MODERN MATHEMATICAL METHODS
1 (2023), No. 1, pp. 22-29
https://modernmathmeth.com/

Research Article

Perov's theorem applied to systems of equations

GABRIELA MOTRONEA, DIANA OTROCOL*, AND IOAN RAŞA

ABSTRACT. In this paper, we consider systems of equations having a linear part and also a nonlinear part. We give sufficient conditions which imply the existence and uniqueness of solutions to the system. Using Perov's theorem, our results extend some results in the literature. An application using the iterative method, numerical experiments and graphics illustrate the main result.

Keywords: Algebraic system, solutions, existence, uniqueness.

2020 Mathematics Subject Classification: 65H10.

1. INTRODUCTION

Consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix},$$

where $a_{ij\geq 0},\ i,j=1,\dots,m$. Let $f_i:[0,\infty)\to[0,\infty), i=1,\dots,m$ be Lipschitz functions, i.e.,

$$|f_i(x)-f_i(y)|\leq l\,|x-y|\,,\ x,y\in[0,\infty), \tag{1.1}$$

where $l>0$ is a given constant. Systems of equations of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} f_1(x_1) \\ \vdots \\ f_m(x_m) \end{pmatrix} \tag{1.2}$$

were investigated in several papers (see [1]-[9], [13]-[15] and the references therein). The existence and the uniqueness of a solution $(x_1,\dots,x_m)\in[0,\infty)^m$ were established using, among other results, Brouwer's theorem and the iterative monotonic convergence method. Such systems appear frequently in applications.

Several real-world problems can be attacked using systems with the above characteristics. The corresponding mathematical models involve also second order Dirichlet problems, Dirichlet problems for partial difference equations, equations with periodic solutions, numerical solutions for differential equations, all of them with important applications to economics. Details can be found in the papers mentioned in our bibliography and in the references therein. In this

Received: 10.10.2023; Accepted: 27.11.2023; Published Online: 15.12.2023
*Corresponding author: Diana Otrocol; diana.otrocol@math.utcluj.ro

paper, we consider systems of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} f_1(x_1) \\ \vdots \\ f_m(x_m) \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}. \tag{1.3}$$

In order to study the existence and the uniqueness of a solution we use Perov's theorem.

2. Perov's theorem

Let (X, d) be a metric space and $A : X \to X$ an operator. In this paper, we use the terminologies and notations from [12]. For the convenience of the reader, we shall recall some of them. Denote by $A^0 := Id_X,\ A^1 := A,\ A^{n+1} := A \circ A^n,\ n \in \mathbb{N}$, the iterate operators of the operator A and by $F_A := \{x \in X |\ A(x) = x\}$ the fixed point set of A.

Definition 2.1. *$A : X \to X$ is called a Picard operator (briefly PO) if: $F_A = \{x^*\}$ and $A^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$.*

Definition 2.2. *$A : X \to X$ is said to be a weakly Picard operator (briefly WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A.*

Definition 2.3. *A matrix $Q \in M_{m\times m}([0, \infty))$ is called a matrix convergent to zero iff $Q^k \to 0$ as $k \to \infty$.*

As concerns matrices which are convergent to zero, we mention the following equivalent characterizations:

Theorem 2.1 ([11])**.** *Let $Q \in M_{m\times m}([0, \infty))$. The following statements are equivalent:*

(i) *Q is a matrix convergent to zero;*
(ii) *$Q^k x \to 0$ as $k \to \infty,\ \forall x \in \mathbb{R}^m$;*
(iii) *$I_m - Q$ is non-singular and $(I_m - Q)^{-1} = I_2 + Q + Q^2 + \cdots$;*
(iv) *$I_m - Q$ is non-singular and $(I_m - Q)^{-1}$ has nonnegative elements;*
(v) *$\lambda \in \mathbb{C},\ \det(Q - \lambda I_m) = 0$ imply $|\lambda| < 1$;*
(vi) *there exists at least one subordinate matrix norm such that $\|Q\| < 1$.*

The matrices convergent to zero were used by Perov [10] to generalize the contraction principle in the case of generalized metric spaces with the metric taking values in the positive cone of $\mathbb{R}^m$.

Definition 2.4 ([10])**.** *Let (X, d) be a complete generalized metric space with $d : X \times X \to [0, \infty)^m$ and $A : X \to X$. The operator A is called a Q-contraction if there exists a matrix $Q \in M_{m\times m}([0, \infty))$ such that:*

(i) *Q is a matrix convergent to zero;*
(ii) *$d(A(x), A(y)) \le Qd(x, y),\ \forall x, y \in X$.*

Theorem 2.2 (Perov's theorem)**.** *Let (X, d) be a complete generalized metric space with $d : X \times X \to [0, \infty)^m$ and $A : X \to X$ be a Q-contraction. Then,*

(i) *A is a Picard operator, $F_A = F_{A^n} = \{x^*\},\ \forall n \in \mathbb{N}^*$;*
(ii) *$d(A^n(x), x^*) \le (I_m - Q)^{-1} Q^n d(x, A(x)), \forall x \in X$.*

3. Main results

Consider again the matrix A with $a_{ij} \geq 0, i, j = 1, \ldots, m$ and the functions $f_i : [0, \infty) \to [0, \infty), i = 1, \ldots, m$ satisfying the Lipschitz condition (1.1). Denote

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}, \; G(x) = A \begin{pmatrix} f_1(x_1) \\ \vdots \\ f_m(x_m) \end{pmatrix}.$$

Then $x \in [0, \infty)^m, \; G(x) \in [0, \infty)^m$. The system (1.3) can be written as

$$G(x) = x. \tag{3.4}$$

For $x, y \in [0, \infty)^m$, let

$$d(x, y) := \begin{pmatrix} |x_1 - y_1| \\ \vdots \\ |x_m - y_m| \end{pmatrix}.$$

Then, d is a generalized metric and $[0, \infty)^m$ is a complete generalized metric space.

Theorem 3.3. *Suppose that the matrix $Q := lA$ is convergent to zero. Then $G : [0, \infty)^m \to [0, \infty)^m$ is a Q-contraction. Moreover, G is a Picard operator, $F_G = F_{G^n} = \{x^*\}$, x^* is the unique solution to the system (3.4) and*

$$d(G^n(x), x^*) \leq (I_m - Q)^{-1} Q^n d(x, G(x)), \; x \in [0, \infty)^m.$$

Proof. Let $x, y \in [0, \infty)^m$. Then

$$d(G(x), G(y)) = \begin{pmatrix} |a_{11}(f_1(x_1) - f_1(y_1)) + \ldots + a_{1m}(f_m(x_m) - f_m(y_m))| \\ \vdots \\ |a_{m1}(f_1(x_1) - f_1(y_1)) + \ldots + a_{mm}(f_m(x_m) - f_m(y_m))| \end{pmatrix}$$
$$\leq \begin{pmatrix} a_{11}l|x_1 - y_1| + \ldots + a_{1m}l|x_m - y_m| \\ \vdots \\ a_{m1}l|x_1 - y_1| + \ldots + a_{mm}l|x_m - y_m| \end{pmatrix},$$

where $\leq$ is understood componentwise. It follows that

$$d(G(x), G(y)) \leq lA \begin{pmatrix} |x_1 - y_1| \\ \vdots \\ |x_m - y_m| \end{pmatrix},$$

and finally,

$$d(G(x), G(y)) \leq Qd(x, y), \; x, y \in [0, \infty)^m.$$

This shows that G is a Q-contraction. We finish the proof by using Perov's theorem. □

Now let us consider the system of equations

$$\begin{cases} x_1 = f_1(a_{11}x_1 + \ldots + a_{1m}x_m + p_1) \\ \vdots \\ x_m = f_m(a_{m1}x_1 + \ldots + a_{mm}x_m + p_m) \end{cases}, \tag{3.5}$$

where, as before, $a_{ij} \geq 0,\ p_i \geq 0,\ i,j = 1,\ldots,m$. Let $x \in [0,\infty)^m$ and

$$H(x) := \begin{pmatrix} f_1\left(a_{11}x_1 + \ldots + a_{1m}x_m + p_1\right) \\ \vdots \\ f_m\left(a_{m1}x_1 + \ldots + a_{mm}x_m + p_m\right) \end{pmatrix}.$$

Then the system (3.5) can be written as

$$H(x) = x. \tag{3.6}$$

With the same distance d as before, we can state:

Theorem 3.4. *If $Q := lA$ is a matrix convergent to zero, then $H : [0,\infty)^m \to [0,\infty)^m$ is a Q-contraction. H is also a Picard operator, $F_H = F_{H^n} = \{x^*\}$ and x^* is the unique solution to (3.6). For each $x \in [0,\infty)^m$, we have*

$$d\left(H^n(x), x^*\right) \leq \left(I_m - Q\right)^{-1} Q^n d(x, H(x)),\ n \geq 1.$$

Proof. For $x, y \in [0,\infty)^m$, we have

$$\begin{aligned} & d(H(x), H(y)) \\ = & \begin{pmatrix} \left|f_1\left(a_{11}x_1 + \ldots + a_{1m}x_m + p_1\right) - f_1\left(a_{11}y_1 + \ldots + a_{1m}y_m + p_1\right)\right| \\ \vdots \\ \left|f_m\left(a_{m1}x_1 + \ldots + a_{mm}x_m + p_m\right) - f_m\left(a_{m1}y_1 + \ldots + a_{mm}y_m + p_m\right)\right| \end{pmatrix} \\ \leq & \begin{pmatrix} a_{11}l\left|x_1 - y_1\right| + \ldots + a_{1m}l\left|x_m - y_m\right| \\ \vdots \\ a_{m1}l\left|x_1 - y_1\right| + \ldots + a_{mm}l\left|x_m - y_m\right| \end{pmatrix} \\ = & lA \begin{pmatrix} |x_1 - y_1| \\ \vdots \\ |x_m - y_m| \end{pmatrix} = lAd(x,y) = Qd(x,y). \end{aligned}$$

Therefore H is a Q-contraction and the rest of the proof follows from Perov's theorem. ☐

Remark 3.1. *In the above considerations, we need $Q := lA$ to be a matrix convergent to zero. Given the matrix A, let $\mu_1,\ldots,\mu_m$ be its eigenvalues and let $M := \max\left\{|\mu_1|,\ldots,|\mu_m|\right\}$. Let $0 < l < M$. Then the eigenvalues of Q are $l\mu_1,\ldots,l\mu_m$, and $|l\mu_j| < 1,\ j = 1,\ldots,m$. This means that Q is a matrix convergent to zero.*

4. Applications

Consider the system of equations

$$\begin{cases} \dfrac{1}{3}\log(x_1+2) + \dfrac{2}{3}\log(x_2+3) = x_1 \\[2ex] \dfrac{3}{5}\log(x_1+2) + \dfrac{2}{5}\log(x_2+3) = x_2 \end{cases}, \tag{4.7}$$

where $x_1,\ x_2 \geq 0$. Then, $m = 2,\ f_1(t) = \log(t+2),\ f_2(t) = \log(t+3),\ t \geq 0$. Since $f_i'(t) \leq \frac{1}{2},\ i = 1,2, t \geq 0$, we can take $l = \frac{1}{2}$. Moreover, the system is of the form (3) with

$$A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$$

and A has the eigenvalues $\mu_1 = 1,\ \mu_2 = -\frac{4}{15}$. Consequently, the matrix

$$Q = lA = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} \\ \\ \frac{3}{10} & \frac{1}{5} \end{pmatrix}$$

has eigenvalues $\frac{1}{2}$ and $-\frac{2}{15}$, according to Theorem 2.1, Q is convergent to zero. With $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, the operator G has the form

$$G(x) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} \log(x_1+2) \\ \\ \log(x_2+3) \end{pmatrix},\ x_1, x_2 \geq 0.$$

According to Theorem 3.3, G is a Q-contraction and its unique fixed point x^* is the unique solution of the system (4.7). Let

$$x^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix},\ x_1^{(0)}, x_2^{(0)} \geq 0$$

be given. Let $x^{(1)} = G(x^{(0)}),\ x^{(2)} = G(x^{(1)}), \ldots$. Then $x^{(n)} = G^n(x^{(0)})$ and

$$d(x^{(n)}, x^*) = d(G^n(x^{(0)}), x^*) \leq (I_2 - Q)^{-1} Q^n d(G(x^{(0)}), x^{(0)}) \underset{n\to\infty}{\rightarrow} 0.$$

In our case,

$$Q^n = \frac{1}{2^n} A^n = \frac{1}{19 \cdot 2^n} \begin{pmatrix} 9 + 10\left(-\frac{4}{15}\right)^n & 10 - 10\left(-\frac{4}{15}\right)^n \\ \\ 9 - 9\left(-\frac{4}{15}\right)^n & 10 + 9\left(-\frac{4}{15}\right)^n \end{pmatrix}$$

and this gives an estimate of the rate of convergence in

$$\lim_{n\to\infty} d(x^{(n)}, x^*) = 0.$$

From $x^{(n+1)} = G(x^{(n)}),\ n \geq 0$, we have

$$\text{(4.8)} \qquad \begin{cases} x_1^{(n+1)} = \frac{1}{3}\log(x_1^{(n)} + 2) + \frac{2}{3}\log(x_2^{(n)} + 3) \\ \\ x_2^{(n+1)} = \frac{3}{5}\log(x_1^{(n)} + 2) + \frac{2}{5}\log(x_2^{(n)} + 3) \end{cases}.$$

Choosing different values for $x_1^{(0)}$ and $x_2^{(0)}$, we get in Figure 1, Figure 2 and Figure 3 the iterations and the representation of solutions.

Iterations	x_1	x_2
1	0.1	0.1
2	1.0015805	1.1120442
3	1.3089932	1.2835545
4	1.3687368	1.310636
5	1.3789029	1.3149648
6	1.3805765	1.3156634
7	1.3808495	1.3157766
8	1.3808939	1.315795
9	1.3809011	1.315798
10	1.3809022	1.3157985
11	1.3809024	1.3157985
12	1.3809025	1.3157986
13	1.3809025	1.3157986

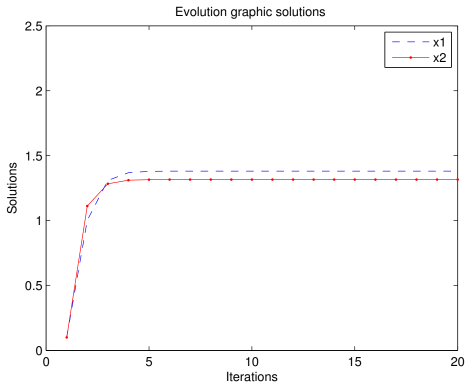

FIGURE 1. Graphical illustration of the iterates for $x_1^{(0)}=0.1,\ x_2^{(0)}=0.1.$

Iterations	x_1	x_2
1	5	1
2	1.572833	1.318533
3	1.3997301	1.3193839
4	1.3833072	1.3165573
5	1.3812567	1.3159317
6	1.380958	1.3158207
7	1.3809114	1.3158022
8	1.3809039	1.3157991
9	1.3809027	1.3157987
10	1.3809025	1.3157986
11	1.3809025	1.3157986

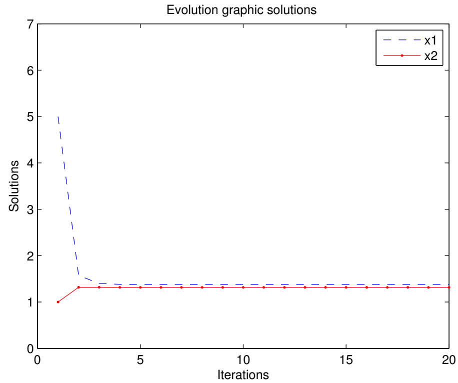

FIGURE 2. Graphical illustration of the iterates for $x_1^{(0)} = 5,\ x_2^{(0)} = 1.$

Iterations	x_1	x_2
1	1	1
2	1.2904003	1.2691233
3	1.3646087	1.3085504
4	1.3781716	1.3146414
5	1.3804543	1.3156118
6	1.3808294	1.3157683
7	1.3808906	1.3157936
8	1.3809005	1.3157978
9	1.3809022	1.3157984
10	1.3809024	1.3157985
11	1.3809025	1.3157986
12	1.3809025	1.3157986

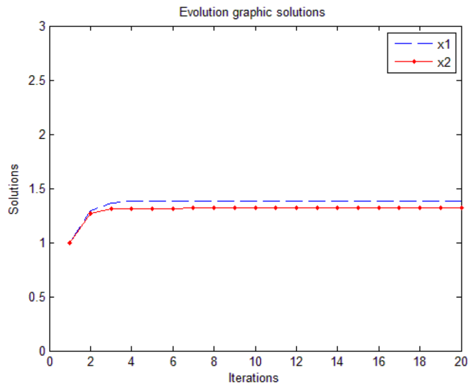

FIGURE 3. Graphical illustration of the iterates for $x_1^{(0)} = 1,\ x_2^{(0)} = 1.$

5. Conclusions and further work

Our paper is devoted to a specific family of algebraic systems, having significant applications to real-world problems. Several papers from the literature are concerned with finding approximate solutions to them. Our approach is based on the Perov's theorem. This allows to estimate componentwise the rate of convergence.

We intend to return to this topic in order to compare our results with other existent ones and to find new applications.

References

[1] A. M. Acu, I. Rașa and A. E. Șteopoaie: *Algebraic systems with positive coefficients and positive solutions*, Mathematics, **10** (8) (2022), Article ID: 1327.

[2] A. Ciurte, S. Nedevschi and I. Rașa: *An algorithm for solving some nonlinear systems with applications to extremum problems*, Taiwanese J. Math., **16** (3) (2012), 1137–1150.

[3] A. Ciurte, S. Nedevschi and I. Rașa: *Systems of nonlinear algebraic equations with unique solution*, Numer. Algorithms, **68** (2015), 367–376.

[4] A. Ciurte, S. Nedevschi and I. Rașa: *Systems of nonlinear algebraic equations, with positive solutions*, J. Inequal. Appl., (2017), Article ID: 178.

[5] Y. Du, G. Zhang and W. Feng: *Existence of positive solutions for a class of nonlinear algebraic systems*, Math. Probl. Eng., (2016), Article ID: 6120169.

[6] I. Gyori, F. Hartung and N. A. Mohamady: *Existence and uniqueness of positive solutions of a system of nonlinear algebraic equations*, Period. Math. Hung., **75** (1) (2017), 114–127.

[7] Y. Jia, Y. Gao, W. Feng and G. Zhang: *Positive solutions of a nonlinear algebraic system with sign-changing coefficient matrix*, Adv. Differ. Equ., (2020), Article ID: 630.

[8] M. Kaykobad: *Positive solutions of positive linear systems*, Linear Algebra Appl., **64** (1985), 133–140.

[9] P. N. Koumantos: *Uniqueness of the solution of a nonlinear algebraic system*, Mat. Vesnik, **74** (4) (2022), 280–288.

[10] A.I. Perov: *On the Cauchy problem for a system of ordinary differential equations*, Priblijen. Metod Res. Dif. Urav Kiev, **2** (1964), 115–134 (in Russian).

[11] R. Precup: *The role of the matrices that are convergent to zero in the study of semilinear operator systems*, Math. Comput. Modelling, **49** (3-4) (2009), 703–708.

[12] I. A. Rus: *Picard operators and applications*, Sci. Math. Jpn., **58** (1) (2003), 191–219.

[13] S. M. Stefanov: *Numerical solution of some systems of nonlinear algebraic equations*, J. Interdiscip. Math., **24** (2021), 1545–1564.

[14] G. Zhang, L. Bai: *Existence of solutions for a nonlinear algebraic system*, Discrete Dyn. Nat. Soc., (2009), Article ID: 785068.

[15] G. Zhang, W. Feng: *On the number of positive solutions of a nonlinear algebraic system*, Linear Algebra Appl., **422** (2007), 404–421.

Gabriela Motronea
Technical University of Cluj-Napoca
Department of Mathematics
Str Memorandumului No 28, 400114, Cluj-Napoca, Romania
Email address: gdenisa19@gmail.com

Diana Otrocol
Technical University of Cluj-Napoca
Department of Mathematics
Str Memorandumului No 28, 400114, Cluj-Napoca, Romania
Romanian Academy
Tiberiu Popoviciu Institute of Numerical Analysis
Str Fântânele No 57, 400110, Cluj-Napoca, Romania
Email address: diana.otrocol@math.utcluj.ro

IOAN RAŞA
TECHNICAL UNIVERSITY OF CLUJ-NAPOCA
DEPARTMENT OF MATHEMATICS
STR MEMORANDUMULUI NO 28, 400114, CLUJ-NAPOCA, ROMANIA
Email address: ioan.rasa@math.utcluj.ro