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Research Article

On the source problem for the diffusion equations with conformable derivative

NGHIEM THI VAN ANH, VU ANH TUAN, LE DINH LONG, AND DOAN VUONG NGUYEN*

ABSTRACT. In this article, we are interested in the problem of finding the source function of the diffusion equations $\partial_t^\alpha - \Delta u = f(x)$, where f as the unknown source function and $\alpha \in (0,1)$. Furthermore, the fractional derivative α of u is defined by the conformable time derivative. This is an ill-posed problem. So, we use the regularized Tikhonov method to construct a regularization solution, and the estimation of convergence is also discussed.

Keywords: Conformable derivative, ill-posed, source function, diffusion equations.

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1. INTRODUCTION

Set $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is bounded domain. We regard the problem

$$\partial_t^\alpha(x,t) - \Delta u(x,t) = f(x), x \in \Omega, \tag{1.1}$$

where $u(x,t)$ satisfies

$$\begin{cases} u(x,t)|_{x\in\partial\Omega} = 0, & t \in (0,T) \\ u(x,0) = u_0(x), & x \in \Omega \\ u(x,T) = g(x), & x \in \Omega \end{cases},$$

∂_t^α is denote of conformable time derivative with $\alpha \in (0,1)$ (see Khalil [6]). With the given function $G : [0,\infty) \to \mathbb{R}$, the conformable derivative with order $\alpha \in (0,1)$ is defined by

$$\partial_t^\alpha G(t) = \lim_{\rho\to 0} \frac{G(t+\rho t^{1-\alpha}) - G(t)}{\rho} \tag{1.2}$$

for all $t > 0$. With $(0,t_0), t_0 > 0$, and exist $\lim_{t\to t_0^+} \partial_t^\alpha G(t)$ then

$$\partial_t^\alpha G(t_0) = \lim_{t\to t_0^+} \partial_t^\alpha G(t).$$

Many models of practical problems with comfortable time derivatives are applied in real life .

There are many kinds of fractional derivatives such as Riemann-Liouville, Caputo, conformable, Grunwald-Letnikov, and so on. Problems with non-integer derivatives have received a lot of attention in recent years due to their good application flexibility [1, 2]. Our problem is ill-posed in the sense of Hadamard (the solution is not continuity on the data), so we need to construct a new approximation solution (is called the regularization solution). We emphasised that input data g is not known, and we only have information that g^δ satisfy $\|g - g^\delta\|_{L^2(\Omega)} \leq \delta$

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*Corresponding author: Doan Vuong Nguyen; doanvuongnguyen@iuh.edu.vn

with the error $\delta > 0$, although the input data error is small, the change of the solutions is large. In literature, there are many results of the problem (find source function of diffusion equation). The techniques used to establish the regularization of solution such as the quasi-reversibility method [11], Quasi-boundary value method [8], Landweber iterative regularization method [12, 13], fractional Landweber method [4], Tikhonov regularized method [10], Fourier truncation method [9]. In this work, our goal is to find the unknown source function of problem by using fractional Tikhonov method [7, 3].

2. Some Priminary Results

2.1. Some function spaces. Denote $\langle\cdot,\cdot\rangle$ inner product $L^2(\Omega)$. There are exists an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ ($\varphi_j \in H_0^1(\Omega) \cap C^\infty(\Omega)$) of $L^2(\Omega)$ satisfy

$$\Delta\varphi_j(x) = -\lambda_j\varphi_j(x), \quad x \in \Omega,$$

where $\{\lambda_j\}_{j=1}^{\infty}$ is set of eigenvalues of Δ satisfy

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \ldots$$

and $\lim_{j\to\infty}\lambda_j = \infty$. Furthermore, for each $m \ge 0$, we defined the space

$$H^m(\Omega) = \Big\{u \in L^2(\Omega) : \sum_{j=1}^{\infty} \lambda_j^{2m} \big|\langle u, \varphi_j\rangle\big|^2 < +\infty\Big\},$$

so $H^m(\Omega)$ is Hilbert space equipped with the norm

$$\|u\|_{H^m(\Omega)} = \Big(\sum_{j=1}^{\infty} \lambda_j^{2m} |(u,\varphi_j)|^2\Big)^{\frac{1}{2}}.$$

2.2. Source function formular. In this section, we introduce a mild solution of initial problem

$$\begin{cases} \partial_t^\alpha u(x,t) - \Delta u(x,t) = f(x), & x \in \Omega \\ u(x,t) = 0, & x \in \partial\Omega, t \in (0,T] \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}. \tag{2.3}$$

Use the method of separation of variables to find the solution of (2.3). We assume that u has Fourier expansions

$$u(x,t) = \sum_{j=1}^{\infty} u_j(t)\varphi_j(x), \quad u_j(t) = \langle u(\cdot,t), \varphi_j\rangle. \tag{2.4}$$

From this, (2.4), we have

$$u_j(t) = \sum_{j=1}^{\infty} \Big[\exp\big(-\lambda_j t^\alpha \alpha^{-1}\big)u_{0,j} + \langle f, \varphi_j\rangle \int_0^t s^{\alpha-1} \exp\big(-\lambda_j\big(t^\alpha - s^\alpha\big)\alpha^{-1}\big)ds\Big]\varphi_j(x).$$

Set $t = T$ and $u_{0,j} = 0$, we take

$$g_j(x) = u_j(T) = \sum_{j=1}^{\infty} \Big[\langle f, \varphi_j\rangle \int_0^T s^{\alpha-1} \exp\big(-\lambda_j(T^\alpha - s^\alpha)\alpha^{-1}\big)ds\Big]\varphi_j(x). \tag{2.5}$$

From (2.5), we have

$$f(x)=\sum_{j=1}^{\infty}\frac{\langle g,\varphi_j\rangle}{\int\limits_0^T s^{\alpha-1}\exp\big(-\lambda_j(T^\alpha-s^\alpha)\alpha^{-1}\big)ds}.$$

Hence

$$f(x)=\sum_{j=1}^{\infty}\frac{\langle g,\varphi_j\rangle\varphi_j(x)}{\int\limits_0^T s^{\alpha-1}\exp\big(-\lambda_j(T^\alpha-s^\alpha)\alpha^{-1}\big)ds}.$$

2.3. **Ill-posedness.** We define the linear operator $T: L^2(\Omega)\to L^2(\Omega):$

$$\begin{aligned}Tf(x)&=\sum_{j=1}^{\infty}\Big[\int\limits_0^T s^{\alpha-1}\exp\big(-\lambda_j(T^\alpha-s^\alpha)\alpha^{-1}\big)ds\Big]\langle f,\varphi_j\rangle\varphi_j(x)\\ &=\int\limits_{\Omega}K(x,\xi)f(\xi)d\xi,\end{aligned}\tag{2.6}$$

here,

$$K(x,\xi)=\sum_{j=1}^{\infty}\Big(\int\limits_0^T s^{\alpha-1}\exp\big(-\lambda_j(T^\alpha-s^\alpha)\alpha^{-1}\big)ds\Big)\varphi_j(x)\varphi_j(\xi).$$

Since $K(x,\xi)=K(\xi,x)$, the T is adjoint operator. Next, we shall show T is compact. Assuming T_N is defined by

$$T_Nf(x)=\sum_{j=1}^{N}\Big(\int\limits_0^T s^{\alpha-1}\exp\big(-\lambda_j(T^\alpha-s^\alpha)\alpha^{-1}\big)ds\Big)\langle f,\varphi_j\rangle\varphi_j(x).\tag{2.7}$$

It is easy to show that T_N is bounded, finite dimension operator. Moreover, from (2.6) and (2.7), we have

$$\big\|T_Nf-Tf\big\|^2_{L^2(\Omega)}=\sum_{j=N+1}^{\infty}\Big(\int\limits_0^T s^{\alpha-1}\exp\big(-\lambda_j(T^\alpha-s^\alpha)\alpha^{-1}\big)ds\Big)^2\big|\langle f,\varphi_j\rangle\big|^2.\tag{2.8}$$

With

$$\mathcal{V}_\alpha=\int\limits_0^{T^\alpha} s^{\alpha-1}\exp\big(-\lambda_j(T^\alpha-s^\alpha)\alpha^{-1}\big)ds,$$

by setting $s^\alpha=\omega$, and using the method of substitution, we obtain

$$\mathcal{V}_\alpha=\frac{1}{\alpha}\int\limits_0^{T^\alpha}\exp\big(-\lambda_j(T^\alpha-s^\alpha)\alpha^{-1}\big)ds\le\frac{1}{\lambda_j}\Big(1-\exp\big(-\lambda_jT^\alpha\alpha^{-1}\big)\Big)\le\frac{1}{\lambda_j}.\tag{2.9}$$

Combining (2.8) and (2.9), we have

$$\big\|T_Nf-Tf\big\|^2_{L^2(\Omega)}\le\sum_{j=N+1}^{\infty}\frac{1}{\lambda_j^2}\big|\langle f,e_j\rangle\big|^2.$$

This means that

$$\big\|T_N f - Tf\big\|_{L^2(\Omega)} \le \frac{1}{\lambda_N}\|f\|_{L^2(\Omega)}.$$

So, $\|T_N - T\|_{L^2(\Omega)} \to 0$ when $N \to \infty$. Hence, T is compact.

Theorem 2.1. (1.1) *is ill-posed.*

Proof. The singular values of linearity compact self adjoint operator T is

$$\Lambda_j = \int_0^T s^{\alpha-1}\exp\big(-\lambda_j(T^\alpha - s^\alpha)\alpha^{-1}\big)ds.$$

On $L^2(\Omega)$, a set of all eigenvalues φ_j is an orthonormal basis. In view of (2.6), we rewrite (1.1) such as

$$Tf(x) = g(x).$$

From the results of Kirsch ([5]), we conclude (1.1) is ill-posed. With $g^k = \frac{\varphi_k}{\sqrt{\lambda_k}}$, we have the source function is

$$f^k(x) = \sum_{j=1}^{\infty}\frac{\langle g^k, \varphi_j\rangle\varphi_j(x)}{\int_0^T s^{\alpha-1}\exp\big(-\lambda_j(T^\alpha - s^\alpha)\alpha^{-1}\big)ds}$$
$$= \frac{\varphi_k(x)}{\sqrt{\lambda_k}\int_0^T s^{\alpha-1}\exp\big(-\lambda_k(T^\alpha - s^\alpha)\alpha^{-1}\big)ds}.$$

If the latter output data is $g = 0$ then $f = 0$, g and g^k, so we have the estimation:

$$\|g^k - g\|_{L^2(\Omega)} = \Big\|\frac{\varphi_k}{\sqrt{\lambda_k}}\Big\|_{L^2(\Omega)} = \frac{1}{\sqrt{\lambda_k}}$$

this derive

$$\lim_{k\to+\infty}\|g^k - g\|_{L^2(\Omega)} = \lim_{k\to+\infty}\frac{1}{\sqrt{\lambda_k}} = 0. \tag{2.10}$$

The error estimator for f^k and f:

$$\big\|f^k - f\big\|_{L^2(\Omega)} = \frac{1}{\sqrt{\lambda_k}}\Bigg(\int_0^T s^{\alpha-1}\exp\big(-\lambda_k(T^\alpha - s^\alpha)\alpha^{-1}\big)ds\Bigg)^{-1}$$
$$\ge \frac{1}{\sqrt{\lambda_k}}\Bigg(\int_0^T s^{\alpha-1}\exp\big(-\lambda_k(T^\alpha - s^\alpha)\alpha^{-1}\big)ds\Bigg)^{-1} = \sqrt{\lambda_k}. \tag{2.11}$$

So, we have $\|f^k - f\|_{L^2(\Omega)} \ge \sqrt{\lambda_k}$. And it follows

$$\lim_{k\to+\infty}\big\|f^k - f\big\|_{L^2(\Omega)} > \lim_{k\to+\infty}\sqrt{\lambda_k} = +\infty.$$

Combining (2.10) , (2.12), we have conclusion . ☐

2.4. Stability condition required for the source function.

Theorem 2.2. *For $s > 0$ and suppose that $f \in H^m(\Omega)$, then*

$$\|f\|_{L^2(\Omega)} \leq C(m, \|f\|_{H^m(\Omega)}) \|g\|_{L^2(\Omega)}^{\frac{m}{m+1}},$$

here,

$$C(m, \|f\|_{H^m(\Omega)}) = \Big(\big|1 - \exp(-\lambda_1 T^\alpha \alpha^{-1})\big| \Big)^{-\frac{m}{m+1}} \|f\|_{H^m(\Omega)}{}^{\frac{1}{m+1}}. \tag{2.12}$$

Proof. By applying Hölder inequality, and in short, we denote,

$$B(\lambda_j, \alpha) = \int_0^T s^{\alpha-1} \exp\big(-\lambda_j (T^\alpha - s^\alpha) \alpha^{-1} \big) ds,$$

and we have

$$\begin{aligned}
\|f\|_{L^2(\Omega)}^2 &= \sum_{j=1}^\infty \Big| \frac{\langle g, \varphi_j \rangle}{B(\lambda_j, \alpha)} \Big|^2 = \sum_{j=1}^\infty \frac{\big|\langle g, \varphi_j \rangle\big|^{\frac{2}{m+1}} \big|\langle g, \varphi_j \rangle\big|^{\frac{2m}{m+1}}}{\big|B(\lambda_j, \alpha)\big|^2} \\
&\leq \Big[\sum_{j=1}^\infty \frac{\big|\langle g, \varphi_j \rangle\big|^2}{\big|B(\lambda_j, \alpha)\big|^{2m+2}} \Big]^{\frac{1}{m+1}} \Big[\sum_{j=1}^\infty \big|\langle g, \varphi_j \rangle\big|^2 \Big]^{\frac{m}{m+1}} \\
&\leq \Big[\sum_{j=1}^\infty \frac{\big|\langle f, \varphi_j \rangle\big|^2}{\big|B(\lambda_j, \alpha)\big|^{2m}} \Big]^{\frac{1}{m+1}} \|g\|_{L^2(\Omega)}^{\frac{2m}{m+1}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\big|B(\lambda_j, \alpha)\big|^{2m} &\geq^{2m} \big|B(\lambda_j, \alpha)\big|^{2m} \\
&\geq^{2m} |\lambda_j|^{-2m} \big|1 - \exp\big(-\lambda_j T^\alpha \alpha^{-1} \big)\big|^{2m},
\end{aligned}$$

this implies that

$$\sum_{j=1}^\infty \frac{\big|\langle f, \varphi_j \rangle\big|^2}{\big|B(\lambda_j, \alpha)\big|^{2m}} \leq \sum_{j=1}^\infty \frac{\lambda_j^{2m} \big|\langle f, \varphi_j \rangle\big|^2}{{}^{2m}\big|1 - \exp(-\lambda_1 T^\alpha \alpha^{-1})\big|^{2m}}.$$

Therefore,

$$\begin{aligned}
\|f\|_{L^2(\Omega)}^2 &\leq \Big(\big|1 - \exp(-\lambda_1 T^\alpha \alpha^{-1})\big| \Big)^{-\frac{2m}{m+1}} \|f\|_{H^m(\Omega)}^{\frac{2}{m+1}} \|g\|_{L^2(\Omega)}^{\frac{2m}{m+1}} \\
&\leq [C(m, \|f\|_{H^m(\Omega)})]^2 \|g\|_{L^2(\Omega)}^{\frac{2m}{m+1}}.
\end{aligned}$$

□

3. Fractional Tikhonov Method

In order to regularize the solution of the problem with input data $g \in L^2(\Omega)$, we shall minimize the functional

$$\min_{f \in L^2(\Omega)} J_{\epsilon(\delta)}(f) = \|Kf - g\|_\xi^2 + \epsilon(\delta) \|f\|_{L^2(\Omega)}^2,$$

where $\epsilon(\delta)$ is regularization parameter, $\|\cdot\|_\xi$ is semi-norm with the weighted and defined by $\|v\|_\xi = \left\|W^{\frac{1}{2}}v\right\|_{L^2(\Omega)}$ for all v and $W = (K^\star K)^{\xi-1}$ $(1/2 \le \xi < 1)$. This minimized problem has a unique solution to satisfy

$$\Big((K^\star K)^{\epsilon(\delta)} + \xi I\Big) f = (K^\star K)^{\xi-1}K^\star g.$$

By using the singular value decomposition of the compact self adjoint operator, we have

$$f_{\epsilon(\delta)}(x) = \sum_{j=1}^{\infty} \frac{\left|B(\lambda_j,\alpha)\right|^{2\xi-1}}{\left[\epsilon(\delta)\right]^2 + \left|B(\lambda_j,\alpha)\right|^{2\xi}} \langle g,\varphi_j\rangle \varphi_j(x), \quad \frac{1}{2} \le \xi \le 1. \tag{3.13}$$

Furthermore, if the measurement data of g is g^δ with erros δ then

$$f^\delta_{\epsilon(\delta)}(x) = \sum_{j=1}^{\infty} \frac{\left|B(\lambda_j,\alpha)\right|^{2\xi-1}}{[\epsilon(\delta)]^2 + \left|B(\lambda_j,\alpha)\right|^{2\xi}} \langle g^\delta(x),\varphi_j(x)\rangle \varphi_j(x), \quad \frac{1}{2} \le \xi \le 1, \tag{3.14}$$

here $\epsilon(\delta)$ is regularization parameter.

The following theorem obtain the estimation for $\|f - f^\delta_{\epsilon(\delta)}\|_{L^2(\Omega)}$ and the order of convergences with suitable regularization parameter. First, we need the lemma.

Lemma 3.1. *For $z \ge \lambda_1$ and $\frac{1}{2} \le \xi \le 1$, then*

$$G_1(z) = \frac{z}{A^{2\xi} + \epsilon z^{2\xi}} \le \overline{B}(\xi, A)\epsilon^{-\frac{1}{2\xi}}, \tag{3.15}$$

here $\overline{B}(\xi, A)$ independent of ϵ, z.

Proof. For $\frac{1}{2} < \xi < 1$, from (3.15), we take the solution of $G_1'(z) = 0$, we get

$$z_0 = A(2\xi - 1)^{-\frac{1}{2\xi}}\epsilon^{-\frac{1}{2\xi}}.$$

And by z_0 , the equation (3.15), has

$$G_1(z) \le G_1(z_0) \le \epsilon^{-\frac{1}{2\xi}} \left(\frac{A^{1-2\xi}(2\xi-1)^{-\frac{1}{2\xi}}}{2\xi} \right).$$

□

Lemma 3.2. *For $z \ge \lambda_1$, $\frac{1}{2} \le \xi \le 1$, then*

$$G_2(z) = \frac{\epsilon^2 z^{2\xi-m}}{A^{2\xi} + \epsilon^2 z^{2\xi}} \le \begin{cases} (2\xi)^{-1}\Big((2\xi-m)^{\frac{2\xi-m}{2\xi}}A^{-m}m^{\frac{m}{2\xi}}\Big)\epsilon^{\frac{m}{\xi}}, & 0 < m < 2\xi \\ \left(A^{2\xi}\lambda_1^{m-2\xi}\right)^{-1}\epsilon^2, & m \ge 2\xi \end{cases}.$$

Proof. This result as the [7], we omit here. □

Theorem 3.3. *Assume that $f \in H^m(\Omega)$ and $g \in L^2(\Omega)$ then*

$$\begin{aligned} \|f - f^\delta_{\epsilon(\delta)}\|_{L^2(\Omega)} &\le C\delta[\epsilon(\delta)]^{-\frac{1}{\xi}} \\ &+ \begin{cases} (2\xi)^{-1}\big((2\xi-m)^{\frac{2\xi-m}{2\xi}}A^{-m}m^{\frac{m}{2\xi}}\big)\|f\|_{H^m(\Omega)}[\epsilon(\delta)]^{\frac{m}{\xi}}, & 0 < m < 2\xi \\ \left(A^{2\xi}\lambda_1^{m-2\xi}\right)^{-1}\|f\|_{H^m(\Omega)}[\epsilon(\delta)]^2, & m \ge 2\xi \end{cases} \end{aligned}$$

with $A = \left|\frac{1}{\lambda_j}\Big(1 - \exp\big(-\lambda_1 T^\alpha \alpha^{-1}\big)\Big)\right|$, C independent of $\epsilon(\delta)$.

Remark 3.1. *This theorem suggests the choice of the following regularization parameter*

- *If* $0 < m < 2\xi$, *we can choose* $\epsilon(\delta) = \Big(\dfrac{\delta}{\|f\|_{H^m(\Omega)}}\Big)^{\frac{\xi}{m+2}}$ *then*

$$\big\|f - f^{\delta}_{\epsilon(\delta)}\big\|_{L^2(\Omega)} \text{ is convergences with order } \delta^{\frac{m}{m+2}}.$$

- *If* $m \geq 2\xi$, *we can choose* $\epsilon(\delta) = \Big(\dfrac{\delta}{\|f\|_{H^m(\Omega)}}\Big)^{\frac{\xi}{\xi+1}}$ *then*

$$\big\|f - f^{\delta}_{\epsilon(\delta)}\big\|_{L^2(\Omega)} \text{ is convergences with order } \delta^{\frac{\xi}{\xi+2}}.$$

Proof. Applying triangle inequality, we have the result

$$\|f - f^{\delta}_{\epsilon(\delta)}\|_{L^2(\Omega)} \leq \underbrace{\|f_{\epsilon(\delta)} - f^{\delta}_{\epsilon(\delta)}\|_{L^2(\Omega)}}_{\mathcal{A}_1} + \underbrace{\|f - f_{\epsilon(\delta)}\|_{L^2(\Omega)}}_{\mathcal{A}_2}$$

with

$$\mathcal{A}_1 = \sum_{j=1}^{\infty} \frac{\big|B(\lambda_j,\alpha)\big|^{2\xi-1}}{[\epsilon(\delta)]^2 + \big|B(\lambda_j,\alpha)\big|^{2\xi}} \langle g^{\delta}(x) - g(x), \varphi_j(x)\rangle \varphi_j(x),$$

$$\mathcal{A}_2 = \sum_{j=1}^{\infty} \left(\frac{\big|B(\lambda_j,\alpha)\big|^{2\xi-1}}{[\epsilon(\delta)]^2 + \big|B(\lambda_j,\alpha)\big|^{2\xi}} - \frac{1}{\big|B(\lambda_j,\alpha)\big|} \right) \langle g, \varphi_j\rangle \varphi_j(x). \tag{3.16}$$

Step 1: First, the estimation of $\|\mathcal{A}_1\|_{L^2(\Omega)}$ will be shown. From

$$\Big|B(\lambda_j,\alpha)\Big| = \left|\frac{1}{\lambda_j}\Big(1 - \exp\big(-\lambda_j T^{\alpha}\alpha^{-1}\big)\Big)\right| \leq \frac{1}{\lambda_j}$$

and the other hand

$$\Big|B(\lambda_j,\alpha)\Big| = \left|\frac{1}{\lambda_j}\Big(1 - \exp\big(-\lambda_j T^{\alpha}\alpha^{-1}\big)\Big)\right| \geq \left|\frac{1}{\lambda_j}\Big(1 - \exp\big(-\lambda_1 T^{\alpha}\alpha^{-1}\big)\Big)\right|,$$

it follows

$$\|\mathcal{Q}_1\|_{L^2(\Omega)} \leq \sum_{j=1}^{\infty} \left| \frac{\Big|\frac{1}{\lambda_j}\Big|^{2\xi-1}}{[\epsilon(\delta)]^2 + \Big|\frac{1}{\lambda_j}\Big(1 - \exp\big(-\lambda_1 T^{\alpha}\alpha^{-1}\big)\Big)\Big|^{2\xi}} \right|^2 \big\langle g^{\delta} - g, \varphi_j\big\rangle^2$$
$$\leq \sum_{j=1}^{\infty} \left| \frac{\lambda_j}{[\epsilon(\delta)]^2 \lambda_j^{2\xi} + \big|1 - \exp\big(-\lambda_1 T^{\alpha}\alpha^{-1}\big)\big|^{2\xi}} \right|^2 \big\langle g^{\delta} - g, \varphi_j\big\rangle^2.$$

Employing Lemma 3.1, we have

$$\|\mathcal{Q}_1\|_{L^2(\Omega)} \leq C\delta(\epsilon(\delta))^{-\frac{1}{2\xi}}. \tag{3.17}$$

Step 2: Next, we have estimates $\|\mathcal{A}_2\|_{L^2(\Omega)}$,

$$
\begin{aligned}
\|\mathcal{A}_2\|^2_{L^2(\Omega)} &\le \sum_{j=1}^{\infty}\left(\frac{\left|B(\lambda_j,\alpha)\right|^{2\xi-1}}{[\epsilon(\delta)]^2+\left|B(\lambda_j,\alpha)\right|^{2\xi}}-\frac{1}{\left|B(\lambda_j,\alpha)\right|}\right)^2\left|\langle g,\varphi_j\rangle\right|^2\\
&\le \sum_{j=1}^{\infty}\left(\frac{[\epsilon(\delta)]^2}{\left|B(\lambda_j,\alpha)\right|\left([\epsilon(\delta)]^2+\left|B(\lambda_j,\alpha)\right|^{2\xi}\right)}\right)^2\left|\langle g,\varphi_j\rangle\right|^2\\
&\le \sum_{j=1}^{\infty}\frac{[\epsilon(\delta)]^4}{\left([\epsilon(\delta)]^2+\left|B(\lambda_j,\alpha)\right|^{2\xi}\right)^2}\frac{\left|\langle g,\varphi_j\rangle\right|^2}{\left|B(\lambda_j,\alpha)\right|^2}\\
&\le \sum_{j=1}^{\infty}\frac{[\epsilon(\delta)]^4\lambda_j^{-2m}}{\left([\epsilon(\delta)]^2+\left|B(\lambda_j,\alpha)\right|^{2\xi}\right)^2}\frac{\lambda_j^{2m}\left|\langle g,\varphi_j\rangle\right|^2}{\left|B(\lambda_j,\alpha)\right|^2}. \qquad (3.18)
\end{aligned}
$$

Since the above estimation, we conclude

$$
\begin{aligned}
\|\mathcal{A}_2\|^2_{L^2(\Omega)} &\le \sup_{j\in\mathbb{N}}\left|G_2(\lambda_j)\right|^2\sum_{j=1}^{\infty}\frac{\lambda_j^{2m}\left|\langle g,\varphi_j\rangle\right|^2}{\left|B(\lambda_j,\alpha)\right|^2}\\
&= \sup_{j\in\mathbb{N}}\left|G_2(\lambda_j)\right|^2\|f\|_{H^m(\Omega)}
\end{aligned}
$$

with

$$
\begin{aligned}
G_2(\lambda_j) &= \frac{[\epsilon(\delta)]^2\lambda_j^{-m}}{[\epsilon(\delta)]^2+\left|B(\lambda_j,\alpha)\right|^{2\xi}}\\
&\le \frac{[\epsilon(\delta)]^2\lambda_j^{2\xi-m}}{[\epsilon(\delta)]^2\lambda_j^{2\xi}+\left|\frac{1}{\lambda_j}\left(1-\exp\left(-\lambda_1T^\alpha\alpha^{-1}\right)\right)\right|^{2\xi}}.
\end{aligned}
$$

Let $A=\left|\frac{1}{\lambda_j}\left(1-\exp\left(-\lambda_1T^\alpha\alpha^{-1}\right)\right)\right|$, applying the Lemma 3.2,

$$
G_2(\lambda_j) \le \begin{cases}(2\xi)^{-1}\left((2\xi-m)^{\frac{2\xi-m}{2\xi}}A^{-m}m^{\frac{m}{2\xi}}\right)[\epsilon(\delta)]^{\frac{m}{\xi}}, & 0<m<2\xi\\ \left(A^{2\xi}\lambda_1^{m-2\xi}\right)^{-1}[\epsilon(\delta)]^2, & m\ge 2\xi\end{cases}. \qquad (3.19)
$$

In view of (3.18) and (3.19), we show that

$$
\|\mathcal{A}_2\|_{L^2(\Omega)} \le \begin{cases}(2\xi)^{-1}\left((2\xi-m)^{\frac{2\xi-m}{2\xi}}A^{-m}m^{\frac{m}{2\xi}}\right)\|f\|_{H^m(\Omega)}[\epsilon(\delta)]^{\frac{m}{\xi}}, & 0<m<2\xi\\ \left(A^{2\xi}\lambda_1^{m-2\xi}\right)^{-1}\|f\|_{H^m(\Omega)}[\epsilon(\delta)]^2, & m\ge 2\xi\end{cases}.
$$

From Step 1 and Step 2, we get

$$
\begin{aligned}
\left\|f-f^{\delta}_{\epsilon(\delta)}\right\|_{L^2(\Omega)} &\le C\delta[\epsilon(\delta)]^{-\frac{1}{\xi}}\\
&\quad+\begin{cases}(2\xi)^{-1}\left((2\xi-m)^{\frac{2\xi-m}{2\xi}}A^{-m}m^{\frac{m}{2\xi}}\right)\|f\|_{H^m(\Omega)}[\epsilon(\delta)]^{\frac{m}{\xi}}, & 0<m<2\xi\\ \left(A^{2\xi}\lambda_1^{m-2\xi}\right)^{-1}\|f\|_{H^m(\Omega)}[\epsilon(\delta)]^2, & m\ge 2\xi\end{cases}.
\end{aligned}
$$

And the regularization parameter $\epsilon(\delta)$ is chosen:

$$\epsilon(\delta) = \begin{cases} \left(\dfrac{\delta}{\|f\|_{H^m(\Omega)}}\right)^{\frac{\xi}{m+2}}, & 0 < m < 2\xi \\ \left(\dfrac{\delta}{\|f\|_{H^m(\Omega)}}\right)^{\frac{\xi}{\xi+1}}, & m \geq 2\xi \end{cases}. \tag{3.20}$$

Thus, with ϵ is confirmed by (3.20), we have show that

Claim 1: If $0 < m \leq 2\xi$, then

$$\begin{aligned} \left\|f - f^{\delta}_{\epsilon(\delta)}\right\|_{L^2(\Omega)} \leq & \delta^{\frac{m}{m+2}} \Big[\delta^{\frac{1}{m+2}} \|f\|^{\frac{1}{m+2}}_{H^m(\Omega)} C \\ & + \|f\|^{\frac{m+1}{m+2}}_{H^m(\Omega)} (2\xi)^{-1} \left((2\xi - m)^{\frac{2\xi - m}{2\xi}} A^{-m} m^{\frac{m}{2\xi}}\right) \Big]. \end{aligned}$$

Claim 2: If $m > 2\xi$, then

$$\left\|f - f^{\delta}_{\epsilon(\delta)}\right\|_{L^2(\Omega)} \leq \delta^{\frac{\xi}{\xi+1}} \left[\|f\|^{\frac{1}{\xi+1}}_{H^m(\Omega)} C + \left(A^{2\xi}\right)^{-1} \delta^{\frac{\xi}{\xi+1}} \|f\|^{\frac{1-\xi}{\xi+1}}_{H^m(\Omega)} \right].$$

□

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NGHIEM THI VAN ANH
INDUSTRIAL UNIVERSITY OF HO CHI MINH CITY
FACULTY OF FUNDAMENTAL SCIENCE
HO CHI MINH CITY, VIETNAM
Email address: nghiemthivananh@iuh.edu.vn

VU ANH TUAN
DONG NAI TECHNOLOGY UNIVERSITY
FACULTY OF FUNDAMENTAL SCIENCES
BIEN HOA CITY, VIETNAM
Email address: vuanhtuan@dntu.edu.vn

LE DINH LONG
FPT UNIVERSITY HCM
FACULTY OF MATH
SAIGON HI-TECH PARK, THU DUC CITY, HO CHI MINH CITY, VIETNAM
Email address: longld13@fe.edu.vn

DOAN VUONG NGUYEN
INDUSTRIAL UNIVERSITY OF HO CHI MINH CITY
FACULTY OF FUNDAMENTAL SCIENCE
HO CHI MINH CITY, VIETNAM
Email address: doanvuongnguyen@iuh.edu.vn