0, \ v = 0, 1, 2,$$

where $\eta(t) = t$*. Then for every* $f \in C_\rho^0(\mathbb{R})$*, we have*

$$\lim_{n\to\infty} \|A_n(f) - f\|_\rho = 0.$$

In [10], the author proved that a theorem of Korovkin type does not hold on the spaces $C_{\rho_1}(\mathbb{R})$ and $B_{\rho_2}(\mathbb{R})$ with different weights ρ_1 and ρ_2. It is shown that if we put some appropriate conditions on the weight functions it holds (see [11]).

Lemma 2.2. *Suppose that for positive linear operators* $A_n : C_{\rho_1}(\mathbb{R}) \to B_{\rho_2}(\mathbb{R})$ *the sequence* $\|A_n\|_{C_{\rho_1}\to B_{\rho_2}}$ *of operator norms is uniformly bounded and it satisfies the following conditions:*

$$\lim_{|x|\to\infty} \frac{\rho_1(x)}{\rho_2(x)} = 0 \tag{2.1}$$

$$\lim_{n\to\infty} \sup_{\|f\|_{\rho_1}=1} \sup_{|x|\leq s} \frac{|A_n(f;x) - f(x)|}{\rho_1(x)} = 0$$

for all $f \in C_{\rho_1}(\mathbb{R})$ *and for any* $s \in \mathbb{R}$*. Then,*

$$\lim_{n\to\infty} \|A_n\|_{C_{\rho_1}\to B_{\rho_2}} = 0.$$

Theorem 2.3. *Let the weight functions* ρ_1 *and* ρ_2 *be as in Lemma 2.2 and for positive linear operators* $A_n : C_{\rho_1}(\mathbb{R}) \to B_{\rho_2}(\mathbb{R})$*, let the sequence* $\|A_n\|_{C_{\rho_1}\to B_{\rho_2}}$ *of operator norms be uniformly bounded. If the equality*

$$\lim_{n\to\infty} |A_n(f;x) - f(x)| = 0$$

holds for all s_0 *with* $|x| \leq s_0$*, then we get*

$$\lim_{n\to\infty} \|A_n(f) - f\|_{\rho_2} = 0$$

for all $f \in C_{\rho_1}(\mathbb{R})$*.*

Remark 2.1. *Let* $A_n : C_{\rho_1}(\mathbb{R}) \to B_{\rho_2}(\mathbb{R})$ *be a sequence of positive linear operators for all* $n \in \mathbb{N}$*. Suppose that there exists* $M > 0$ *such that for all* $x \in \mathbb{R}$ *we have* $\rho_1(x) < M\rho_2(x)$*. If*

$$\lim_{n\to\infty} \|A_n(\rho_1) - \rho_1\|_{\rho_2} = 0,$$

then the sequence $(A_n)_{n\in\mathbb{N}}$ *is uniformly bounded.*

Let φ_1 and φ_2 be two monotonically increasing and continuous functions on $\mathbb{R}$ such that $\lim_{x\to\pm\infty} \varphi_1(x) = \lim_{x\to\pm\infty} \varphi_2(x) = \pm\infty$ and $\rho_k(x) = 1 + \varphi_k^2(x)$, $k = 1, 2$.

Theorem 2.4. *If the positive linear operators sequence* $A_n : C_{\rho_1}(\mathbb{R}) \to B_{\rho_2}(\mathbb{R})$ *satisfies the following three conditions*

$$\lim_{n\to\infty} \|A_n(\varphi_1^v) - \varphi_1^v\|_{\rho_2} = 0, \ v = 0, 1, 2,$$

and the condition expressed in equation (6.26), then we get

$$\lim_{n\to\infty} \|A_n(f) - f\|_{\rho_2} = 0$$

for all $f \in C_{\rho_1}(\mathbb{R})$.

In [9], the authors presented some ideas about approximation of functions in weighted spaces and explained some unsolved problems in weighted approximation theory. Some of these problems can be explained as follows:

1. Let $\mathcal{F}$ be a linear subspace of $I \subseteq \mathbb{R}$ and $A_n : \mathcal{F} \to C(I)$ a sequence of linear positive operators. For which weights ρ, does A_n map $C\rho(I) \cap \mathcal{F}$ onto $C_\rho(I)$ with uniformly bounded norms?
2. For which functions $f \in C_\rho(I)$ do we have $\|A_n f - f\|_\rho \to 0$, as $n \to \infty$?
3. Which moduli of smoothness are appropriate for weighted approximation?

In [29], A. Holhoş presented some answers to these problems. For a given positive linear operators A_n and belonging to functions $C_\rho(I)$, A. Holhoş showed that $\|A_n f - f\|_\rho \to 0$ as $n \to \infty$ for all the weights ρ. Also, for $f \in B_\rho(I)$ and $\delta > 0$, the authors introduced a suitable modulus of continuity given by

$$\omega_\varphi(f,\delta) := \sup_{\substack{|\varphi(t)-\varphi(x)|\le\delta \\ t,x\in I}} |f(t) - f(x)|, \tag{2.2}$$

where $\varphi : I \to J \subset \mathbb{R}$ is a differentiable bijective function with $\rho'(x) > 0$ for all $x \in I$. The modulus of continuity given in (2.2) has the following properties:

1. $\omega_\varphi(f,\delta) = \omega\left(f \circ \varphi^{-1}, \delta\right)$, where ω is the usual modulus of continuity,
2. $\lim_{\delta\to 0} \omega_\varphi(f,\delta) \to 0$ for every uniformly continuous function $f \circ \varphi^{-1}$ on J.

Now, we give the main result presented in the article in order.

Theorem 2.5. *Let $A_n : C_\rho(I) \to B_\rho(I)$ be a sequence of poisitive linear operators reproducing constant functions and satisfying the conditions:*

$$\sup_{x\in I} A_n(|\varphi(t) - \varphi(x)|, x) = a_n \to 0, \ (n \to \infty)$$
$$\sup_{x\in I} \frac{A_n(|\rho(t) - \rho(x)|, x)}{\rho(x)} = b_n \to 0, \ (n \to \infty).$$

If $A_n(f,x)$ is continuously differentiable and for every $x \in I$, it satisfies

$$\frac{\left|(A_n f)'(x)\right|}{\varphi'(x)} \le K(f,\rho,n)\,\rho(x),$$

where $K(f,\rho,n)$ is a constant and ρ, φ are such that there exists a constant $\alpha > 0$

$$\frac{\rho'(x)}{\varphi'(x)} \le \alpha\rho(x).$$

Then, we have that the following statements are equilavent:

(i)
$$\|A_n f - f\|_\rho \to 0 \text{ as } n \to \infty.$$

(ii) $\frac{f}{\rho} \circ \varphi^{-1}$ *is uniformly continuous on J.*

Moreover, from these, we get

$$\|A_n f - f\|_\rho \le b_n \|f\|_\rho + 2\omega_\varphi\left(\frac{f}{\rho}, a_n\right)$$

for every $n \ge 1$.

We can obtain the result on convergence of Szász-Mirakjan operators $S_n : C_\rho[0,\infty) \to C_\rho[0,\infty)$ defined by

$$S_n(f,x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),\ x\in[0,\infty).$$

Corollary 2.1. *For every $\alpha>0$ and $\rho(x)=e^{\alpha\sqrt{x}}$, we have*

$$\|S_n\|_\rho = \sup_{x\geq 0}\frac{S_n(\rho,x)}{\rho(x)} \leq C_\alpha,$$

where C_α is a constant that only depends on α.

Corollary 2.2. *Let $\alpha>0$ and $\rho(x)=e^{\alpha\sqrt{x}}$. If $f\left(x^2\right)e^{-\alpha x}$ is uniformly continuous on $[0,\infty)$, then we obtain*

$$\|S_n f - f\|_\rho \to 0,\ \text{as } n\to\infty.$$

Furthermore, for $f\in C_\rho([0,\infty))$, we get

$$\|S_n f - f\|_\rho \leq \|f\|_\rho \frac{\alpha C}{\sqrt{n}} + 2\omega\left(f\left(t^2\right)e^{-\alpha t},\frac{1}{\sqrt{n}}\right) \text{ for every } n\geq 1,$$

where $C=\sup_{n\in\mathbb{N}}\frac{1}{2}\sqrt{\|S_n\rho^2\|_{\rho^2}+2\|S_n\rho\|_\rho+1}$ is constant that only depends on α.

2.2. **Korovkin type Theorems in H_ω.** In [22], the authors present some Korovkin type theorems on uniform approximation of some subclass of continuous and bounded functions by linear positive operators on all positive semi real axis $\mathbb{R}^+$ by using the test functions $\left(\frac{x}{x+1}\right)^v$, $v=0,1,2$.

Let ω be a function of the type of modulus of continuity. The basic properties of this type functions are the following:

1. ω is non-negative increasing function on $\mathbb{R}^+$,
2. $\omega(\delta_1+\delta_2)\leq\omega(\delta_1)+\omega(\delta_2)$ for any $\delta_1,\delta_2>0$,
3. $\lim_{\delta\to 0}\omega(\delta)=0$.

By H_ω, we denote the space of all real-valued functions defined on $\mathbb{R}^+$ which satisfy the following condition:

$$|f(x)-f(y)| \leq \omega\left(\left|\frac{x}{1+x}-\frac{y}{1+y}\right|\right) \tag{2.3}$$

for any $x,y\in\mathbb{R}^+$. Also, by $C_B(\mathbb{R}^+)$, we denote the space of all bounded functions $f\in C(\mathbb{R}^+)$ with the usual sup-norm

$$\|f\|_{C_B} = \sup_{x\in\mathbb{R}^+}|f(x)|.$$

Considering the property 3. of modulus of continuity, it is obvious that any function in H_ω satisfies the inequality

$$|f(x)| \leq f(0)+\omega(1),x\in\mathbb{R}^+$$

and therefore it is bounded on $\mathbb{R}^+$. So $H_\omega\subset C_B(\mathbb{R}^+)$. Some examples of the functions, belonging to H_ω are the following [22] :

$$f_1(x)=\sum_{k=0}^{\infty}c_k\left(\frac{x}{1+x}\right)^k,$$

where $\sum_{k=1}^{\infty} k\left|c_k\right| < \infty$ with $\omega(t) = 2t^{\alpha} \sum_{k=1}^{\infty} k\left|c_k\right|, 0 < \alpha \leq 1$ and

$$f_2(x) = \frac{1+2x}{1+x}$$

with $\omega(t) = t$. In the case of $\omega(t) = Mt^{\alpha}$, $0 < \alpha \leq 1$, H_{α} is used instead of H_{ω}. In this case, it follows from (2.3) that

$$|f(x) - f(y)| \leq M \frac{|x-y|^{\alpha}}{(1+x)^{\alpha}(1+y)^{\alpha}}$$

and therefore, $H_{\alpha} \subset Lip_M \alpha$.

Following result is the Korovkin type theorem on the conditions of sequence linear positive operators to functions in H_{ω}. Note that this type theorem can not be obtained neither from classical Korovkin's theorem nor from the Korovkin's theorem concerning Chebyhev's system since both of them are devoted the problem of approximation by positive operators on finite intervals. It can not be obtained from weigted Korovkin's type theorem in [18] and [19], since all the test functions in these theorems connected with the weight functions $\rho(x) \geq 1$.

Theorem 2.6. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators, acting from H_{ω} to $C_B(\mathbb{R}^+)$ and satisfying the conditions*

$$\lim_{n\to\infty} \left\| A_n\left((\sigma)^v\right) - (\sigma)^v \right\|_{C_B} = 0, \ v = 0, 1, 2, \tag{2.4}$$

where $\sigma(t) = \frac{t}{1+t}$. Then, for any function in H_{ω}, we get

$$\lim_{n\to\infty} \|A_n f - f\|_{C_B} = 0.$$

Bleimann-Butzer-Hahn operator is given by

$$L_n(f;x) = \frac{1}{(1+x)^n} \sum_{k=0}^{\infty} f\left(\frac{k}{n-k+1}\right) \binom{n}{k} x^k, \ x \in \mathbb{R}^+. \tag{2.5}$$

Corollary 2.3. *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators defined in (2.5). Then for any function in H_{ω}, we have*

$$\lim_{n\to\infty} \|L_n f - f\|_{C_B} = 0.$$

Proof. By Theorem 2.6, it is sufficient to verify the conditions (2.4). It is obvious that $L_n(1;x) = 1$. Then,

$$L_n(\frac{t}{1+t};x) = \frac{n}{n+1}\left(\frac{x}{1+x}\right)$$

and therefore,

$$\|L_n(\sigma) - \sigma\|_{C_B} \leq \frac{1}{n+1}.$$

Moreover, we have

$$L_n(\left(\frac{t}{1+t}\right)^2;x) = \frac{n(n-1)}{(n+1)^2}\left(\frac{x}{1+x}\right)^2 + \frac{n}{(n+1)^2}\frac{x}{1+x}.$$

Then, we get

$$\left\| L_n\left((\sigma)^2\right) - (\sigma)^2 \right\|_{C_B} \leq \frac{3n+2}{(n+1)^2}.$$

This proves the theorem. □

3. Korovkin Type Theorems in Weighted Space of Lebesgue Measurable Functions

In this section, we give Korovkin type theorems in weighted Lebesgue spaces.

3.1. Korovkin type Theorems in $L_{p,\tilde{\omega}}(\mathbb{R})$.

In [26], the authors studied Korovkin type theorems in the weighted Lebesgue spaces considering a positive continuous function $\tilde{\omega}$ on the whole real axis which satisfies the condition

$$\int_{\mathbb{R}} t^{2p}\tilde{\omega}(t)\,dt < \infty,\ p \in [1,\infty). \tag{3.6}$$

By $L_{p,\tilde{\omega}}(\mathbb{R})$, we will denote the linear space of measurable, $p-$absolutely integrable functions on $\mathbb{R}$ with respect to the weight function $\tilde{\omega}$, i.e.

$$L_{p,\tilde{\omega}}(\mathbb{R}) := \left\{ f:\mathbb{R}\to\mathbb{R} \,\middle|\, \|f\|_{p,\tilde{\omega}} := \left(\int_{\mathbb{R}} |f(t)|^p\,\tilde{\omega}(t)\,dt\right)^{\frac{1}{p}} < \infty \right\}.$$

In [26], the author proved following theorem:

Theorem 3.7. *Let $L_n : L_{p,\tilde{\omega}}(\mathbb{R}) \to L_{p,\tilde{\omega}}(\mathbb{R})$ be a uniformly bounded sequence of positive linear operators which satisfy the conditions*

$$\lim_{n\to\infty} \|L_n(\tilde{e}_i) - \tilde{e}_i\|_{p,\tilde{\omega}} = 0,\ i = 0,1,2,$$

where $\tilde{e}_i(t) = t^i$. Then for every $f \in L_{p,\tilde{\omega}}(\mathbb{R})$, we get

$$\lim_{n\to\infty} \|L_n f - f\|_{p,\tilde{\omega}} = 0.$$

Furthermore, the authors established an analogue of Theorem 3.7 for the space of function of multivariable. For $1 \le p < \infty$, let Ω be a positive continuous function in $\mathbb{R}^n$ which satisfies the condition

$$\int_{\mathbb{R}^n} |t|^{2p}\,\Omega(t)\,dt < \infty,$$

and let

$$L_{p,\Omega}(\mathbb{R}^n) = \left\{ f:\mathbb{R}^n\to\mathbb{R} \,\middle|\, \|f\|_{p,\Omega} = \left(\int_{\mathbb{R}^n} |f(t)|^p\,\Omega(t)\,dt\right)^{\frac{1}{p}} < \infty \right\}.$$

Theorem 3.8. *Let $(L_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of positive linear operators from $L_{p,\Omega}(\mathbb{R}^n)$ into itself, satisfying the conditions*

$$\lim_{n\to\infty} \|L_n(1;x) - 1\|_{p,\Omega} = 0$$
$$\lim_{n\to\infty} \|L_n(t_i;x) - x_i\|_{p,\Omega} = 0,\ i = 1,\dots,n,$$
$$\lim_{n\to\infty} \left\| L_n\left(|t|^2;x\right) - |x|^2 \right\|_{p,\Omega} = 0.$$

Then for every $f \in L_{p,\Omega}(\mathbb{R}^n)$, we have

$$\lim_{n\to\infty} \|L_n f - f\|_{p,\Omega} = 0.$$

4. Quantitative Type Theorems in Certain Weigted Spaces

In this section, we give some quantitative type theorems in certain weighted spaces.

4.1. **Quantitative type Theorems in $C_\rho(\mathbb{R}^+)$.** In [27], the authors constructed a new type of modulus of continuity in weighted spaces of continuous functions. While constructing the new type of modulus of continuity, they considered the following conditions:

(i) ρ is a continuously differentiable function on $\mathbb{R}^+$ with $\rho(0)=1$,
(ii) $\inf_{x\geq 0}\rho'(x)\geq 1$.

For each $f\in C_\rho(\mathbb{R}^+)$ and $\delta>0$, the new type weighted modulus of continuity is defined by

$$\Omega_\rho(f;\delta)_{\mathbb{R}^+}=\sup_{\substack{|\rho(t)-\rho(x)|\leq\delta\\ t,x\in\mathbb{R}^+}}\frac{|f(t)-f(x)|}{[|\rho(t)-\rho(x)|+1]\,\rho(x)}. \tag{4.7}$$

Let $C_\rho^k(\mathbb{R}^+)$ be the subspace of all function $f\in C_\rho(\mathbb{R}^+)$ for which $\lim\limits_{x\to\infty}\dfrac{f(x)}{\rho(x)}=k\in\mathbb{R}$.

The modulus of continuity given in (4.7) has the some properties that are similar to properties of the usual modulus of continuity. For example, for any $f\in C_\rho^k(\mathbb{R}^+)$, we have $\lim_{\delta\to 0}\Omega_\rho(f;\delta)_{\mathbb{R}^+}=0$. For more details, see [27]. In the same paper, the authors introduced an analogy of the classical Lipschitz space $Lip_M\alpha$.

Definition 4.1. *Let $\rho(x)$ satisfy the conditions (i) and (ii), $0<\alpha\leq 1$ and $M>0$. By $Lip_M(\rho(x);\alpha)$, we denote the set of all functions which satisfy the inequality*

$$|f(t)-f(x)|\leq M\,|\rho(t)-\rho(x)|^\alpha,\ x,t\geq 0.$$

We can immediately see that

$$Lip_M\alpha\subset Lip_M(\rho(x);\alpha)$$

and

$$Lip_M\alpha=Lip_M(1+x;\alpha).$$

Using (4.7) and Definition 4.1, we get

$$\Omega_\rho(f;\delta)_{\mathbb{R}^+}\leq M\delta^\alpha. \tag{4.8}$$

Now, we give the following quantitative result:

Theorem 4.9. *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators which satisfies the conditions*

$$\|L_n1-1\|_\rho=\alpha_n\to 0,\text{as } n\to\infty, \tag{4.9}$$

$$\|L_n\rho-\rho\|_\rho=\beta_n\to 0,\text{as } n\to\infty, \tag{4.10}$$

$$\left\|L_n\rho^2-\rho^2\right\|_{\rho^2}=\gamma_n\to 0,\text{as } n\to\infty. \tag{4.11}$$

Then for all $f\in C_\rho^k(\mathbb{R}^+)$, we have

$$\|L_nf-f\|_{\rho^4}\leq 16\Omega_\rho\left(f;\sqrt{\alpha_n+2\beta_n+\gamma_n}\right)_{\mathbb{R}^+}+\|f\|_\rho\,\alpha_n$$

for sufficiently large n.

Using the inequality (4.8), we can give the following corollary.

Corollary 4.4. *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators which satisfies the conditions (4.9)-(4.11). For some $\alpha\in(0,1]$, if $f\in Lip_M(\rho(x);\alpha)$, then we have*

$$\|L_nf-f\|_{\rho^4}\leq 16M(\alpha_n+2\beta_n+\gamma_n)^{\frac{\alpha}{2}}+\|f\|_\rho\,\alpha_n$$

for sufficiently large n, where $M>0$ is a constant independent of n.

The theorem presented below establishes the convergence of sequences for positive linear operators within the weighted space $C_\rho^k(\mathbb{R}^+)$, with the convergence interval expanding as $n \to \infty$.

Theorem 4.10. *Under the assumptions of Theorem 4.9, if the sequence of positive real numbers η_n satisfies the conditions*

$$\lim_{n\to\infty} \eta_n = \infty$$
$$\lim_{n\to\infty} \rho^{\frac{3}{2}}(\eta_n)\,\delta_n = 0,$$

then, we get

$$\sup_{0\le x\le \eta_n} \frac{|L_n(f;x)-f(x)|}{\rho(x)} \le 16\Omega_\rho\left(f;\rho^{\frac{3}{2}}(\eta_n)\,\delta_n\right)_{\mathbb{R}^+} + \|f\|_\rho\,\rho^{\frac{3}{2}}(\eta_n)\,\delta_n$$

for each $f \in C_\rho^k(\mathbb{R}^+)$ and sufficiently large n.

Finally, we give a more general result of Theorem 4.9. For given functions ψ_1 and ψ_2, let $\psi_1(x) \le \psi(x)$ and $\psi_2(x) \le \psi(x)$ for all $x \ge 0$, where $\psi(x) = \max(\psi_1(x), \psi_2(x))$.

Theorem 4.11. *Let $\rho(x) \le \psi_k(x)$, $k = 0,1,2,3$. If $(L_n)_{n\in\mathbb{N}}$ a sequence of positive linear operators satisfying the conditions*

$$\begin{aligned} \|L_n 1 - 1\|_{\psi_1} &= \alpha_n \to 0, \text{ as } n\to\infty, \\ \|L_n\rho - \rho\|_{\psi_2} &= \beta_n \to 0, \text{ as } n\to\infty, \\ \left\|L_n\rho^2 - \rho^2\right\|_{\psi_3} &= \gamma_n \to 0, \text{ as } n\to\infty. \end{aligned}$$

Then for any function $f \in C_\rho^k(\mathbb{R}^+)$, we obtain inequality

$$\|L_n f - f\|_{\psi\rho^2} \le 16\Omega_\rho\left(f;\sqrt{\alpha_n + 2\beta_n + \gamma_n}\right)_{\mathbb{R}^+} + \|f\|_\rho\,\alpha_n$$

for sufficiently large n.

In [28], for $f \in C_\rho(\mathbb{R}^+)$ and $\delta > 0$, the author presented the modulus of continuity given by

$$\omega_\varphi(f,\delta) = \sup_{\substack{|\varphi(x)-\varphi(y)|\le\delta \\ x,y\ge 0}} \frac{|f(x)-f(y)|}{\rho(x)+\rho(y)}. \tag{4.12}$$

Now, let us define the space

$$U_\rho(\mathbb{R}^+) := \left\{ f \in C_\rho(\mathbb{R}^+) : \frac{f}{\rho} \text{ is uniformly continuous} \right\}.$$

The modulus of continuity given in (4.12) has some properties that are similar to properties of the classical modulus of continuity. For example, for every $f \in U_\rho(\mathbb{R}^+)$, we have $\lim_{\delta\to 0}\omega_\varphi(f,\delta) = 0$. For more details, we refer the reader to [28]. Now, we give the main results presented in the same paper.

Theorem 4.12. *Let $L_n : C_\rho(\mathbb{R}^+) \to B_\rho(\mathbb{R}^+)$ be a sequence of positive linear operators satisfying the conditions*

$$\left\|L_n\varphi^0 - \varphi^0\right\|_{\rho^0} = a_n \to 0, \text{ as } n \to \infty,$$
$$\left\|L_n\varphi - \varphi\right\|_{\rho^{\frac{1}{2}}} = b_n \to 0, \text{ as } n \to \infty,$$
$$\left\|L_n\varphi^2 - \varphi^2\right\|_{\rho} = c_n \to 0, \text{ as } n \to \infty,$$
$$\left\|L_n\varphi^3 - \varphi^3\right\|_{\rho^{\frac{3}{2}}} = d_n \to 0, \text{ as } n \to \infty.$$

Then for all $f \in C_\rho(\mathbb{R}^+)$, we get

$$\left\|L_n f - f\right\|_{\rho^{\frac{3}{2}}} \le (7 + 4a_n + 2c_n)\,\omega_\varphi(f, \delta_n) + \|f\|_\rho\, a_n,$$

where $\delta_n = 2\sqrt{(a_n + 2b_n + c_n)(1 + a_n)} + a_n + 3b_n + 3c_n + d_n$.

Remark 4.2. *Under the conditions of Theorem 4.12, we obtain*

$$\lim_{n\to\infty} \left\|L_n f - f\right\|_{\rho^{\frac{3}{2}}} = 0$$

for every $f \in C^k_{\rho^{\frac{3}{2}}}(\mathbb{R}^+)$.

Corollary 4.5. *Let $L_n : C_\rho(\mathbb{R}^+) \to B_\rho(\mathbb{R}^+)$ be a sequence of positive linear operators satisfying the conditions*

$$\left\|L_n\varphi^0 - \varphi^0\right\|_{\rho^0} = a_n \to 0, \text{ as } n \to \infty,$$
$$\left\|L_n\varphi - \varphi\right\|_{\rho^{\frac{1}{2}}} = b_n \to 0, \text{ as } n \to \infty,$$
$$\left\|L_n\varphi^2 - \varphi^2\right\|_{\rho} = c_n \to 0, \text{ as } n \to \infty,$$
$$\left\|L_n\varphi^3 - \varphi^3\right\|_{\rho^{\frac{3}{2}}} = d_n \to 0, \text{ as } n \to \infty.$$

Let η_n be a sequence of real numbers such that

$$\lim_{n\to\infty} \eta_n = \infty,$$
$$\lim_{n\to\infty} \rho^{\frac{1}{2}}(\eta_n)\,\delta_n = 0,$$

where $\delta_n = 2\sqrt{(a_n + 2b_n + c_n)(1 + a_n)} + a_n + 3b_n + 3c_n + d_n$. Then we have

$$\sup_{0\le x\le \eta_n} \frac{|L_n f(x) - f(x)|}{\rho(x)} \le (7 + 4a_n + 2c_n)\,\omega_\varphi\left(f, \rho^{\frac{1}{2}}(\eta_n)\,\delta_n\right) + \|f\|_\rho\, a_n$$

for all $f \in C_\rho(\mathbb{R}^+)$.

Now, we give an example of linear positive operators as an applications of the theorems given in this subsection (see [27]). In what follows, we consider that $\omega : \mathbb{R} \to \mathbb{R}$, $\omega(x) = 1 + x^2$.

Definition 4.2. *We consider a sequence of linear positive operators $(D_n)_{n\in\mathbb{N}}$ given by*

$$D_n(f; x) = \rho^2(x) \sum_{k=0}^{\infty} \frac{f\left(\frac{k}{n}\right)}{\rho^2\left(\frac{k}{n}\right)} a_{k,n}(x),$$

where $a_{k,n}$, $n = 1, 2, \ldots, x \in \mathbb{R}^+$ are nonnegative functions which satisfy the following conditions:

(i)

$$\sum_{k=0}^{\infty} a_{k,n}(x) = 1,$$

(ii)

$$\lim_{n\to\infty}\left\|\sum_{k=0}^{\infty}\left(\frac{k}{n}\right)^{i}a_{k,n}-e_i\right\|_{\omega},\ i=1,2.$$

By straightforward calculation, we get

$$D_n(1;x)-1=\rho^2(x)\left[\sum_{k=0}^{\infty}\frac{1}{\rho^2\left(\frac{k}{n}\right)}a_{k,n}(x)-\frac{1}{\rho^2(x)}\right],$$

$$D_n(\rho;x)-\rho(x)=\rho^2(x)\left[\sum_{k=0}^{\infty}\frac{1}{\rho\left(\frac{k}{n}\right)}a_{k,n}(x)-\frac{1}{\rho(x)}\right],$$

$$D_n\left(\rho^2;x\right)-\rho^2(x)=0.$$

Before presenting the results, we consider the following theorem:

Theorem 4.13 ([27]). *Let $B_n : C_\omega(\mathbb{R}^+)\to B_\omega(\mathbb{R}^+)$ be a sequence of positive linear operators. If*

$$\lim_{n\to\infty}\|B_n(e_i;\cdot)-e_i\|_{\omega}=0 \text{ for } i=0,1,2,$$

then, we have

$$\lim_{n\to\infty}\|B_nf-f\|_{\omega}=0$$

for all functions $f\in C_\omega^k(\mathbb{R}^+)$.

Since $\frac{1}{\rho}$ and $\frac{1}{\rho^2}$ are bounded functions, thanks to Theorem 4.13 we have

$$\lim_{n\to\infty}\left\|\sum_{k=0}^{\infty}\frac{1}{\rho^2\left(\frac{k}{n}\right)}a_{k,n}-\frac{1}{\rho^2}\right\|_{\omega}=0$$

$$\lim_{n\to\infty}\left\|\sum_{k=0}^{\infty}\frac{1}{\rho\left(\frac{k}{n}\right)}a_{k,n}-\frac{1}{\rho}\right\|_{\omega}=0.$$

Hence, we get

$$\alpha_n=\lim_{n\to\infty}\|D_n(1;\cdot)-1\|_{\rho^2\omega}=0,$$

$$\beta_n=\lim_{n\to\infty}\|D_n(\rho;\cdot)-\rho\|_{\rho^2\omega}=0,$$

$$\gamma_n=\lim_{n\to\infty}\left\|D_n\left(\rho^2;\cdot\right)-\rho^2\right\|_{\rho^2\omega}=0.$$

Then, under the assumptions Theorem 4.11, we obtain for each $f\in C_\rho^k(\mathbb{R}^+)$

$$\|D_nf-f\|_{\rho^4\omega}\le C(f)\,\Omega_\rho\left(f;\sqrt{\alpha_n+2\beta_n+\gamma_n}\right),$$

where $C(f)$ is the constant depending on f.

4.2. **Quantitative type Theorems in $L_{p,\tilde{\omega}}(\mathbb{R})$.** In [2], the authors explored the approximation properties of the operators in some weighted spaces $L_p, 1\le p<\infty$. They considered the weight function

$$\nu:\mathbb{R}\to(0,1],\ \nu(x)=\left(1+x^{2m}\right)^{-p},$$

where $m>1$ is a fixed integer. Now, by $L_{p,\nu}(\mathbb{R}^+)$, we will denote the linear space of $p-$ absolutely integrable functions on $\mathbb{R}$ with respect to weight ν, that is

$$L_{p,\nu}(\mathbb{R}):=\left\{f:\mathbb{R}\to\mathbb{R}: f\nu^{\frac{1}{p}}\in L_p(\mathbb{R}),1\le p<\infty\right\}.$$

The norm on $L_{p,\nu}$ denoted by $\|f\|_{p,\nu}$ is defined by

$$\|f\|_{p,\nu} := \left\|f\nu^{\frac{1}{p}}\right\|_{L_p} = \left(\int_{\mathbb{R}} |f(t)|^p \nu(t)\,dt\right)^{\frac{1}{p}}.$$

Remark 4.3. *If we consider $\tilde{\omega} = \nu$, since $p \geq 1$ and $m > 1$, this weight function satisfies the condition* (3.6).

In the same paper, the authors introduced a new type modulus of continuity for the functions $f \in L_{p,\nu}(\mathbb{R})$. The new type modulus of continuity is defined by

$$\omega_{p,m}(f;\delta) = \sup_{0\leq h\leq\delta}\left(\int_{\mathbb{R}}\left|\frac{f(x+h)-f(x)}{1+(|x|+h)^{2m}}\right|^p dx\right)^{\frac{1}{p}}, \delta \geq 0. \tag{4.13}$$

The modulus of continuity given in (4.13) has some properties that are similar to properties of the classical modulus of continuity. For example, for every $f \in L_{p,\nu}(\mathbb{R})$, we have $\lim_{\delta\to 0^+}\omega_{p,m}(f;\delta) = 0$. For more details, see [2].

Now for $\alpha \geq 1$, $\lambda = \frac{1}{2\alpha} \in \left(0, \frac{1}{2}\right]$ and for each $n \in \mathbb{N}$, we consider the positive linear operators defined by

$$(L_n f)(x) := \frac{1}{a_n}\int_{x-1}^{x} f(t+\tau_{2,n})\left(1-(x-t)^{2\alpha}\right)^n dt,\ x \in \mathbb{R}, \tag{4.14}$$

where

$$a_n = \int_0^1 \left(1-y^{2\alpha}\right)^n dy,$$
$$\tau_{l,n} = \frac{B(n+1,l\lambda)}{B(n+1,\lambda)},\ p \in \mathbb{N}.$$

Here, B is the usual Beta function. Now, we give the main results presented in [2].

Theorem 4.14. *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators as defined in* (4.14). *Then for all $f \in L_{p,\nu}(\mathbb{R})$, we have*

$$\|L_n f - f\|_{p,\nu} \leq 2\omega_{p,m}\left(f; \sqrt{\tau_{3,n} - \tau_{2,n}^2}\right).$$

In the following theorem, the authors presented global smoothness preservation property.

Theorem 4.15. *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators as defined in* (4.14). *Then for every $f \in L_{p,\nu}(\mathbb{R})$ and $\delta > 0$, the inequality*

$$\omega_{p,m}(L_n f;\delta) \leq 2^{2m-1}\left(1+\sqrt{\tau_{3,n}-\tau_{2,n}^2}\right)\omega_{p,m}(f;\delta)$$

holds.

5. Applications

In this section, we establish general Durrmeyer type operators which occur to approximate Lebesgue integrable function on finite and infinite interval in approximation by linear positive operators. Ibragimov and Gadjiev [17] introduced operators $\{G_n\}$ by

$$G_n(f;x) = \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n^2\psi_n(0)}\right) K_n^{(\nu)}(x,t,u)\frac{\left(-\alpha_n\psi_n(0)\right)^{\nu}}{\nu!}, \tag{5.15}$$

where $\left(K_n^{(\nu)}(x,t,u)\right)_{n\in\mathbb{N}} := \frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\big|_{u=\alpha_n\psi_n(t),t=0}$ is a sequence of functions of trivariate x,t,u, such that $x,t \in [0,A]$ and $u \geq 0$, satisfying following conditions:

1. for each $x,t\in[0,A]$ and for each $n\in\mathbb{N}=\{1,2,3,...\}$, K_n is entire function with respect to variable u,
2. $K_n(x,0,0)=1$, $x\in[0,A]$, $n\in\mathbb{N}$,
3. $\left[(-1)^{\nu}\frac{\partial^{\nu}}{\partial u^{\nu}}K_n(x,t,u)\Big|_{u=u_1,t=0}\right]\geq 0$ for $\nu=0,1,...,n\in\mathbb{N}$ and $x\in[0,A]$,
4. $\frac{\partial^{\nu}}{\partial u^{\nu}}K_n(x,t,u)\Big|_{u=u_1,t=0}=-nx\left[\frac{\partial^{\nu-1}}{\partial u^{\nu-1}}K_{m+n}(x,t,u)\Big|_{u=u_1,t=0}\right]$ for all $x\in[0,A]$ and $n,\nu\in\mathbb{N}$, m is a number such that $m+n=0$ or a natural number.

Here, $(\varphi_n(t))_{n\in\mathbb{N}}$, $(\psi_n(t))_{n\in\mathbb{N}}$ are sequences of functions in $C[0,\infty)$, which is the space of continuous function on $[0,\infty)$, such that $\varphi_n(0)=0$ and $\lim\limits_{n\to\infty}\frac{1}{n^2}\psi_n(0)=0$. Also let $(\alpha_n)_{n\in\mathbb{N}}$ denote a sequence of positive numbers satisfying the conditions:

$$\lim_{n\to\infty}\frac{\alpha_n}{n}=\mathbf{1}\text{ and }\lim_{n\to\infty}\alpha_n\psi_n(0)=l_1,\ l_1\geq 0. \tag{5.16}$$

The operator G_n is defined by for $x\in\mathbb{R}^+$ and any function f defined on the interval $\mathbb{R}^+$. Now, these sequences of positive linear operators are called Ibragimov-Gadjiev operators. The authors studied uniform convergence and some shape preserving properties of the operators G_n. As $\{G_n\}$ contains well-known operators in the special cases (see [14]), this sequence of linear positive operators has been studied extensively. In [13, 14], some generalization and order of approximation of unbounded functions were obtained. Also, weighted approximation results were presented in the papers [4, 21].

In the year 1967, Durrmeyer [15] introduced a modification of the Bernstein polynomials with the aim of approximating Lebesgue integrable functions on $[0,1]$. These operators, called by Bernstein-Durrmeyer operators, are defined by

$$D_n(f,x)=(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)\,dt,\ x\in[0,1], \tag{5.17}$$

where $p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$.

In the present paper, we introduce Durrmeyer modification of the operators (5.15). Firstly, we give some auxiliary results to construct the new operators and calculate moments for these operators. In last section, we obtain local approximation results for new Durrmeyer operators using second order modulus of smoothness and modulus of continuity of f belongs to space of functions bounded and continuous on $[0,\infty)$. However, this condition can be replaced by weaker conditions in some special cases of operators. We also study on asymptotic formulas. It is well-known in approximation theory by linear positive operators that the classical Bernstein operator $B_n:C[0,1]\to C[0,1]$ is given by

$$B_n(f,x)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k},\ x\in[0,1]$$

and Voronovskaya theorem proved in [32] as: *If f is bounded on $[0,1]$, differentiable in some neighborhood of x and has second derivative f'' for some $x\in[0,1]$, then*

$$\lim_{n\to\infty}n[B_n(f,x)-f(x)]=\frac{x(1-x)}{2}f''(x). \tag{5.18}$$

This type results were also obtained for other operators as Szász and Baskakov operators and their generalizations. We firstly obtain Voronovskaya type theorem for new Durrmeyer operators. We finally give a quantitative Voronovskaya result for new Durrmeyer operators.

5.1. **Construction of Ibragimov-Gadjiev-Durrmeyer operators.** With similar consideration constructed by Ibragimov and Gadjiev, we purpose to define a general Durrmeyer type operators including well-known Durrmeyer operators. In order to achieve this, additionally to four conditions mentioned in introduction we also assume the following condition:

5. $\int_0^A K_n^{(\nu)}(x,t,u)\,dx = (-1)^\nu \frac{\nu!}{(n-m)u_1^{\nu+1}}$.

Now we can give new generalized Durrmeyer operators:

$$M_n(f;x) = (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty} K_n^{(\nu)}(x,t,u)\,\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}$$

$$\times\int_0^A f(y)\,K_n^{(\nu)}(y,t,u)\,\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy. \tag{5.19}$$

We call these new operators as Ibragimov-Gadjiev-Durrmeyer operators. The family of operators $M_n(f;x)$ is linear and positive.

5.2. **Some auxilary results.** We note that in this paper we study generalized operators (5.19) with $A=\infty$. Using the assumptions on $K_n(x,t,u)$, since $K_n(x,t,u)$ is entire functions respectively the variable u, we can write for any $u_1\in\mathbb{R}$ the following Taylor expansion

$$K_n(x,t,u) = \sum_{\nu=0}^{\infty}\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\bigg|_{u=u_1}\frac{(u-u_1)^\nu}{\nu!}.$$

Replacing $u=\varphi_n(t)$, $u=\alpha_n\varphi_n(t)$ and $t=0$, where (α_n) is the sequence defined in (5.16),

$$K_n(x,0,0) = \sum_{\nu=0}^{\infty}\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\bigg|_{u=\alpha_n\varphi_n(t),t=0}\frac{(-\alpha_n\varphi_n(0))^\nu}{\nu!}$$

is obtained by the condition $\varphi_n(0)=0$. Taking into account that $K_n(x,0,0)=1$ by the condition (2), we have

$$\sum_{\nu=0}^{\infty}\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\bigg|_{u=\alpha_n\varphi_n(t),t=0}\frac{(-\alpha_n\varphi_n(0))^\nu}{\nu!}=1.$$

Also we assume that the following three conditions satisfied:

a) $K_n(0,0,u)=1$ for any $u\geq 0$, $x\in[0,A]$, $p\in\mathbb{N}$ and
$$\lim_{x\to\infty} x^p K_n^{(\nu)}(x,t,u)=0,$$
b) $\frac{d}{dx}\left.K_n(x,t,u)\right|_{u=u_1,t=0} = -nu_1\left.K_{m+n}(x,t,u)\right|_{u=u_1,t=0}$,
c) $\frac{n+\nu m}{1+u_1 m x}K_n^{(\nu)}(x,t,u) = nK_{n+m}^{(\nu)}(x,t,u)$.

Lemma 5.3. *The condition* $b)$ *is equivalent to the following equality*

$$\frac{d}{dx}K_n^{(\nu)}(x,t,u) = \frac{\nu}{x}K_n^{(\nu)}(x,t,u) - nu_1K_{n+m}^{(\nu)}(x,t,u).$$

Proof. By ν-multiple application of condition (4), we obtain

$$K_n^{(\nu)}(x,t,u) = (-1)^\nu n(n+m)\ldots(n+(\nu-1)m)\,x^\nu\left.K_{n+\nu m}(x,t,u)\right|_{u=u_1,t=0}. \tag{5.20}$$

Applying condition $b)$, we get

$$(-1)^\nu\frac{d}{dx}K_n^{(\nu)}(x,t,u) = n(n+m)\ldots(n+(\nu-1)m)$$

$$\times\left\{\nu x^{\nu-1}\,K_{n+\nu m}\left(x,t,u\right)\big|_{u=u_1,t=0}-x^{\nu}\left(n+\nu m\right)u_1\,K_{n+(\nu+1)m}\left(x,t,u\right)\big|_{u=u_1,t=0}\right\}.$$

Using (5.20), we get desired result. □

Using condition $c)$ and Lemma 5.3, we obtain:

Conclusion 1. *We have*

$$x\left(1+u_1mx\right)\frac{d}{dx}K_n^{(\nu)}\left(x,t,u\right)=\left(\nu-xu_1n\right)K_n^{(\nu)}\left(x,t,u\right).$$

Proposition 5.1. *If the conditions* $1.-4.$ *and* $a)$, $b)$ *are satisfied, we have*

$$\int_0^{\infty}K_n^{(\nu)}\left(x,t,u\right)dx=\left(-1\right)^{\nu}\frac{\nu!}{\left(n-m\right)u_1^{\nu+1}}.$$

Proof. Using partial integration and condition $2.$, we have

$$\int_0^{\infty}K_n^{(\nu)}\left(x,t,u\right)dx=-\int_0^{\infty}x\frac{d}{dx}K_n^{(\nu)}\left(x,t,u\right)dx.$$

Using Lemma 5.3, we get

$$\int_0^{\infty}K_n^{(\nu)}\left(x,t,u\right)dx=-\nu\int_0^{\infty}K_n^{(\nu)}\left(x,t,u\right)dx+nu_1\int_0^{\infty}xK_{n+m}^{(\nu)}\left(x,t,u\right)dx.$$

Using condition $4.$, we have

$$\int_0^{\infty}K_n^{(\nu)}\left(x,t,u\right)dx=-\nu\int_0^{\infty}K_n^{(\nu)}\left(x,t,u\right)dx-u_1\int_0^{\infty}K_n^{(\nu+1)}\left(x,t,u\right)dx.$$

We can write

$$\int_0^{\infty}K_n^{(\nu)}\left(x,t,u\right)dx=\frac{-u_1}{\nu+1}\int_0^{\infty}K_n^{(\nu+1)}\left(x,t,u\right)dx.$$

By ν- times application of above equality and using condition $a)$ and $b)$, we get

$$\begin{aligned}\int_0^{\infty}K_n^{(\nu)}\left(x,t,u\right)dx &= -\frac{\nu}{u_1}\int_0^{\infty}K_n^{(\nu-1)}\left(x,t,u\right)dx\\ &\;\vdots\\ &= \left(-1\right)^{\nu}\frac{\nu!}{u_1^{\nu}}\int_0^{\infty}K_n\left(x,t,u\right)\big|_{u=u_1,t=0}dx\\ &= \frac{\left(-1\right)^{\nu+1}\nu!}{\left(n-m\right)u_1^{\nu+1}}\int_0^{\infty}\frac{d}{dx}K_{n-m}\left(x,t,u\right)\big|_{u=u_1,t=0}dx\\ &= \left(-1\right)^{\nu}\frac{\nu!}{\left(n-m\right)u_1^{\nu+1}}.\end{aligned}\tag{5.21}$$

□

5.3. **Moments of Ibragimov-Gadjiev-Durrmeyer operators.** Firstly, we give the moments of Ibragimov-Gadjiev operators.

Lemma 5.4. *Let* $\nu,n\in\mathbb{N}$. *For any natural number* r, *we have*

$$\int_0^{\infty}x^rK_n^{(\nu)}\left(x,t,u\right)dx=\frac{\left(-1\right)^{\nu}\left(\nu+r\right)!}{\left(n-m\right)\left(n-2m\right)...\left(n-pm\right)\left(n-\left(r+1\right)m\right)u_1^{\nu+r+1}}.\tag{5.22}$$

Proof. Using the condition (4) recursively ν- times, we get

$$\begin{aligned}\int_0^\infty x^r K_n^{(\nu)}(x,t,u)\,dx &= -\frac{1}{n-m}\int_0^\infty x^{r-1}K_{n-m}^{(\nu+1)}(x,t,u)\,dx.\\ &=\frac{1}{(n-m)(n-2m)}\int_0^\infty x^{r-2}K_{n-2m}^{(\nu+2)}(x,t,u)\,dx\\ &\vdots\\ &=\frac{(-1)^r}{(n-m)(n-2m)\dots(n-rm)}\int_0^\infty K_{n-rm}^{(\nu+r)}(x,t,u)\,dx.\end{aligned}$$

Using (5.21), we have

$$\begin{aligned}&\int_0^\infty x^p K_n^{(\nu)}(x,t,u)\,dx\\ =&\frac{(-1)^p}{(n-m)(n-2m)\dots(n-pm)}\frac{(-1)^{\nu+p}(\nu+p)!}{(n-(p+1)m)\,u_1^{\nu+p+1}}\\ =&\frac{(-1)^{\nu}(\nu+p)!}{(n-m)(n-2m)\dots(n-pm)(n-(p+1)m)\,u_1^{\nu+p+1}}.\end{aligned}$$

□

Lemma 5.5. *Let $\nu, n\in\mathbb{N}$. For any natural number r, we have*

$$\begin{aligned}M_n(t^r;x) &= \frac{n^{2r}}{(n-2m)\dots(n-pm)(n-(r+1)m)(\alpha_n)^r(n^2\psi_n(0))^r}\\ &\times\sum_{j=0}^{r} n(n+m)\dots(n+(j-1)m)\,C_{j,r}\,[\alpha_n\psi_n(0)]^j x^j,\end{aligned}$$

where $C_{j,r}=\frac{r!}{j!}\binom{r}{j}$. Also,

(5.23)
$$M_n(1;x)=1,\quad M_n(t;x)=\frac{n^2}{(n-2m)\alpha_n}\left(\frac{\alpha_n}{n}x+\frac{1}{n^2\psi_n(0)}\right),$$

(5.24)
$$M_n(t^2;x)=\frac{n^4}{(n-2m)(n-3m)\alpha_n^2}\left(\left(\frac{\alpha_n}{n}x\right)^2\frac{(m+n)}{n}+\frac{\alpha_n}{n}\frac{4}{n^2\psi_n(0)}x+\frac{2}{(n^2\psi_n(0))^2}\right).$$

Proof. Using (5.19), we obtain

$$\begin{aligned}M_n(t^r;x) &= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,t,u)\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\\ &\times\int_0^\infty y^r K_n^{(\nu)}(y,t,u)\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy.\end{aligned}$$

Using (5.22), we have

$$M_n(t^r;x) = (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty} K_n^{(\nu)}(x,t,u)\,\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}$$
$$\times \frac{(-1)^{\nu}(\nu+r)!}{(n-m)(n-2m)\dots(n-rm)(n-(r+1)m)(\alpha_n\psi_n(0))^{\nu+r+1}}\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}.$$
$$= \sum_{\nu=0}^{\infty} K_n^{(\nu)}(x,t,u)\,\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}$$
$$\times \frac{1}{(n-2m)\dots(n-rm)(n-(r+1)m)(\alpha_n\psi_n(0))^{r}}(\nu+r)\dots(\nu+1),$$

where

$$(\nu+r)\dots(\nu+1) = \sum_{j=0}^{r} C_{j,r}\prod_{l=0}^{j-1}(\nu-l)$$

and $C_{j,r} = \frac{r!}{j!}\binom{r}{j}$. Using (5.21), we have

$$M_n(t^r;x) = \sum_{\nu=0}^{\infty} K_n^{(\nu)}(x,t,u)\,\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}$$
$$\times \frac{1}{(n-2m)\dots(n-rm)(n-(r+1)m)(\alpha_n\varphi_n(0))^{r}}\sum_{j=0}^{r} C_{j,r}\prod_{l=0}^{j-1}(\nu-l)$$
$$= \frac{1}{(n-2m)\dots(n-rm)(n-(r+1)m)(\alpha_n\psi_n(0))^{r}}$$
$$\times \sum_{j=0}^{r} C_{j,r}\sum_{\nu=0}^{\infty}\prod_{l=0}^{j-1}(\nu-l)\,K_n^{(\nu)}(x,t,u)\,\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}$$
$$= \frac{1}{(n-2m)\dots(n-rm)(n-(r+1)m)(\alpha_n\psi_n(0))^{r}}$$
$$\times \sum_{j=0}^{r} C_{j,r}\sum_{\nu=j}^{\infty} K_n^{(\nu)}(x,t,u)\,\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu-j)!}$$
$$= \frac{1}{(n-2m)\dots(n-rm)(n-(r+1)m)\,u_1^{r+1}}$$
$$\times \sum_{j=0}^{r} C_{j,r}x^{j}\sum_{\nu=0}^{\infty} K_n^{(\nu)}(x,t,u)\,\frac{(-1)^{j}[-\alpha_n\psi_n(0)]^{\nu+j}}{(\nu)!}.$$

Hence

$$M_n(t^r;x) = \frac{n^{2r}}{(n-2m)\dots(n-rm)(n-(r+1)m)(\alpha_n)^{r}(n^2\psi_n(0))^{r}}$$
$$\times \sum_{j=0}^{r} n(n+m)\dots(n+(j-1)m)\,C_{j,r}\,[\alpha_n\psi_n(0)]^{j}\,x^{j}.$$

□

Lemma 5.6. *For each $x \geq 0$ and $n > 3m$, we have*

(i) $M_n(t-x;x)=\frac{2mx}{(n-2m)}+\frac{1}{(n-2m)\alpha_n\psi_n(0)}$,
(ii) $M_n\left((t-x)^2;x\right)=x^2\left[\frac{m(2n+6m)}{(n-2m)(n-3m)}\right]+\frac{(2n+6m)\alpha_n\psi_n(0)x+2}{(n-2m)(n-3m)\alpha_n^2\psi_n^2(0)}$,
(iii) $M_n\left((t-x)^r;x\right)=\mathcal{O}\left((n\alpha_n\psi_n(0))^{-\left[\frac{r+1}{2}\right]}\right)$, where $[.]$ is integral part of $(r+1)/2$.

Proof. Proof is clear from the Lemma 5.5. □

Lemma 5.7. *For each $x\geq 0$ and $n>3m$, we have:*

$$M_n\left((t-x)^2;x\right)\leq\frac{C}{(n-2m)\alpha_n\psi_n(0)}\left[\varphi^2(x)+\frac{1}{(n+3m)\alpha_n\psi_n(0)}\right], \tag{5.25}$$

where $\varphi(x):=\sqrt{x(1+xm\alpha_n\psi_n(0))}$ *and* $C=\sup_{n\in\mathbb{N}}\left\{\frac{2n+6m}{(n-3m)}\right\}$.

Proof. If we consider the equalities (5.23) and (5.24), we can write

$$\begin{aligned}
&M_n\left((t-x)^2;x\right)\\
=\;&\frac{n^4}{(n-2m)(n-3m)\alpha_n^2}\left(\left(\frac{\alpha_n}{n}x\right)^2\frac{(m+n)}{n}+\frac{\alpha_n}{n}\frac{4}{n^2\psi_n(0)}x+\frac{2}{(n^2\psi_n(0))^2}\right)\\
-2x\;&\left[\frac{n^2}{(n-2m)\alpha_n}\left(\frac{\alpha_n}{n}x+\frac{1}{n^2\psi_n(0)}\right)\right]+x^2\\
=\;&x^2\left[\frac{n(m+n)}{(n-2m)(n-3m)}-\frac{2n}{(n-2m)}+1\right]\\
+x\;&\left[\frac{4n}{(n-2m)(n-3m)\alpha_n\psi_n(0)}-\frac{2}{(n-2m)\alpha_n\psi_n(0)}\right]\\
+\;&\frac{2}{(n-2m)(n-3m)\alpha_n^2\psi_n^2(0)}\\
=\;&x^2\left[\frac{m(2n+6m)\alpha_n\psi_n(0)}{(n-2m)(n-3m)\alpha_n\psi_n(0)}\right]+x\left[\frac{2n+6m}{(n-2m)(n-3m)\alpha_n\psi_n(0)}\right]\\
+\;&\frac{2}{(n-2m)(n-3m)\alpha_n^2\psi_n^2(0)}\\
=\;&\left[\frac{(2n+6m)}{(n-2m)(n-3m)\alpha_n\psi_n(0)}\right]\left[x(1+xm\alpha_n\psi_n(0))+\frac{1}{(n+3m)\alpha_n\psi_n(0)}\right]
\end{aligned}$$

and if we choose $C=\sup_{n\in\mathbb{N}}\left\{\frac{2n+6m}{(n-3m)}\right\}$, we have

$$M_n\left((t-x)^2;x\right)\leq\frac{C}{(n-2m)\alpha_n\psi_n(0)}\left[\varphi^2(x)+\frac{1}{(n+3m)\alpha_n\psi_n(0)}\right],$$

which is desired. □

6. VORONOVSKAYA TYPE RESULTS

This section will be dedicated to results on Voronovskaya type theorems. Let $\rho_1(x)=1+x^2$ and $\rho_2(x)=1+x^4$. From (5.23) and (5.24), we have

$$\sup_{x\in\mathbb{R}^+}\frac{|M_n(\rho_1;x)|}{\rho_2(x)}\leq 1+\frac{n^4}{(n-2m)(n-3m)\alpha_n^2}\left(\left(\frac{\alpha_n}{n}\right)^2\frac{(m+n)}{n}+\frac{\alpha_n}{n}\frac{4}{n^2\psi_n(0)}+\frac{2}{(n^2\psi_n(0))^2}\right).$$

Since the right hand side of above inequality tends to zero, we get $M_n : C_{\rho_1} \to C_{\rho_2}$ and

$$\lim_{n\to\infty} \left\| M_n\left(t^v; x\right) - x^v \right\|_{\rho_2} = 0,\ v = 0, 1, 2.$$

Then, we have:

Theorem 6.16. *For all $f \in C_{\rho_1}(\mathbb{R}^+)$,*

$$\lim_{n\to\infty} \left\| M_n\left(f\right) - f \right\|_{\rho_2} = 0.$$

Theorem 6.17. *Let $f \in C_{\rho_1}\left(\mathbb{R}^+\right)$. Suppose that the first and second derivative $f'$and f'' exist at a point $x \in [0, \infty)$, then we have*

$$\lim_{n\to\infty} n\alpha_n\psi_n\left(0\right)\left[M_n\left(f;x\right) - f\left(x\right)\right] = \left(2xml_1 + 1\right) f'\left(x\right) + x\left(xml_1 + 1\right) f''\left(x\right).$$

Theorem 6.18. *For all $f \in C_{\rho_1}\left(\mathbb{R}^+\right)$, we have*

$$\left\| M_n\left(f\right) - f \right\|_{\rho_2} \le 8\left(1 + \sqrt{A}\right)\Omega_{\rho_1}\left(f; \sqrt{\frac{C}{\left(n-2m\right)\alpha_n\psi_n\left(0\right)}\left[\alpha_n\psi_n\left(0\right) + \frac{1}{\left(n+3m\right)\alpha_n\psi_n\left(0\right)}\right]}\right)_{\mathbb{R}^+}.$$

Proof. From the definitions of the operators (5.19), we have

$$\left| M_n\left(f;x\right) - f\left(x\right) \right| \le 8\rho_1\left(x\right)\left(1 + \frac{1}{\delta^2} M_n\left(\left(\rho_1\left(t\right) - \rho_1\left(x\right)\right)^2; x\right)\right)\Omega_{\rho_1}\left(f;\delta\right)_{\mathbb{R}^+}$$

and $M_n\left(\left(\rho_1\left(t\right) - \rho_1\left(x\right)\right)^2; x\right) \le \left[M_n\left(\left(t-x\right)^2; x\right)\right]^{1/2}\left[M_n\left(\left(t+x\right)^2; x\right)\right]^{1/2}$. Then, we get

$$\text{(6.26)}\quad \frac{M_n\left(\left(t-x\right)^2; x\right)}{\sqrt{\rho_2\left(x\right)}} \le \frac{C}{\left(n-2m\right)\alpha_n\psi_n\left(0\right)}\left[\frac{\varphi^2\left(x\right)}{\sqrt{\rho_2\left(x\right)}} + \frac{1}{\left(n+3m\right)\alpha_n\psi_n\left(0\right)}\right] \le \frac{C}{\left(n-2m\right)\alpha_n\psi_n\left(0\right)}\left[\alpha_n\psi_n\left(0\right) + \frac{1}{\left(n+3m\right)\alpha_n\psi_n\left(0\right)}\right]$$

and

(6.27)

$$\frac{M_n\left(\left(t+x\right)^2; x\right)}{\sqrt{\rho_2\left(x\right)}} \le \frac{n^4}{\left(n-2m\right)\left(n-3m\right)\alpha_n^2}\left(\left(\frac{\alpha_n}{n}\right)^2\frac{\left(m+n\right)}{n} + \frac{\alpha_n}{n}\frac{4}{n^2\psi_n\left(0\right)} + \frac{2}{\left(n^2\psi_n\left(0\right)\right)^2}\right) + 2\left(\frac{n^2}{\left(n-2m\right)\alpha_n}\left(\frac{\alpha_n}{n} + \frac{1}{n^2\psi_n\left(0\right)}\right)\right) + 1,$$

where $\varphi\left(x\right) := \sqrt{x\left(1 + xm\alpha_n\psi_n\left(0\right)\right)}$ and $C = \sup_{n\in\mathbb{N}}\left\{\frac{2n+6m}{\left(n-3m\right)}\right\}$. The right hand side of (6.27) is convergent, it follows that $\exists\ A > 0$ such that

$$\frac{M_n\left(\left(t+x\right)^2; x\right)}{\sqrt{\rho_2\left(x\right)}} < A$$

for all $n \in \mathbb{N}$. Choosing $\delta = \sqrt{\frac{C}{\left(n-2m\right)\alpha_n\psi_n\left(0\right)}\left[\alpha_n\psi_n\left(0\right) + \frac{1}{\left(n+3m\right)\alpha_n\psi_n\left(0\right)}\right]}$, we obtain

$$\frac{\left| M_n\left(f;x\right) - f\left(x\right) \right|}{\rho_2\left(x\right)} \le 8\left(1 + \sqrt{A}\right)\Omega_{\rho_1}\left(f;\delta\right)_{\mathbb{R}^+}.$$

It completes the proof. ☐

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Ali Aral
Kirikkale University
Department of Mathematics
71450, Yahsihan, Kirikkale, Turkey
Email address: aliaral73@yahoo.com