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*Research Article*

# Trigonometric derived rate of convergence of various smooth singular integral operators

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ABSTRACT. In this article, we continue the study of approximation of various smooth singular integral operators. This time the foundation of our research is a trigonometric Taylor's formula. We establish the convergence of our operators to the unit operator with rates via Jackson type inequalities engaging the first modulus of continuity. Of interest here is a residual appearing term. Note that our operators are not positive. Our results are pointwise and uniform. The studied operators here are of the following types: Gauss-Weierstrass, Poisson-Cauchy and trigonometric.

**Keywords:** Singular integral, Gauss-Weierstrass, Poisson-Cauchy and trigonometric operator, modulus of continuity, trigonometric Taylor formula.

**2020 Mathematics Subject Classification:** 26A15, 26D15, 41A17, 41A35.

## 1. INTRODUCTION

We are motivated by [2, 3, chapters 10-14], and [1, 4]. We use a trigonometric new Taylor formula from [1], see also [4]. Here, we consider some very general operators, the smooth Gauss-Weierstrass, Poisson-Cauchy and trigonometric singular integral operators over the real line and we study further their convergencnce properties quantitatively. We establish related inequalities involving the first modulus of continuity with respect to uniform norm and the estimates are pointwise and uniform. We demonstrate detailed proofs.

## 2. RESULTS

By [1, 4], for $f \in C^2(\mathbb{R})$ and $a, x \in \mathbb{R}$, we have by trigonometric Taylor formula

$$f(x) - f(a) = f'(a)\sin(x-a) + 2f''(a)\sin^2\left(\frac{x-a}{2}\right) + \int_a^x [(f''(t) + f(t)) - (f''(a) + f(a))]\sin(x-t)\,dt. \tag{2.1}$$

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we set

$$\alpha_j := \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1, \ldots, r, \\ 1 - \sum\limits_{j=1}^{r} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 0, \end{cases}$$

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that is

$$\sum_{j=0}^{r} \alpha_j = 1.$$

$C_U(\mathbb{R})$ denotes the space of uniformly continuous functions on $\mathbb{R}$, and $C_B(\mathbb{R})$ denotes the space of bounded continuous functions on $\mathbb{R}$. Here we consider both $f, f'' \in C_U(\mathbb{R}) \cup C_B(\mathbb{R})$. Denote by

$$\omega_1(f, \delta) := \sup_{\substack{x,y \in \mathbb{R} \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0,$$

the first modulus of continuity of $f$.

I) We define the smooth Gauss-Weierstrass singular integral operators ([5]). Let $f \in C^2(\mathbb{R})$, we define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral

$$W_{r,\xi}(f; x) = \frac{1}{\sqrt{\pi \xi}} \int_{-\infty}^{\infty} \left( \sum_{j=0}^{r} \alpha_j f(x + jt) \right) e^{-\frac{t^2}{\xi}} dt.$$

We assume that $W_{r,\xi}(f; x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. We will use also that

$$W_{r,\xi}(f; x) = \frac{1}{\sqrt{\pi \xi}} \sum_{j=0}^{r} \alpha_j \left( \int_{-\infty}^{\infty} f(x + jt) e^{-\frac{t^2}{\xi}} dt \right),$$

notice by $\frac{1}{\sqrt{\pi \xi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{\xi}} dt = 1$ that $W_{r,\xi}(c; x) = c$, $c$ is constant and

$$W_{r,\xi}(f; x) - f(x) = \frac{1}{\sqrt{\pi \xi}} \left( \sum_{j=0}^{r} \alpha_j \int_{-\infty}^{\infty} (f(x + jt) - f(x)) e^{-\frac{t^2}{\xi}} dt \right).$$

We set

$$\Delta_2(x) := W_{r,\xi}(f; x) - f(x) - f''(x) \sum_{j=0}^{r} \alpha_j \left( 1 - e^{-\frac{j^2}{4}\xi} \right), \quad x \in \mathbb{R}; \tag{2.2}$$

$j = 0, 1, ..., r \in \mathbb{N}$.

II) We define the smooth Poisson-Cauchy singular integral operators ([5]). Let $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$ and $f \in C^2(\mathbb{R})$. We define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral

$$M_{r,\xi}(f; x) = W \int_{-\infty}^{\infty} \frac{\sum_{j=0}^{r} \alpha_j f(x + jt)}{(t^{2\alpha} + \xi^{2\alpha})^{\beta}} dt,$$

where the constant is defined as

$$W = \frac{\Gamma(\beta)\, \alpha \beta^{2\alpha\beta - 1}}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)}.$$

We assume that $M_{r,\xi}(f; x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. We will use also that

$$M_{r,\xi}(f; x) = W \sum_{j=0}^{r} \alpha_j \left( \int_{-\infty}^{\infty} f(x + jt) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{\beta}} dt \right).$$

We notice by $W\int_{-\infty}^{\infty}\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^{\beta}}dt=1$ that $M_{r,\xi}(c;x)=c$, $c$ constant and

$$M_{r,\xi}(f;x)-f(x)=W\left(\sum_{j=0}^{r}\alpha_j\int_{-\infty}^{\infty}[f(x+jt)-f(x)]\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^{\beta}}dt\right).$$

We set

$$\Delta_3(x):=M_{r,\xi}(f;x)-f(x)-4f''(x)\frac{\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\sum_{j=0}^{r}\alpha_j\int_0^{\infty}\frac{\sin^2\left(\frac{j\xi}{2}t\right)}{(1+t^{2\alpha})^{\beta}}dt, \tag{2.3}$$

$\xi>0$, $x\in\mathbb{R}$; $\beta>\frac{1}{2\alpha}$, $\alpha\in\mathbb{N}$; $j=0,1,...,r\in\mathbb{N}$.

III) We define the smooth trigonometric singular integral operators ([5]) as follows. Let $\xi>0$, $f\in C^2(\mathbb{R})$, $x\in\mathbb{R}$, $\beta\in\mathbb{N}$; we set

$$T_{r,\xi}(f;x):=\frac{1}{\lambda}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+jt)\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt,$$

where

$$\lambda:=\int_{-\infty}^{\infty}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt=2\xi^{1-2\beta}\int_0^{\infty}\left(\frac{\sin t}{t}\right)^{2\beta}dt$$

(by [6, p. 210, item 1033])

$$\lambda=2\xi^{1-2\beta}\pi(-1)^{\beta}\beta\sum_{k=1}^{\beta}(-1)^k\frac{k^{2\beta-1}}{(\beta-k)!(\beta+k)!}.$$

Denote

$$\lambda_1:=2\pi(-1)^{\beta}\beta\sum_{k=1}^{\beta}(-1)^k\frac{k^{2\beta-1}}{(\beta-k)!(\beta+k)!},$$

that is

$$\lambda=\lambda_1\xi^{1-2\beta}.$$

We suppose that $T_{r,\xi}(f;x)\in\mathbb{R}$ for all $x\in\mathbb{R}$. Clearly, again it is

$$\frac{1}{\lambda}\int_{-\infty}^{\infty}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt=1$$

and $T_{r,\xi}(c;x)=c$, $c$ is constant. We set

$$\Delta_4(x):=T_{r,\xi}(f;x)-f(x)-4f''(x)\lambda_1^{-1}\sum_{j=0}^{r}\alpha_j\int_0^{\infty}\sin^2\left(\frac{j\xi}{2}t\right)\left(\frac{\sin t}{t}\right)^{2\beta}dt, \tag{2.4}$$

where $\xi>0$, $x\in\mathbb{R}$; $\beta\in\mathbb{N}-\{1,2\}$; $j=0,1,...,r\in\mathbb{N}$. Next, we present uniform approximation results regarding the smooth singular operators $W_{r,\xi}$, $M_{r,\xi}$ and $T_{r,\xi}$.

**Theorem 2.1.** *It holds*

$$|\Delta_2(x)|\le\|\Delta_2\|_{\infty}\le\omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|j^2\left[\frac{\xi}{4}+\frac{j\sqrt{\xi}}{6\sqrt{\pi}}\right]=:A_2;\quad\xi>0,\ x\in\mathbb{R}. \tag{2.5}$$

*And* $\|\Delta_2\|_\infty \to 0$, *as* $\xi \to 0$. *If* $f''(x) = 0$, *then* $|W_{r,\xi}(f,x) - f(x)| \le A_2$ *and* $W_{r,\xi}(f,x) \to f(x)$, *as* $\xi \to 0$.

*Proof.* By (2.1), we get that

$$\begin{aligned} f(x+jt) - f(x) &= f'(x)\sin(jt) + 2f''(x)\sin^2\left(\frac{jt}{2}\right) \\ &+ \int_x^{x+jt} \left[(f''(s) + f(s)) - (f''(x) + f(x))\right]\sin(x+jt-s)\,ds, \end{aligned}$$

or better

$$\begin{aligned} f(x+jt) - f(x) &= f'(x)\sin(jt) + 2f''(x)\sin^2\left(\frac{jt}{2}\right) \\ &+ \int_0^{jt} \left[(f''(x+z) + f(x+z)) - (f''(x) + f(x))\right]\sin(jt-z)\,dz. \end{aligned}$$

Furthermore, it holds

$$\begin{aligned} \sum_{j=0}^{r} \alpha_j \left[f(x+jt) - f(x)\right] &= f'(x)\sum_{j=0}^{r}\alpha_j \sin(jt) + 2f''(x)\sum_{j=0}^{r}\alpha_j \sin^2\left(\frac{jt}{2}\right) \\ &+ \sum_{j=0}^{r}\alpha_j \int_0^{jt} \left[(f''(x+z) + f(x+z)) - (f''(x) + f(x))\right]\sin(jt-z)\,dz, \end{aligned}$$

or better

$$\begin{aligned} &\sum_{j=0}^{r} \alpha_j \left[f(x+jt) - f(x)\right] \\ =& f'(x)\sum_{j=0}^{r}\alpha_j \sin(jt) + 2f''(x)\sum_{j=0}^{r}\alpha_j \sin^2\left(\frac{jt}{2}\right) \\ +& \sum_{j=0}^{r}\alpha_j j \int_0^{t} \left[(f''(x+jw) + f(x+jw)) - (f''(x) + f(x))\right]\sin j(t-w)\,dw. \end{aligned}$$

Call

$$R := R(t) := \sum_{j=0}^{r}\alpha_j j \int_0^{t} \left[(f''(x+jw) + f(x+jw)) - (f''(x) + f(x))\right]\sin j(t-w)\,dw,$$

$\forall\, t \in \mathbb{R}$. Then, for $t \geq 0$,

$$\begin{aligned}
|R| &\leq \sum_{j=0}^{r} |\alpha_j|\, j \int_0^t |(f''(x+jw) + f(x+jw)) - (f''(x) + f(x))|\, |\sin j(t-w)|\, dw \\
&\leq \sum_{j=0}^{r} |\alpha_j|\, j \int_0^t \omega_1(f''+f, jw)\, j(t-w)\, dw \quad (\xi > 0) \\
&= \sum_{j=0}^{r} |\alpha_j|\, j^2 \int_0^t \omega_1\left(f''+f, \frac{\xi j w}{\xi}\right)(t-w)\, dw \\
&\leq \sum_{j=0}^{r} |\alpha_j|\, j^2 \omega_1(f''+f, \xi) \int_0^t \left(1 + \frac{jw}{\xi}\right)(t-w)\, dw \\
&= \omega_1(f''+f, \xi) \sum_{j=0}^{r} |\alpha_j|\, j^2 \left[\int_0^t (t-w)\, dw + \frac{j}{\xi}\int_0^t w^{2-1}(t-w)^{2-1}\, dw\right] \\
&= \omega_1(f''+f, \xi) \sum_{j=0}^{r} |\alpha_j|\, j^2 \left[\left.\frac{(t-w)^2}{2}\right|_t^0 + \frac{j}{\xi}\frac{1}{6}t^3\right] \\
&= \omega_1(f''+f, \xi) \sum_{j=0}^{r} |\alpha_j|\, j^2 \left[\frac{t^2}{2} + j\frac{t^3}{6\xi}\right].
\end{aligned}$$

Hence ($t \geq 0$)

$$|R| \leq \omega_1(f''+f, \xi) \sum_{j=0}^{r} |\alpha_j|\, j^2 \left[\frac{t^2}{2} + j\frac{t^3}{6\xi}\right].$$

Let now $t < 0$, then

$$\begin{aligned}
|R| &\leq \sum_{j=0}^{r} |\alpha_j|\, j \left|\int_0^t [(f''(x+jw) + f(x+jw)) - (f''(x) + f(x))] \sin j(t-w)\, dw\right| \\
&\leq \sum_{j=0}^{r} |\alpha_j|\, j \int_t^0 |(f''(x+jw) + f(x+jw)) - (f''(x) + f(x))|\, |\sin j(t-w)|\, dw \\
&\leq \sum_{j=0}^{r} |\alpha_j|\, j \int_t^0 \omega_1(f''+f, -jw)\, j(w-t)\, dw \\
&= \sum_{j=0}^{r} |\alpha_j|\, j \int_t^0 \omega_1\left(f''+f, -jw\frac{\xi}{\xi}\right) j(w-t)\, dw \\
&= \omega_1(f''+f, \xi) \sum_{j=0}^{r} |\alpha_j|\, j^2 \int_t^0 \left(1 - \frac{j}{\xi}w\right)(w-t)\, dw \\
&= \omega_1(f''+f, \xi) \sum_{j=0}^{r} |\alpha_j|\, j^2 \left[\int_t^0 (w-t)\, dw + \frac{j}{\xi}\int_t^0 (0-w)^{2-1}(w-t)^{2-1}\, dw\right]
\end{aligned}$$

$$= \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \left. \frac{(w-t)^2}{2} \right|_t^0 + \frac{j}{\xi} \frac{1}{6} (-t)^3 \right]$$

$$= \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \frac{t^2}{2} + j \frac{(-t)^3}{6\xi} \right].$$

We found that ($t < 0$)

$$|R| \le \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \frac{t^2}{2} + j \frac{(-t)^3}{6\xi} \right].$$

Consequently, for $t \in \mathbb{R}$, we obtain

$$|R(t)| \le \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \frac{t^2}{2} + \frac{j}{6\xi} |t|^3 \right], \quad \xi > 0.$$

So, we have

$$\sum_{j=0}^{r} \alpha_j [f(x + jt) - f(x)] - f'(x) \sum_{j=0}^{r} \alpha_j \sin(jt) - 2 f''(x) \sum_{j=0}^{r} \alpha_j \sin^2 \left( \frac{jt}{2} \right) = R(t). \tag{2.6}$$

We call

$$\begin{aligned} \overline{\Delta}_2 (x) &:= W_{r,\xi} (f, x) - f(x) - f'(x) \sum_{j=0}^{r} \alpha_j \frac{1}{\sqrt{\pi \xi}} \left( \int_{-\infty}^{\infty} \sin(jt) e^{-\frac{t^2}{\xi}} dt \right) \\ &- 2 f''(x) \sum_{j=0}^{r} \alpha_j \frac{1}{\sqrt{\pi \xi}} \left( \int_{-\infty}^{\infty} \sin^2 \left( \frac{jt}{2} \right) e^{-\frac{t^2}{\xi}} dt \right) \\ &= \frac{1}{\sqrt{\pi \xi}} \int_{-\infty}^{\infty} R(t) e^{-\frac{t^2}{\xi}} dt. \end{aligned} \tag{2.7}$$

Hence, we get that

$$\begin{aligned} \left| \overline{\Delta}_2 (x) \right| &\le \frac{1}{\sqrt{\pi \xi}} \int_{-\infty}^{\infty} R(t) e^{-\frac{t^2}{\xi}} dt \\ &\le \frac{1}{\sqrt{\pi \xi}} \int_{-\infty}^{\infty} \left[ \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \frac{t^2}{2} + \frac{j}{6\xi} |t|^3 \right] \right] e^{-\frac{t^2}{\xi}} dt \\ &= \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \frac{1}{\sqrt{\pi \xi}} \int_{-\infty}^{\infty} \left[ \frac{t^2}{2} + \frac{j}{6\xi} |t|^3 \right] e^{-\frac{t^2}{\xi}} dt \\ &= \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \frac{1}{2\sqrt{\pi \xi}} \int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{\xi}} dt + \frac{j}{6\xi \sqrt{\pi \xi}} \int_{-\infty}^{\infty} |t|^3 e^{-\frac{t^2}{\xi}} dt \right] \\ &= \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \frac{1}{\sqrt{\pi \xi}} \int_{0}^{\infty} t^2 e^{-\frac{t^2}{\xi}} dt + \frac{j}{3\xi \sqrt{\pi \xi}} \int_{0}^{\infty} t^3 e^{-\frac{t^2}{\xi}} dt \right] \end{aligned}$$

$$= \omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j| j^2\left[\frac{(\sqrt{\xi})^3}{\sqrt{\pi\xi}}\int_0^\infty \left(\frac{t}{\sqrt{\xi}}\right)^2 e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2} d\left(\frac{t}{\sqrt{\xi}}\right)\right.$$

$$\left.+\frac{j(\sqrt{\xi})^4}{3\xi\sqrt{\pi\xi}}\int_0^\infty \left(\frac{t}{\sqrt{\xi}}\right)^3 e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2} d\left(\frac{t}{\sqrt{\xi}}\right)\right]$$

$$= \omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j| j^2\left[\frac{\xi}{\sqrt{\pi}}\int_0^\infty z^2 e^{-z^2}dz + \frac{j(\sqrt{\xi})^3\xi^{-1}}{3\sqrt{\pi}}\int_0^\infty z^3 e^{-z^2}dz\right]$$

$$= \omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j| j^2\left[\frac{\xi}{\sqrt{\pi}}\frac{\sqrt{\pi}}{4} + \frac{j\sqrt{\xi}}{3\sqrt{\pi}}\frac{1}{2}\right]$$

$$= \omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j| j^2\left[\frac{\xi}{4} + \frac{j\sqrt{\xi}}{6\sqrt{\pi}}\right].$$

We have proved that

$$\left|\overline{\Delta}_2(x)\right| \le \omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j| j^2\left[\frac{\xi}{4} + \frac{j\sqrt{\xi}}{6\sqrt{\pi}}\right]; \ \ \xi>0, x\in\mathbb{R}.$$

Notice that $\overline{\Delta}_2(x) \to 0,$ as $\xi \to 0$. Next, we simplify the left hand side of (2.7). We observe that: clearly, it is

$$\int_{-\infty}^{\infty}\sin(jt)\, e^{-\frac{t^2}{\xi}}dt = 0, j = 0, 1, ..., r.$$

Furthermore, we have that

$$\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{t^2}{\xi}}dt = 2\int_0^\infty \sin^2\left(\frac{jt}{2}\right)e^{-\frac{t^2}{\xi}}dt$$

$$= 2\sqrt{\xi}\int_0^\infty \sin^2\left(\left(\frac{j\sqrt{\xi}}{2}\right)\frac{t}{\sqrt{\xi}}\right)e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2}d\frac{t}{\sqrt{\xi}} \ \ \left(\frac{t}{\sqrt{\xi}} =: x \text{ and } \frac{j\sqrt{\xi}}{2} =: \beta_1\right)$$

$$= 2\sqrt{\xi}\int_0^\infty \sin^2(\beta_1 x)\, e^{-x^2}dx$$

$$= \sqrt{\xi}\frac{1}{2}\sqrt{\pi}e^{-\beta_1^2}\left(e^{\beta_1^2}-1\right)$$

$$= \frac{\sqrt{\xi}\sqrt{\pi}}{2}e^{-\frac{j^2}{4}\xi}\left(e^{\frac{j^2}{4}\xi}-1\right).$$

Consequently, we derive

$$\overline{\Delta}_2(x) \overset{(2.7)}{=} W_{r,\xi}(f,x) - f(x) - f''(x)\sum_{j=0}^{r}\alpha_j\left(1-e^{-\frac{j^2}{4}\xi}\right) = \Delta_2(x).$$

The theorem is proved. □

We derive the following.

**Corollary 2.1** (to Theorem 2.1)**.** *Additionally, assume $f'' \in C_B(\mathbb{R})$. It follows ($\xi > 0$, $x \in \mathbb{R}$)*

$$\|W_{r,\xi}(f) - f\|_\infty \leq \omega_1(f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[\frac{\xi}{4} + \frac{j\sqrt{\xi}}{6\sqrt{\pi}}\right] + \|f''\|_\infty \left(\sum_{j=0}^{r} |\alpha_j| \left(1 - e^{-\frac{j^2}{4}\xi}\right)\right) \to 0, \text{ as } \xi \to 0.$$

*Proof.* By (2.5) and (2.2). □

We continue with the following result.

**Theorem 2.2.** *It holds*

$$|\Delta_3(x)| \leq \|\Delta_3\|_\infty \leq \frac{\omega_1(f'' + f, \xi)\xi^2}{2\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta - \frac{1}{2\alpha}\right)} \times \sum_{j=0}^{r} |\alpha_j| j^2 \left[\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta - \frac{3}{2\alpha}\right) + \frac{j}{3}\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\beta - \frac{2}{\alpha}\right)\right] =: A_3; \tag{2.8}$$

*$\xi > 0$, $x \in \mathbb{R}$; and $\beta > \frac{2}{\alpha}$, $\alpha \in \mathbb{N}$. Notice that $\|\Delta_3\|_\infty \to 0$, as $\xi \to 0$. If $f''(x) = 0$, then $|M_{r,\xi}(f,x) - f(x)| \leq A_3$ and $M_{r,\xi}(f,x) \to f(x)$, as $\xi \to 0$.*

*Proof.* The proof is the same as the proof of Theorem 2.1 from start till equation (2.6). We call

$$\overline{\Delta}_3(x) := M_{r,\xi}(f,x) - f(x) - f'(x) \sum_{j=0}^{r} \alpha_j W \left(\int_{-\infty}^{\infty} \sin(jt) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt\right) - 2f''(x) \sum_{j=0}^{r} \alpha_j W \left(\int_{-\infty}^{\infty} \sin^2\left(\frac{jt}{2}\right) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt\right) = W \int_{-\infty}^{\infty} R(t) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt. \tag{2.9}$$

Hence, we get

$$\begin{aligned}
&\left|\overline{\Delta}_3(x)\right| \\
\leq &W \int_{-\infty}^{\infty} |R(t)| \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \\
\leq &W \int_{-\infty}^{\infty} \left[\omega_1(f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[\frac{t^2}{2} + \frac{j}{6\xi} |t|^3\right]\right] \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \\
= &\omega_1(f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 W \int_{-\infty}^{\infty} \left[\frac{t^2}{2} + \frac{j}{6\xi} |t|^3\right] \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \\
= &\omega_1(f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[\frac{W}{2} \int_{-\infty}^{\infty} t^2 \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt + \frac{jW}{6\xi} \int_{-\infty}^{\infty} |t|^3 \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt\right] \\
= &\omega_1(f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[W \int_{0}^{\infty} t^2 \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt + \frac{jW}{3\xi} \int_{0}^{\infty} t^3 \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt\right]
\end{aligned}$$

$$=\omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|j^2\left[W\xi^{-2\alpha\beta}\xi^2\xi\int_0^\infty\left(\frac{t}{\xi}\right)^2\frac{1}{\left(\left(\frac{t}{\xi}\right)^{2\alpha}+1\right)^\beta}d\left(\frac{t}{\xi}\right)\right.$$

$$\left.+\frac{jW}{3\xi}\xi^{-2\alpha\beta}\xi^4\int_0^\infty\left(\frac{t}{\xi}\right)^3\frac{1}{\left(\left(\frac{t}{\xi}\right)^{2\alpha}+1\right)^\beta}d\left(\frac{t}{\xi}\right)\right]$$

$$=\omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|j^2\left[W\xi^{3-2\alpha\beta}\int_0^\infty\frac{z^2}{(z^{2\alpha}+1)^\beta}dz+\frac{jW}{3}\xi^{3-2\alpha\beta}\int_0^\infty\frac{z^3}{(z^{2\alpha}+1)^\beta}dz\right]$$

(by [7, p. 397, formula 595])

$$=\omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|j^2W\xi^{3-2\alpha\beta}\left[\frac{\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta-\left(\frac{3}{2\alpha}\right)\right)}{2\alpha\Gamma(\beta)}+\frac{j}{3}\frac{\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\beta-\frac{2}{\alpha}\right)}{2\alpha\Gamma(\beta)}\right]$$

$$=\omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|j^2\frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}\xi^{3-2\alpha\beta}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\left[\frac{\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta-\left(\frac{3}{2\alpha}\right)\right)}{2\alpha\Gamma(\beta)}+\frac{j}{3}\frac{\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\beta-\frac{2}{\alpha}\right)}{2\alpha\Gamma(\beta)}\right]$$

$$=\frac{\omega_1(f''+f,\xi)}{2}\sum_{j=0}^{r}|\alpha_j|j^2\frac{\xi^2}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\left[\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta-\frac{3}{2\alpha}\right)+\frac{j}{3}\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\beta-\frac{2}{\alpha}\right)\right].$$

We have proved that

$$\left|\overline{\Delta}_3(x)\right|\leq\frac{\omega_1(f''+f,\xi)\xi^2}{2\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\sum_{j=0}^{r}|\alpha_j|j^2\left[\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta-\frac{3}{2\alpha}\right)+\frac{j}{3}\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\beta-\frac{2}{\alpha}\right)\right];$$

$\xi>0$, $x\in\mathbb{R}$; and $\beta>\frac{2}{\alpha}$, $\alpha\in\mathbb{N}$. Notice that $\overline{\Delta}_3(x)\to0$, as $\xi\to0$. Next we simplify the left hand side of (2.9). We observe that: clearly, it is

$$\int_{-\infty}^{\infty}\sin(jt)\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^\beta}dt=0,\ j=0,1,...,r.$$

Furthermore, we have that

$$W\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^\beta}dt$$

$$=W2\int_0^\infty\sin^2\left(\frac{jt}{2}\right)\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^\beta}dt$$

$$=W2\xi^{-2\alpha\beta}\xi\int_0^\infty\sin^2\left(\left(\frac{j\xi}{2}\right)\frac{t}{\xi}\right)\frac{1}{\left(\left(\frac{t}{\xi}\right)^{2\alpha}+1\right)^\beta}d\left(\frac{t}{\xi}\right)\quad\left(\frac{t}{\xi}=:x\text{ and }\frac{j\xi}{2}=:\gamma\right)$$

$$
\begin{aligned}
&=2W\xi^{1-2\alpha\beta}\int_0^{\infty}\sin^2(\gamma x)\frac{1}{(x^{2\alpha}+1)^{\beta}}dx\\
&=2\frac{\Gamma(\beta)\,\alpha\xi^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\xi^{1-2\alpha\beta}\int_0^{\infty}\frac{\sin^2(\gamma x)}{(x^{2\alpha}+1)^{\beta}}dx\\
&=\frac{2\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\int_0^{\infty}\frac{\sin^2(\gamma x)}{(x^{2\alpha}+1)^{\beta}}dx\\
&\leq\frac{2\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\int_0^{\infty}\frac{dx}{(1+x^{2\alpha})^{\beta}}\qquad\text{(by [7], p. 397, formula 595)}\\
&=\frac{2\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\frac{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}{2\alpha\Gamma(\beta)}=1
\end{aligned}
$$

for $\beta>\frac{1}{2\alpha}$, $\alpha\in\mathbb{N}$. Therefore it holds ($j=0,1,...,r$)

$$
W\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^{\beta}}dt\leq 1,
$$

where $\beta>\frac{1}{2\alpha}$, $\alpha\in\mathbb{N}$. Consequently, we get that

$$
\begin{aligned}
\overline{\Delta}_3(x)&\overset{(2.9)}{=}M_{r,\xi}(f,x)-f(x)-4f''(x)\frac{\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\sum_{j=0}^{r}\alpha_j\int_0^{\infty}\frac{\sin^2\left(\frac{j\xi}{2}t\right)}{(t^{2\alpha}+1)^{\beta}}dt\\
&=\Delta_3(x),
\end{aligned}
$$

$\xi>0$, $x\in\mathbb{R}$; $\beta>\frac{1}{2\alpha}$, $\alpha\in\mathbb{N}$. The theorem is proved. □

We give the following

**Corollary 2.2** (to Theorem 2.2)**.** *Additionally, assume $f''\in C_B(\mathbb{R})$. It follows ($\xi>0$, $x\in\mathbb{R}$)*

$$
\begin{aligned}
\|M_{r,\xi}(f)-f\|_{\infty}\leq{}&\frac{\omega_1(f''+f,\xi)\,\xi^2}{2\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\\
&\times\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta-\frac{3}{2\alpha}\right)+\frac{j}{3}\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\beta-\frac{2}{\alpha}\right)\right]\\
&+4\|f''\|_{\infty}\frac{\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\sum_{j=0}^{r}|\alpha_j|\int_0^{\infty}\frac{\sin^2\left(\frac{j\xi}{2}t\right)}{(1+t^{2\alpha})^{\beta}}dt\to 0,\ \text{as } \xi\to 0,
\end{aligned}
$$

$\xi>0$, $\beta>\frac{1}{2\alpha}$, $\alpha\in\mathbb{N}$.

*Proof.* By (2.8) and (2.3), we notice that

$$
\left|\sin\left(\frac{j\xi}{2}t\right)\right|\leq\frac{j\xi}{2}t,\ \forall\, t\in\mathbb{R}_+.
$$

Hence, we get

$$
\sin^2\left(\frac{j\xi}{2}t\right)\leq\frac{j^2\xi^2}{4}t^2,\ \forall\, t\in\mathbb{R}_+.
$$

Consequently, it holds

$$
\int_0^{\infty}\frac{\sin^2\left(\frac{j\xi}{2}t\right)}{(1+t^{2\alpha})^{\beta}}dt\leq\frac{j^2\xi^2}{4}\int_0^{\infty}\frac{t^2}{(1+t^{2\alpha})^{\beta}}dt=\frac{j^2\xi^2}{8}\frac{\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta-\left(\frac{3}{2\alpha}\right)\right)}{\alpha\Gamma(\beta)}\to 0,\ \text{as } \xi\to 0.
$$

□

The last main result comes next.

**Theorem 2.3.** *It holds*

$$|\Delta_4(x)| \leq \|\Delta_4\|_\infty \leq \frac{\omega_1(f''+f,\xi)}{\lambda_1}\xi^2$$
$$\times\left\{\sum_{j=0}^{r}|\alpha_j|j^2\left[\frac{\pi(-1)^{\beta-1}(2\beta)!}{8(2\beta-3)!}\left(\sum_{k=1}^{\beta}(-1)^k\frac{k^{2\beta-3}}{(\beta-k)!(\beta+k)!}\right)\right.\right.$$
$$\left.\left.+\frac{j}{3}\left(\frac{-(2\beta)!}{8(2\beta-4)!}\left(\sum_{k=1}^{\beta}\frac{(-1)^{\beta-k}k^{2\beta-k}\ln(2k)}{(\beta-k)!(\beta+k)!}\right)\right)\right]\right\} =: A_4; \tag{2.10}$$

*$\xi>0$, $x\in\mathbb{R}$ and $\beta\in\mathbb{N}-\{1,2\}$.*

*Notice that $\|\Delta_4\|_\infty \to 0$, as $\xi\to 0$. If $f''(x)=0$, then $|T_{r,\xi}(f,x)-f(x)|\leq A_4$ and $T_{r,\xi}(f,x)\to f(x)$, as $\xi\to 0$.*

*Proof.* The proof is the same as the proof of Theorem 2.1 from start till equation (2.6). We call

$$\overline{\Delta}_4(x) := T_{r,\xi}(f,x)-f(x)-f'(x)\sum_{j=0}^{r}\alpha_j\lambda^{-1}\left(\int_{-\infty}^{\infty}\sin(jt)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)$$
$$-2f''(x)\sum_{j=0}^{r}\alpha_j\lambda^{-1}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right) \tag{2.11}$$
$$=\lambda^{-1}\int_{-\infty}^{\infty}R(t)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt.$$

Hence, we get

$$\left|\overline{\Delta}_4(x)\right|$$
$$\leq\lambda^{-1}\int_{-\infty}^{\infty}|R(t)|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt$$
$$\leq\lambda^{-1}\int_{-\infty}^{\infty}\left[\omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|j^2\left[\frac{t^2}{2}+\frac{j}{6\xi}|t|^3\right]\right]\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt$$
$$=\omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|j^2\lambda^{-1}\int_{-\infty}^{\infty}\left[\frac{t^2}{2}+\frac{j}{6\xi}|t|^3\right]\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt$$
$$=\omega_1(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|j^2\left[\frac{\lambda^{-1}}{2}\int_{-\infty}^{\infty}t^2\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt+\frac{j\lambda^{-1}}{6\xi}\int_{-\infty}^{\infty}|t|^3\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right]$$

$$=\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\lambda^{-1}\int_0^{\infty}t^2\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt+\frac{j\lambda^{-1}}{3\xi}\int_0^{\infty}t^3\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right]$$

$$=\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\lambda^{-1}\xi^{3-2\beta}\int_0^{\infty}\left(\frac{t}{\xi}\right)^2\left(\frac{\sin\left(\frac{t}{\xi}\right)}{\left(\frac{t}{\xi}\right)}\right)^{2\beta}d\left(\frac{t}{\xi}\right)\right.$$

$$\left.+\frac{j\lambda^{-1}}{3\xi}\xi^{4-2\beta}\int_0^{\infty}\left(\frac{t}{\xi}\right)^3\left(\frac{\sin\left(\frac{t}{\xi}\right)}{\left(\frac{t}{\xi}\right)}\right)^{2\beta}d\left(\frac{t}{\xi}\right)\right]$$

$$=\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\lambda^{-1}\xi^{3-2\beta}\int_0^{\infty}x^2\left(\frac{\sin x}{x}\right)^{2\beta}dx+\frac{j\lambda^{-1}}{3}\xi^{3-2\beta}\int_0^{\infty}x^3\left(\frac{\sin x}{x}\right)^{2\beta}dx\right]$$

(by [6, p. 210, item 1033])

$$=\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\lambda_1^{-1}\xi^2\left(\frac{\pi\left(-1\right)^{\beta-1}\left(2\beta\right)!}{8\left(2\beta-3\right)!}\left(\sum_{k=1}^{\beta}\left(-1\right)^k\frac{k^{2\beta-3}}{\left(\beta-k\right)!\left(\beta+k\right)!}\right)\right)\right.$$

$$\left.+\frac{j\lambda_1^{-1}}{3}\xi^2\left(\frac{-\left(2\beta\right)!}{8\left(2\beta-4\right)!}\left(\sum_{k=1}^{\beta}\frac{\left(-1\right)^{\beta-k}k^{2\beta-k}\ln\left(2k\right)}{\left(\beta-k\right)!\left(\beta+k\right)!}\right)\right)\right]$$

$$=\frac{\omega_1\left(f''+f,\xi\right)}{\lambda_1}\xi^2\left\{\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\frac{\pi\left(-1\right)^{\beta-1}\left(2\beta\right)!}{8\left(2\beta-3\right)!}\left(\sum_{k=1}^{\beta}\left(-1\right)^k\frac{k^{2\beta-3}}{\left(\beta-k\right)!\left(\beta+k\right)!}\right)\right.\right.$$

$$\left.\left.+\frac{j}{3}\left(\frac{-\left(2\beta\right)!}{8\left(2\beta-4\right)!}\left(\sum_{k=1}^{\beta}\frac{\left(-1\right)^{\beta-k}k^{2\beta-4}\ln\left(2k\right)}{\left(\beta-k\right)!\left(\beta+k\right)!}\right)\right)\right]\right\}.$$

We have proved that

$$\left|\overline{\Delta}_4\left(x\right)\right|\leq\frac{\omega_1\left(f''+f,\xi\right)}{\lambda_1}\xi^2\left\{\sum_{j=0}^{r}\left|\alpha_j\right|j^2\right.$$

$$\times\left[\frac{\pi\left(-1\right)^{\beta-1}\left(2\beta\right)!}{8\left(2\beta-3\right)!}\left(\sum_{k=1}^{\beta}\left(-1\right)^k\frac{k^{2\beta-3}}{\left(\beta-k\right)!\left(\beta+k\right)!}\right)\right.$$

$$\left.\left.+\frac{j}{3}\left(\frac{-\left(2\beta\right)!}{8\left(2\beta-4\right)!}\left(\sum_{k=1}^{\beta}\frac{\left(-1\right)^{\beta-k}k^{2\beta-4}\ln\left(2k\right)}{\left(\beta-k\right)!\left(\beta+k\right)!}\right)\right)\right]\right\},$$

$\xi>0$, $x\in\mathbb{R}$; and $\beta\in\mathbb{N}-\{1,2\}$. Notice that $\overline{\Delta}_4\left(x\right)\rightarrow0$, as $\xi\rightarrow0$. Next, we simplify the left hand side of (2.11). We observe that: clearly, it is

$$\int_{-\infty}^{\infty}\sin\left(jt\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt=0,\ j=0,1,...,r.$$

Furthermore, we have that

$$\lambda^{-1}\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt$$

$$=2\lambda_1^{-1}\xi^{2\beta-1}\xi^{1-2\beta}\int_0^{\infty}\sin^2\left(\left(\frac{j\xi}{2}\right)\frac{t}{\xi}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{\frac{t}{\xi}}\right)^{2\beta}d\left(\frac{t}{\xi}\right)\quad (\frac{t}{\xi}=:x \text{ and } \frac{j\xi}{2}=:\gamma)$$

$$=2\lambda_1^{-1}\int_0^{\infty}\sin^2(\gamma x)\left(\frac{\sin x}{x}\right)^{2\beta}dx$$

$$\leq 2\lambda_1^{-1}\int_0^{\infty}\left(\frac{\sin x}{x}\right)^{2\beta}dx \qquad \text{(by [6, p. 210, item 1033])}$$

$$=2\lambda_1^{-1}\pi(-1)^{\beta}\beta\left(\sum_{k=1}^{\beta}(-1)^k\frac{k^{2\beta-1}}{(\beta-k)!(\beta+k)!}\right)=1, \text{ for } \beta\in\mathbb{N}.$$

Therefore it holds ($j=0,1,...,r$)

$$\lambda^{-1}\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\leq 1,$$

where $\beta\in\mathbb{N}$. Consequently, we get that

$$\overline{\Delta}_4(x)\overset{(2.11)}{=}T_{r,\xi}(f,x)-f(x)-4f''(x)\lambda_1^{-1}\sum_{j=0}^{r}\alpha_j\int_0^{\infty}\sin^2\left(\frac{j\xi}{2}t\right)\left(\frac{\sin t}{t}\right)^{2\beta}dt=\Delta_4(x),$$

where $\xi>0$, $x\in\mathbb{R}$; $\beta\in\mathbb{N}-\{1,2\}$. The theorem is proved. □

We finish with the following.

**Corollary 2.3** (to Theorem 2.3)**.** *Additionally, assume $f''\in C_B(\mathbb{R})$. It follows ($\xi>0$, $x\in\mathbb{R}$)*

$$\|T_{r,\xi}(f)-f\|_{\infty}\leq\frac{\omega_1(f''+f,\xi)}{\lambda_1}\xi^2$$
$$\times\left\{\sum_{j=0}^{r}|\alpha_j|j^2\left[\frac{\pi(-1)^{\beta-1}(2\beta)!}{8(2\beta-3)!}\left(\sum_{k=1}^{\beta}(-1)^k\frac{k^{2\beta-3}}{(\beta-k)!(\beta+k)!}\right)\right.\right.$$
$$\left.\left.+\frac{j}{3}\left(\frac{-(2\beta)!}{8(2\beta-4)!}\left(\sum_{k=1}^{\beta}\frac{(-1)^{\beta-k}k^{2\beta-4}\ln(2k)}{(\beta-k)!(\beta+k)!}\right)\right)\right]\right\}$$
$$+4\|f''\|_{\infty}\lambda_1^{-1}\sum_{j=0}^{r}|\alpha_j|\int_0^{\infty}\sin^2\left(\frac{j\xi}{2}t\right)\left(\frac{\sin t}{t}\right)^{2\beta}dt\to 0,\ \text{as } \xi\to 0,$$

*where $\xi>0$, $\beta\in\mathbb{N}-\{1,2\}$.*

*Proof.* By (2.10) and (2.4), and from

$$\sin^2\left(\frac{j\xi}{2}t\right)\leq\frac{j^2\xi^2}{4}t^2,\ \forall\, t\in\mathbb{R}_+,$$

we obtain that

$$\int_0^\infty \sin^2\left(\frac{j\xi}{2}t\right)\left(\frac{\sin t}{t}\right)^{2\beta} dt$$
$$\leq \frac{j^2\xi^2}{4}\int_0^\infty t^2\left(\frac{\sin t}{t}\right)^{2\beta} dt$$
$$= \frac{j^2\xi^2}{4}\left(\frac{\pi(-1)^{\beta-1}(2\beta)!}{8(2\beta-3)!}\left(\sum_{k=1}^{\beta}(-1)^k \frac{k^{2\beta-3}}{(\beta-k)!(\beta+k)!}\right)\right) \to 0, \text{ as } \xi \to 0.$$

□

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