

Cesàro and barycentric approximation schemes for coincidence points of g -nonexpansive mappings

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ABSTRACT. We study quantitative approximation schemes for coincidence points of two mappings satisfying a g -nonexpansiveness condition. Starting from the approximate coincidence sequence considered in [3], we develop two complementary procedures for constructing approximate coincidence points and identify conditions under which their weak cluster points are genuine coincidence points. The first procedure is based on Cesàro averaging and applies when the mappings are affine and weak-to-weak continuous. The second is based on a barycentric construction associated with the nonexpansive mapping induced on the range of g . It applies when this range has suitable closedness, boundedness, and convexity properties, without requiring affinity. Thus, the two methods address different geometric settings: the first does not require convexity of the range of g , whereas the second does not require the mappings to be affine. As an application, we consider nonlinear Hammerstein operators on $L^2([a, b])$. Under a natural Lipschitz condition, these operators are nonexpansive. For the quantitative existence and approximation theorem we additionally require the nonlinearity to be affine; this ensures that the range $g(C)$ is closed, bounded and convex. Thus the Hammerstein application below does not cover general nonlinear Lipschitz Nemytskii terms under the stated hypotheses alone. An explicit rank-one example illustrates the construction and shows that the coincidence set associated with the canonical pair (g^2, g) can be strictly larger than the fixed point set of g .

Keywords: Coincidence point, g -nonexpansive mapping, barycentric approximation, Hammerstein integral equation.

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1. INTRODUCTION

The existence of fixed points for nonexpansive self-maps of closed, bounded, convex subsets of a Hilbert space was established independently, and in the same year, by Browder [4], Göhde [7], and Kirk [8]. A constructive, approximation-based proof of this fact was subsequently given by Browder and Petryshyn [5], and it is this constructive version that we take as our starting point.

In [3], we extended the Browder-Petryshyn theorem to a coincidence theorem for two mappings $f, g: C \rightarrow C$, where C is a subset of a Hilbert space: if $f(C)$ is a closed, convex, bounded subset of $g(C)$ and f is g -nonexpansive, meaning $\|f(u) - f(v)\| \leq \|g(u) - g(v)\|$ for all $u, v \in C$, then f and g have a coincidence point. We recall this theorem formally, together with two of its preliminary consequences, in Section 2 below.

This coincidence-point formulation belongs to a broader line of research on generalized nonexpansive-type conditions between pairs of mappings and their associated iterative or approximation schemes. In a related direction, we studied coincidence phenomena in [2] for relatively expansive maps, a natural counterpart of the relatively nonexpansive setting considered here.

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The goal of this paper is to develop explicit approximation schemes that complement the existence result established in our earlier work [3]. We give two complementary approaches, each with different additional assumptions and different quantitative rates.

The first approach uses Cesàro averaging of the approximate coincidence sequence furnished by the proof of the main theorem of [3]. If f and g are affine and weak-to-weak continuous, the Cesàro means s_n of this sequence satisfy $\|f(s_n) - g(s_n)\| = \mathcal{O}((\log n)/n)$, and every weak cluster point of (s_n) is a coincidence point. This scheme requires affinity of f and g but does not require convexity of $g(C)$.

The second approach uses an induced nonexpansive map and p -barycenters. We observe that any g -nonexpansive pair induces a nonexpansive map $A: g(C) \rightarrow f(C)$ defined by $A(g(x)) = f(x)$. This is well-defined because $g(x) = g(y)$ implies $f(x) = f(y)$. Moreover, $f(x) = g(x)$ iff $A(g(x)) = g(x)$. Thus the coincidence problem for (f, g) is equivalent to a fixed point problem for the nonexpansive map A . Under the additional geometric hypothesis that $M := g(C)$ is a closed, bounded, convex subset of a uniformly convex space satisfying a p -uniform convexity condition, we apply a quantitative p -barycenter asymptotic regularity theorem to A . This yields $\|B_n^{(p)} - AB_n^{(p)}\| = \mathcal{O}(n^{-1/p})$, which pulls back to $\|g(x_n) - f(x_n)\| = \mathcal{O}(n^{-1/p})$. This scheme does not require affinity of f and g , but it does require the additional convexity of $g(C)$.

Both approaches are explicit and quantitative; they do not claim algorithmic computability but provide indexed sequences with known rates of residual decay. The Cesàro scheme is better suited when f and g are affine and $g(C)$ is not necessarily convex; the barycentric scheme is better suited when $g(C)$ is convex but f and g are not affine. Section 3 develops both schemes and closes with a direct comparison of their hypotheses and rates. Section 4 demonstrates the barycentric scheme on a class of nonlinear Hammerstein integral equations. Under a Lipschitz condition on the kernel and the nonlinearity, the Hammerstein operator g is nonexpansive on a suitable closed, bounded, convex set $C \subset L^2([a, b])$, and $f(C) \subseteq g(C)$ together with g -nonexpansiveness of $f := g \circ g$ then come for free from the nonexpansiveness of g alone. For the existence theorem via the coincidence-point theorem of [3] and for the barycentric approximation, we impose the additional assumption that the nonlinearity is affine, which ensures that $g(C)$ is closed, bounded and convex; under this assumption, existence of a coincidence point follows and the barycentric approximation then provides explicit approximate coincidence points with an $\mathcal{O}(n^{-1/p})$ residual rate. A concrete rank-one example illustrates the construction and shows that the coincidence set of (f, g) can be strictly larger than the fixed point set of g , with an explicit residual decaying faster than the general rate.

2. PRELIMINARIES

This section recalls the existence theorem proved in our earlier work [3], the approximate sequence furnished by its proof, and the induced-map observation that underlies both approximation schemes developed in Section 3.

2.1. The Existence Theorem.

Theorem 2.1 ([3]). *Let C be a subset of a Hilbert space and let $f, g: C \rightarrow C$ satisfy:*

- (1) $f(C)$ is closed, convex, and bounded;
- (2) $f(C) \subseteq g(C)$;
- (3) f is g -nonexpansive:

$$\|f(u) - f(v)\| \leq \|g(u) - g(v)\|, \quad u, v \in C.$$

Then, there exists $x_0 \in C$ such that $f(x_0) = g(x_0)$.

Remark 2.1. Hypothesis (3) is genuinely load-bearing: Theorem 2.1 can fail without it, even when the range inclusion in (2) holds. Take $C = [0, 1]$, $f(x) = x$, and define $g: [0, 1] \rightarrow [0, 1]$ by

$$g(x) = \begin{cases} 1, & x = 0, \\ x + \frac{1}{2}, & 0 < x < \frac{1}{2}, \\ x - \frac{1}{2}, & \frac{1}{2} \leq x < 1, \\ 0, & x = 1. \end{cases}$$

Then $f(C) = g(C) = [0, 1]$, so $f(C) \subseteq g(C)$. But g has no fixed point: $g(0) = 1$, $g(1) = 0$, $g(x) = x + 1/2 \neq x$ for $0 < x < 1/2$, and $g(x) = x - 1/2 \neq x$ for $1/2 \leq x < 1$. Hence f and g have no coincidence point. Moreover f is not g -nonexpansive, since for $u = 1/2$ and $v = 1$,

$$|f(u) - f(v)| = \frac{1}{2}, \quad |g(u) - g(v)| = |0 - 0| = 0.$$

Thus inclusion of ranges alone, without the distance comparison in (3), is not enough to guarantee a coincidence point.

2.2. The Approximate Coincidence Sequence. Let H be a Hilbert space, $C \subset H$ nonempty, and $f, g: C \rightarrow C$ satisfy the hypotheses of Theorem 2.1. Its proof yields the following approximate coincidence construction. Fix $v_0 \in C$ and, for $\lambda \in (0, 1)$, define

$$h_\lambda(x) := \lambda f(x) + (1 - \lambda)f(v_0).$$

Since $f(C)$ is convex, $h_\lambda(C) \subset f(C) \subseteq g(C)$. Moreover,

$$\|h_\lambda(x) - h_\lambda(y)\| = \lambda \|f(x) - f(y)\| \leq \lambda \|g(x) - g(y)\|.$$

The contraction-type coincidence theorem used in the proof of Theorem 2.1 therefore yields $u_\lambda \in C$ such that

$$(1) \quad g(u_\lambda) = h_\lambda(u_\lambda).$$

We take $\lambda_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$ for $n \geq 1$, and set $u_n := u_{\lambda_n}$. Then

$$(2) \quad g(u_n) = \frac{n}{n+1} f(u_n) + \frac{1}{n+1} f(v_0).$$

Lemma 2.1. The sequence (u_n) satisfies

$$\|f(u_n) - g(u_n)\| \leq \frac{D}{n+1},$$

where $D := \text{diam}(f(C))$.

Proof. From (2),

$$f(u_n) - g(u_n) = \frac{1}{n+1} (f(u_n) - f(v_0)).$$

Since $v_0 \in C$, we have $f(v_0) \in f(C)$. Since $f(C)$ is bounded, $\|f(u) - f(v_0)\| \leq D$ for all $u \in C$. The claim follows. $\square$

2.3. The Induced Nonexpansive Mapping. The following observation is fundamental and does not require convexity of $g(C)$; it is the conceptual bridge between the coincidence problem for (f, g) and a fixed point problem for a single nonexpansive map, and it underlies both approximation schemes of Section 3.

Proposition 2.1. *Let X be a normed space and let C be a set. Suppose that $f, g: C \rightarrow X$ satisfy*

$$\|f(x) - f(y)\| \leq \|g(x) - g(y)\|, \quad x, y \in C.$$

Then, the formula

$$A: g(C) \rightarrow f(C), \quad A(g(x)) := f(x),$$

defines a well-defined nonexpansive mapping. Moreover,

$$f(x) = g(x) \iff A(g(x)) = g(x).$$

Consequently, if $M := g(C)$ is closed, bounded and convex and $f(C) \subseteq g(C)$, then $A: M \rightarrow M$ is a nonexpansive self-map, and coincidence points of f and g correspond exactly to fixed points of A .

Proof. If $g(x) = g(y)$, then $\|f(x) - f(y)\| \leq \|g(x) - g(y)\| = 0$, so $f(x) = f(y)$. Thus A is well-defined. For $u = g(x)$ and $v = g(y)$,

$$\|A(u) - A(v)\| = \|f(x) - f(y)\| \leq \|g(x) - g(y)\| = \|u - v\|.$$

So A is nonexpansive. Finally, $A(g(x)) = g(x)$ iff $f(x) = g(x)$. □

Remark 2.2. *The additional assumption that $g(C)$ is convex does not follow from the hypotheses of [3]. Thus, the barycentric approximation developed below applies to the subclass of pairs satisfying the hypotheses of [3] for which $g(C)$ is a closed, bounded, and convex subset of the ambient space.*

Remark 2.3. *More generally, the induced-map construction applies to any S -nonexpansive pair (T, S) , i.e., a pair satisfying $\|Tx - Ty\| \leq \|Sx - Sy\|$. The induced map $A: S(C) \rightarrow T(C)$ is nonexpansive. Whenever $S(C)$ is a closed, bounded, convex subset of a uniformly convex space and $T(C) \subseteq S(C)$, the barycentric approximation theorem applies to A .*

3. MAIN RESULTS

We now develop the two approximation schemes announced in the Introduction: a Cesàro scheme for affine pairs, and a p -barycentric scheme for pairs with convex range, transferred to the coincidence setting via Proposition 2.1. We close the section with a direct comparison of the two schemes.

3.1. Cesàro Approximation. We now assume that C is closed, bounded, and convex, and that f, g are affine and weak-to-weak continuous.

Definition 3.1. *A mapping $f: C \rightarrow H$ is affine if*

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y), \quad x, y \in C, t \in [0, 1].$$

Weak-to-weak continuity means: if $x_n \rightharpoonup x$, then $f(x_n) \rightharpoonup f(x)$.

Theorem 3.2. *Let C be a nonempty closed, bounded, convex subset of a Hilbert space H , and let $f, g: C \rightarrow C$ be affine and weak-to-weak continuous maps satisfying the hypotheses of Theorem 2.1. Let (u_n) be as in (2), and define the Cesàro means*

$$s_n := \frac{1}{n} \sum_{j=1}^n u_j.$$

Then:

- (1) $\|f(s_n) - g(s_n)\| \leq \frac{D(H_{n+1} - 1)}{n} \leq \frac{D(1 + \log(n + 1))}{n}$;
 (2) Every weak cluster point p of (s_n) satisfies $f(p) = g(p)$.

Proof. Affinity gives

$$f(s_n) - g(s_n) = \frac{1}{n} \sum_{j=1}^n (f(u_j) - g(u_j)).$$

Using Lemma 2.1,

$$\|f(s_n) - g(s_n)\| \leq \frac{D}{n} \sum_{j=1}^n \frac{1}{j+1} = \frac{D(H_{n+1} - 1)}{n}.$$

Since $H_{n+1} - 1 \leq 1 + \log(n + 1)$, the first claim follows. Now let $s_{n_k} \rightharpoonup p$. Weak-to-weak continuity gives

$$f(s_{n_k}) \rightharpoonup f(p), \quad g(s_{n_k}) \rightharpoonup g(p).$$

But $\|f(s_{n_k}) - g(s_{n_k})\| \rightarrow 0$, so $f(s_{n_k}) - g(s_{n_k}) \rightarrow 0$. Hence $f(p) = g(p)$. $\square$

Remark 3.4. In Theorem 3.2, the points u_n are furnished existentially by the contraction-type coincidence theorem used in the proof of Theorem 2.1. The theorem gives a residual estimate for any such selection satisfying (2). It does not assert algorithmic computability of u_n , nor does the residual decay alone imply strong convergence of u_n or of the Cesàro means s_n .

Since C is weakly compact, (s_n) has at least one weak cluster point, and the theorem ensures that every such point is a coincidence point.

Example 3.1. Let $H = \mathbb{R}$, $C = [0, 1]$, $f(x) = x/2$, $g(x) = x$. Then $f(C) = [0, 1/2] \subset [0, 1] = g(C)$, and f is g -nonexpansive since

$$|f(u) - f(v)| = \frac{1}{2}|u - v| \leq |u - v| = |g(u) - g(v)|.$$

For $v_0 = 1$, the construction of [3] gives $u_n = \frac{1}{n+2}$. The Cesàro mean satisfies

$$0 \leq s_n = \frac{1}{n} \sum_{j=1}^n \frac{1}{j+2} \leq \frac{1}{n} \sum_{j=1}^n \frac{1}{j} = \frac{H_n}{n} \rightarrow 0,$$

so $s_n \rightarrow 0$, and $0 = f(0) = g(0)$, consistent with Theorem 3.2.

3.2. The p -Barycentric Approximation. Let E be a uniformly convex Banach space satisfying the following p -uniform convexity inequality for some $p \geq 2$ and $c_p > 0$:

$$(UC_p) \quad \left\| \frac{x+y}{2} \right\|^p \leq \frac{1}{2} \|x\|^p + \frac{1}{2} \|y\|^p - c_p \|x-y\|^p, \quad x, y \in E.$$

For L^p and ℓ^p , $p \geq 2$, Clarkson's inequality [6] gives (UC_p) with $c_p = 2^{-p}$. In Hilbert spaces, it holds with $p = 2$ and $c_2 = 1/4$. The barycentric scheme developed below may be viewed as a finite-orbit, quantitative analogue, in spirit, of the averaging approach in Baillon's non-linear ergodic theorem [1], which guarantees weak convergence of Cesàro means of a nonexpansive orbit to a fixed point in Hilbert space; our construction replaces the Cesàro mean by a p -barycenter of the finite orbit and yields an explicit convergence rate, though the two results are not directly comparable, since one concerns weak convergence of an infinite Cesàro sequence and the other concerns the asymptotic regularity of finite-orbit minimizers.

Let $M \subset E$ be nonempty, closed, bounded, and convex, and let $A: M \rightarrow M$ be nonexpansive. Fix $z_0 \in M$. Define the functional

$$\Phi_n(z) := \frac{1}{n} \sum_{j=0}^{n-1} \|z - A^j z_0\|^p, \quad z \in M.$$

Since E is uniformly convex, it is reflexive, so the closed, bounded, convex set M is weakly compact. Each map $z \mapsto \|z - A^j z_0\|^p$ is weakly lower semicontinuous (norms are weakly lower semicontinuous, and this is preserved under the continuous increasing map $t \mapsto t^p$), so Φ_n , being a finite sum of such maps, is weakly lower semicontinuous on the weakly compact set M and therefore attains its minimum. Uniqueness of the minimizer needs less than the full strength of (UC_p) : since $p > 1$ and E is strictly convex (uniform convexity implies strict convexity), each map $z \mapsto \|z - z_j\|^p$ is strictly convex, hence so is the finite sum Φ_n , and the minimizer is unique. We denote it by $B_n^{(p)} \in M$ (the p -barycenter of the finite orbit); the stronger standing hypothesis $p \geq 2$ is used only in the rate estimate of the next theorem, not in this existence and uniqueness argument.

Theorem 3.3 (Barycentric asymptotic regularity). *Under (UC_p) with $p \geq 2$,*

$$\|B_n^{(p)} - AB_n^{(p)}\| \leq \left(\frac{2^{p-1}}{c_p} \right)^{1/p} R n^{-1/p},$$

where $R := \sup_{x \in M} \|x\|$. In particular, $\|B_n^{(p)} - AB_n^{(p)}\| \rightarrow 0$.

Proof. Write $b_n := B_n^{(p)}$ and $z_j := A^j z_0$ for $j = 0, \dots, n-1$, so that

$$\Phi_n(z) = \frac{1}{n} \sum_{j=0}^{n-1} \|z - z_j\|^p.$$

- *Step 1 (shift estimate).* By nonexpansiveness of A , for $k = 0, \dots, n-1$,

$$\|z_{k+1} - Ab_n\| = \|Az_k - Ab_n\| \leq \|z_k - b_n\|.$$

Raising to the p -th power and summing over $k = 0, \dots, n-1$,

$$\sum_{k=0}^{n-1} \|z_{k+1} - Ab_n\|^p \leq \sum_{k=0}^{n-1} \|z_k - b_n\|^p = n \Phi_n(b_n).$$

The left-hand side sums $\|z_j - Ab_n\|^p$ over $j = 1, \dots, n$, which is the same as summing over $j = 0, \dots, n-1$ (that is, $n \Phi_n(Ab_n)$) after removing the $j = 0$ term and adding the $j = n$ term:

$$\sum_{k=0}^{n-1} \|z_{k+1} - Ab_n\|^p = n \Phi_n(Ab_n) - \|z_0 - Ab_n\|^p + \|z_n - Ab_n\|^p.$$

Combining the last two displays and discarding the nonnegative term $\|z_n - Ab_n\|^p$,

$$n \Phi_n(Ab_n) - n \Phi_n(b_n) \leq \|z_0 - Ab_n\|^p - \|z_n - Ab_n\|^p \leq \|z_0 - Ab_n\|^p \leq (2R)^p.$$

Thus

$$(*) \quad \Phi_n(Ab_n) - \Phi_n(b_n) \leq \frac{(2R)^p}{n}.$$

Here, we used $z_0, Ab_n \in M$ and $\|z_0 - Ab_n\| \leq \|z_0\| + \|Ab_n\| \leq 2R$.

• *Step 2 (minimizer property).* Since b_n minimizes Φ_n over M , and $(b_n + Ab_n)/2 \in M$ by convexity of M ,

$$\Phi_n(b_n) \leq \Phi_n\left(\frac{b_n + Ab_n}{2}\right).$$

• *Step 3 (uniform convexity).* Applying (UC_p) with $x = b_n - z_j$, $y = Ab_n - z_j$, for each j ,

$$\left\|\frac{b_n + Ab_n}{2} - z_j\right\|^p \leq \frac{1}{2}\|b_n - z_j\|^p + \frac{1}{2}\|Ab_n - z_j\|^p - c_p\|b_n - Ab_n\|^p.$$

Averaging over $j = 0, \dots, n-1$,

$$\Phi_n\left(\frac{b_n + Ab_n}{2}\right) \leq \frac{1}{2}\Phi_n(b_n) + \frac{1}{2}\Phi_n(Ab_n) - c_p\|b_n - Ab_n\|^p.$$

Combining with Step 2,

$$\Phi_n(b_n) \leq \frac{1}{2}\Phi_n(b_n) + \frac{1}{2}\Phi_n(Ab_n) - c_p\|b_n - Ab_n\|^p,$$

which rearranges to

$$c_p\|b_n - Ab_n\|^p \leq \frac{1}{2}(\Phi_n(Ab_n) - \Phi_n(b_n)).$$

• *Step 4 (conclusion).* By $(*)$,

$$c_p\|b_n - Ab_n\|^p \leq \frac{(2R)^p}{2n} = \frac{2^{p-1}R^p}{n},$$

so

$$\|b_n - Ab_n\|^p \leq \frac{2^{p-1}R^p}{c_p n}, \quad \text{i.e.} \quad \|b_n - Ab_n\| \leq \left(\frac{2^{p-1}}{c_p}\right)^{1/p} R n^{-1/p}. \quad \square$$

Remark 3.5. For $p = 2$ in a Hilbert space, $c_2 = 1/4$, and $B_n^{(2)} = \frac{1}{n} \sum_{j=0}^{n-1} A^j z_0$, so the estimate becomes $\|B_n^{(2)} - AB_n^{(2)}\| \leq 2\sqrt{2} R n^{-1/2}$.

Corollary 3.1 (Weak cluster points are fixed points). *Let E satisfy (UC_p) , let $M \subset E$ be nonempty, closed, bounded and convex, and let $A: M \rightarrow M$ be nonexpansive. Assume that A is weak-to-weak continuous. Let $B_n^{(p)}$ be the p -barycenters of the orbit of some $z_0 \in M$. Then $(B_n^{(p)})$ has at least one weak cluster point: Since E is reflexive and M is nonempty, closed, bounded and convex, M is weakly compact. By the Eberlein-Šmulian theorem, every sequence in M has a weakly convergent subsequence. Moreover, every weak cluster point of $(B_n^{(p)})$ is a fixed point of A . In particular, every weak cluster point is a solution of*

$$z = Az.$$

Proof. By Theorem 3.3,

$$\|B_n^{(p)} - AB_n^{(p)}\| \longrightarrow 0.$$

Let $B_{n_k}^{(p)} \rightharpoonup z^*$. Since A is weak-to-weak continuous,

$$AB_{n_k}^{(p)} \rightharpoonup Az^*.$$

On the other hand,

$$B_{n_k}^{(p)} - AB_{n_k}^{(p)} \longrightarrow 0$$

strongly and hence weakly. Therefore

$$z^* - Az^* = 0,$$

so $Az^* = z^*$. □

3.3. Approximation of Coincidence Points. Combining the induced-map construction of Proposition 2.1 with Theorem 3.3 transfers the barycentric estimate directly to the coincidence problem for (f, g) .

Corollary 3.2 (Barycentric approximation of coincidence points). *Let E be a Banach space satisfying (UC_p) for some $p \geq 2$. Let C be a nonempty set and let $f, g: C \rightarrow E$ satisfy*

$$\|f(x) - f(y)\| \leq \|g(x) - g(y)\|, \quad x, y \in C, \quad \text{and} \quad f(C) \subseteq g(C).$$

Assume in addition that $M := g(C)$ is closed, bounded and convex, and let $A: M \rightarrow M$, $A(g(x)) := f(x)$, be the induced map of Proposition 2.1. For $z_0 = g(v_0) \in M$, let $B_n^{(p)}$ be the p -barycenter of the orbit $z_0, Az_0, \dots, A^{n-1}z_0$, and let $R = \sup_{z \in M} \|z\|$. Then there exists $x_n \in C$ with $g(x_n) = B_n^{(p)}$, and

$$\|g(x_n) - f(x_n)\| \leq \left(\frac{2^{p-1}}{c_p} \right)^{1/p} R n^{-1/p}.$$

Thus (x_n) is an approximate coincidence sequence with quantitative rate $\mathcal{O}(n^{-1/p})$.

Proof. By Proposition 2.1, A is a well-defined nonexpansive self-map of M , so Theorem 3.3 applies and gives

$$\|B_n^{(p)} - AB_n^{(p)}\| \leq \left(\frac{2^{p-1}}{c_p} \right)^{1/p} R n^{-1/p}.$$

Since $B_n^{(p)} \in M = g(C)$, there exists $x_n \in C$ with $B_n^{(p)} = g(x_n)$, and then $AB_n^{(p)} = A(g(x_n)) = f(x_n)$, so $\|g(x_n) - f(x_n)\| = \|B_n^{(p)} - AB_n^{(p)}\|$, giving the stated bound. $\square$

Remark 3.6. The point x_n in Corollary 3.2 is any preimage of $B_n^{(p)}$ under g . Its existence is guaranteed because $B_n^{(p)} \in M = g(C)$, but the corollary does not provide a constructive selection rule. The estimate is residual only; it does not by itself imply strong convergence of the selected points x_n .

Remark 3.7. Corollary 3.2 does not assume weak-to-weak continuity of A , f , or g . Consequently, it guarantees an approximate coincidence sequence with the stated residual rate, but it does not by itself ensure that every weak cluster point of a selected preimage sequence (x_n) is a coincidence point. A sufficient additional condition for this conclusion is that f and g are weak-to-weak continuous, as assumed in Corollary 3.1, provided the other hypotheses of that corollary hold.

Remark 3.8. Corollary 3.2 does not require f or g to be affine, nor weak-to-weak continuous, and it does not require $f(C)$ itself to be closed or convex; only the geometric condition that $M = g(C)$ is closed, bounded, and convex, together with the p -uniform convexity condition on E . In this sense, the barycentric scheme is logically independent of the closedness/convexity of $f(C)$ demanded by the existence theorem (Theorem 2.1); it needs only the geometry of $g(C)$, because the induced map A is a self-map of $M = g(C)$ alone.

3.4. Comparison of the Two Schemes. The Cesàro and barycentric schemes trade one set of hypotheses for another, and are best viewed as complementary rather than competing:

Scheme	Main assumption	Geometry required	Rate
Cesàro	f, g affine, weak-to-weak continuous	C closed, bounded, convex	$\mathcal{O}(\log n/n)$
p -barycentric	no affinity or weak continuity	$g(C)$ closed, bounded, convex	$\mathcal{O}(n^{-1/p})$

Neither scheme dominates the other: the Cesàro scheme needs no convexity of $g(C)$ but does need f, g affine, while the barycentric scheme needs no affinity but does need $M = g(C)$ convex. In the Hilbert-space case $p = 2$, the barycentric rate specializes to $\mathcal{O}(n^{-1/2})$ and the barycenter itself reduces to the ordinary Cesàro mean of the orbit $z_0, Az_0, \dots, A^{n-1}z_0$ (Remark after Theorem 3.3), which is the sense in which the barycentric scheme may be regarded as a finite-orbit relative of Baillon's ergodic theorem, as noted above.

4. APPLICATION TO HAMMERSTEIN INTEGRAL EQUATIONS

We now apply the theory developed above to a class of nonlinear Hammerstein integral equations, with a general Lipschitz nonlinearity at the nonexpansiveness stage and an affine nonlinearity in the existence and approximation theorem. Unlike a naive application in which one mapping is taken to be the identity, we build here a pair of distinct operators $f \neq g$, giving a nontrivial two-map coincidence formulation of the underlying Hammerstein fixed point problem, solved via the coincidence-point theorem (Theorem 2.1) rather than via the fixed point theorem of Browder and Petryshyn. Let

$$H = L^2([a, b])$$

with its usual Hilbert space norm

$$\|x\|_2 = \left(\int_a^b |x(t)|^2 dt \right)^{1/2}.$$

Let $C \subset H$ be a nonempty closed, bounded, convex set. Fix

$$h \in L^2([a, b]),$$

a kernel $K \in L^2([a, b]^2)$, and a Lipschitz nonlinearity $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ with Lipschitz constant L_Φ , that is,

$$(H1) \quad |\Phi(r) - \Phi(s)| \leq L_\Phi |r - s|, \quad r, s \in \mathbb{R}.$$

The nonexpansiveness results below are valid for general Lipschitz Φ . However, the existence theorem and the barycentric approximation theorem in this section are proved under the additional assumption that Φ is affine, which is needed to ensure that $g(C)$ is closed, bounded and convex.

4.1. Nonexpansive Hammerstein Operators. Let

$$(\mathcal{K}y)(t) = \int_a^b K(t, s)y(s) ds,$$

a bounded linear operator on $L^2([a, b])$ with operator norm $\|\mathcal{K}\|$. Define the Hammerstein operator

$$(H2) \quad g(x)(t) = h(t) + \int_a^b K(t, s)\Phi(x(s)) ds.$$

As in the scalar Nemytskii estimate used previously, (H1) gives $\|\Phi(x) - \Phi(y)\|_2 \leq L_\Phi \|x - y\|_2$ pointwise, and consequently

$$(H3) \quad \|g(x) - g(y)\|_2 = \|\mathcal{K}(\Phi(x) - \Phi(y))\|_2 \leq L_\Phi \|\mathcal{K}\| \|x - y\|_2, \quad x, y \in H.$$

Proposition 4.2. *Let $g: C \rightarrow C$ be given by (H2). If*

$$(H4) \quad L_\Phi \|\mathcal{K}\| \leq 1,$$

then g is nonexpansive on C :

$$\|g(x) - g(y)\|_2 \leq \|x - y\|_2, \quad x, y \in C.$$

Proof. Immediate from (H3) and (H4). □

The strict inequality $L_\Phi \|\mathcal{K}\| < 1$ puts the problem in the classical contraction setting. The genuinely nonexpansive case considered here includes the borderline value $L_\Phi \|\mathcal{K}\| = 1$, where the Banach contraction theorem is no longer available.

4.2. The Canonical Coincidence Pair. The key step that turns the fixed point problem $g(x) = x$ into a nontrivial two-map coincidence formulation is the following elementary but useful observation, which applies to any nonexpansive self-map, not only to Hammerstein operators.

Proposition 4.3 (Canonical coincidence pair). *Let C be a nonempty set in a normed space and let $g: C \rightarrow C$ be nonexpansive. Define*

$$f := g \circ g, \quad f(x) := g(g(x)), \quad x \in C.$$

Then:

- (1) $f(C) \subseteq g(C)$;
- (2) f is g -nonexpansive: $\|f(x) - f(y)\| \leq \|g(x) - g(y)\|$ for all $x, y \in C$;
- (3) $x \in C$ is a coincidence point of (f, g) if and only if $g(x)$ is a fixed point of g . In particular,

$$\text{Coin}(f, g) = g^{-1}(\text{Fix}(g)),$$

so every fixed point of g is a coincidence point of (f, g) , but the coincidence set of (f, g) may be strictly larger than $\text{Fix}(g)$.

Proof. (1) For $x \in C$, $f(x) = g(g(x))$; since $g(x) \in C$ (as $g: C \rightarrow C$), $f(x) = g(g(x)) \in g(C)$.

(2) By nonexpansiveness of g , applied to the pair $u = g(x)$, $v = g(y)$,

$$\|f(x) - f(y)\| = \|g(g(x)) - g(g(y))\| \leq \|g(x) - g(y)\|.$$

(3) $f(x) = g(x)$ means $g(g(x)) = g(x)$, i.e., the point $y := g(x)$ satisfies $g(y) = y$. Conversely, if $g(y) = y$ for $y = g(x)$, then $f(x) = g(g(x)) = g(y) = y = g(x)$, so x is a coincidence point. □

Remark 4.9. Proposition 4.3 requires no additional hypothesis on g beyond nonexpansiveness of the self-map $g: C \rightarrow C$: the inclusion $f(C) \subseteq g(C)$ and the g -nonexpansiveness of f hold automatically. This is exactly the mechanism used, in the linear case $g(x)(t) = \int_a^b K(t, s)x(s) ds$, $f = g \circ g$, in the integral-equation example in [3]; here we extend it to the nonlinear Hammerstein setting.

Applying Proposition 4.3 to the Hammerstein operator g of (H2), set

$$(H5) \quad f(x) := g(g(x)), \quad \text{i.e.} \quad f(x)(t) = h(t) + \int_a^b K(t, s) \Phi \left(h(s) + \int_a^b K(s, r) \Phi(x(r)) dr \right) ds.$$

The coincidence equation $f(x) = g(x)$ reads, explicitly,

$$(H6) \quad h(t) + \int_a^b K(t, s) \Phi \left(h(s) + \int_a^b K(s, r) \Phi(x(r)) dr \right) ds = h(t) + \int_a^b K(t, s) \Phi(x(s)) ds,$$

where $t \in [a, b]$. By Proposition 4.3 (3), (H6) holds precisely when $y := g(x)$ solves the original Hammerstein equation $g(y) = y$; that is, (H6) is equivalent to $g(x) \in \text{Fix}(g)$. Every fixed point of g trivially solves (H6), but (H6) also admits solutions x that are not themselves fixed points

of g -it suffices that g map x onto some fixed point. Equation (H6) is therefore a nontrivial two-map coincidence formulation of the original fixed point problem, whose solution set can be considerably larger than $\text{Fix}(g)$, even though, as we make precise below, its solvability is equivalent to that of the original fixed point equation. Below, we solve (H6) as a coincidence-point problem rather than as a fixed point problem.

4.3. Existence and Barycentric Approximation. To invoke Theorem 2.1 and Corollary 3.2 for the pair (f, g) , we additionally need $M := g(C)$ to be closed, bounded, and convex-convexity of $f(C)$ is automatic from part (1) of Proposition 4.3 once $g(C)$ itself is known to be convex. Since g is in general a nonlinear map, $g(C)$ need not be convex even when C is; the following lemma isolates a natural situation in which it is.

Lemma 4.2 (Image lemma). *Suppose in addition that Φ is affine, $\Phi(r) = \mu r + \nu$ for constants $\mu, \nu \in \mathbb{R}$ (so that $L_\Phi = |\mu|$), and that $g: C \rightarrow C$ is as in (H2), with C closed, bounded and convex. Then for every closed, bounded, convex set $D \subseteq C$, the image $g(D)$ is closed, bounded, and convex. In particular, taking $D = C$, $M := g(C)$ is closed, bounded and convex.*

Proof. Since $\Phi(r) = \mu r + \nu$, we have

$$g(x) = h + \mathcal{K}(\mu x + \nu e),$$

so g is affine. Moreover, the bounded linear operator $\mathcal{K}$ is weak-to-weak continuous, and hence so is g .

Since $H = L^2([a, b])$ is reflexive and D is closed, bounded and convex, D is weakly compact. Therefore $g(D)$ is weakly compact. In particular, $g(D)$ is norm-closed and bounded. Finally, because g is affine and D is convex, $g(D)$ is convex. $\square$

Theorem 4.4 (Existence of a coincidence point). *Let $C \subset L^2([a, b])$ be nonempty, closed, bounded, and convex, let $g: C \rightarrow C$ be given by (H2) with Φ affine, and suppose (H4) holds. Let $f := g \circ g$. Then f and g have a coincidence point $x^* \in C$, that is,*

$$h(t) + \int_a^b K(t, s) \Phi(g(x^*)(s)) ds = h(t) + \int_a^b K(t, s) \Phi(x^*(s)) ds, \quad t \in [a, b].$$

Proof. By Proposition 4.2, g is nonexpansive on C , so by Proposition 4.3, $f = g \circ g$ is g -nonexpansive with $f(C) \subseteq g(C)$. By Lemma 4.2 applied with $D = C$, $M := g(C)$ is closed, bounded and convex; in particular $M \subseteq C$ (since $g: C \rightarrow C$), so M is itself a closed, bounded, convex subset of C . Applying Lemma 4.2 a second time, now with $D = M$, the image

$$g(M) = g(g(C)) = f(C)$$

is closed, bounded and convex. Since also $f(C) \subseteq M = g(C)$, the pair (f, g) satisfies all hypotheses of Theorem 2.1 on C : $f(C)$ is a closed, convex, bounded subset of $g(C)$, and f is g -nonexpansive. Consequently, f and g have a coincidence point $x^* \in C$. $\square$

Remark 4.10. *This is the point where we depart from the earlier, fixed point-based treatment of the Hammerstein equation: existence here is obtained from the coincidence-point theorem (Theorem 2.1) applied to the genuinely distinct pair $(f, g) = (g \circ g, g)$, rather than from the Browder-Petryshyn fixed point theorem applied to g alone. This is a different route to essentially the same existence question, not a weaker one: by Proposition 4.3 (3), $\text{Coin}(f, g) = g^{-1}(\text{Fix}(g))$, and this set is nonempty if and only if $\text{Fix}(g)$ itself is nonempty (if $y \in \text{Fix}(g)$ then $y = g(y) \in g(C)$, so $y \in g^{-1}(\text{Fix}(g))$; conversely any element of $g^{-1}(\text{Fix}(g))$ exhibits a fixed point of g). Theorem 4.4 therefore gives an alternative proof that g has a fixed point, obtained from Theorem 2.1 rather than from Browder-Petryshyn. What is genuinely new is not the existence question but the size and shape of the solution set: $\text{Coin}(f, g)$ can be*

substantially larger than $\text{Fix}(g)$, since it contains every point mapped by g onto a fixed point, not only the fixed points themselves. The rank-one example below makes this precise.

We now approximate a coincidence point using Corollary 3.2. A pleasant simplification occurs here: since $A: M \rightarrow M$ is defined by $A(g(x)) := f(x) = g(g(x))$, for any $z = g(x) \in M$ we have

$$A(z) = g(g(x)) = g(z),$$

so A is simply the restriction of g to $M = g(C)$. Consequently, for $v_0 \in C$ and $z_0 := g(v_0)$, the orbit

$$z_0, Az_0, A^2z_0, \dots$$

used to build the barycenter coincides with the tail of the Picard orbit of g started at v_0 :

$$(H7) \quad z_j = A^j z_0 = g^{j+1}(v_0), \quad j = 0, 1, 2, \dots$$

Theorem 4.5. Assume the hypotheses of Theorem 4.4, and suppose in addition that $H = L^2([a, b])$ satisfies (UC_p) for some $p \geq 2$ (in particular $p = 2$ always works in Hilbert space). Fix $v_0 \in C$ and let $B_n^{(p)}$ be the p -barycenter of the orbit (H7), i.e., of $g(v_0), g^2(v_0), \dots, g^n(v_0)$. Let

$$R := \sup_{z \in M} \|z\|_2, \quad M = g(C).$$

Then there exists $x_n \in C$ with $g(x_n) = B_n^{(p)}$, and

$$(H8) \quad \|g(x_n) - g(g(x_n))\|_2 \leq \left(\frac{2^{p-1}}{c_p} \right)^{1/p} R n^{-1/p}.$$

Note that $f(x_n) = g(g(x_n))$, so the left-hand side of (H8) is $\|g(x_n) - f(x_n)\|_2$. In the Hilbert-space case $p = 2$ ($c_2 = 1/4$), the barycenter is the ordinary Cesàro mean of the shifted orbit,

$$(H9) \quad B_n^{(2)} = \frac{1}{n} \sum_{k=1}^n g^k(v_0),$$

and (H8) becomes

$$\|g(x_n) - g(g(x_n))\|_2 \leq 2\sqrt{2} R n^{-1/2}.$$

Proof. By Theorem 4.4, $A: M \rightarrow M$ is a well-defined nonexpansive self-map, and as noted above $A = g|_M$. Corollary 3.2 applied to (f, g) on C yields $x_n \in C$ with $g(x_n) = B_n^{(p)}$ and

$$\|g(x_n) - f(x_n)\|_2 = \|B_n^{(p)} - AB_n^{(p)}\|_2 \leq \left(\frac{2^{p-1}}{c_p} \right)^{1/p} R n^{-1/p}.$$

Since $f(x_n) = g(g(x_n))$, this is exactly (H8). The identification (H9) of $B_n^{(2)}$ as the arithmetic mean of the orbit (H7) follows from the Hilbert-space case of Theorem 3.3 (Remark after that theorem), and substituting $c_2 = 1/4$, $p = 2$ into the general rate gives $2\sqrt{2} R n^{-1/2}$. $\square$

Remark 4.11. Unlike the residual $\|g(x_n) - x_n\|$ of a fixed point iteration, the quantity controlled here, $\|g(x_n) - g(g(x_n))\|_2$, measures how far $g(x_n)$ is from being a fixed point of g ; it can be small even when x_n itself is far from any fixed point of g , consistent with Proposition 4.3 (3).

4.4. A Rank-One Example. We revisit the rank-one kernel used earlier, now to illustrate the coincidence problem (H6) rather than a fixed point problem. Let $H = L^2([0, 1])$, let $e \equiv 1$ on $[0, 1]$ (so $\|e\|_2 = 1$), and let

$$K(t, s) = \alpha, \quad 0 < \alpha < 1, \quad h = 0, \quad \Phi(r) = r.$$

Then, as computed previously, $g(x)(t) = \alpha \int_0^1 x(s) ds = \alpha \langle x, e \rangle e$, so that

$$g(x) = \alpha \langle x, e \rangle e.$$

Since Φ is affine (indeed linear) with $L_\Phi = 1$ and $L_\Phi \|K\| = \alpha < 1$, (H4) holds. Let

$$C = B_R := \{x \in L^2([0, 1]) : \|x\|_2 \leq R\}, \quad R \geq 1,$$

so that $v_0 = e \in C$ (recall $\|e\|_2 = 1$). As before, g is (in fact) a contraction on C , and $g(C) \subset C$. Now compute $f = g \circ g$:

$$f(x) = g(g(x)) = g(\alpha \langle x, e \rangle e) = \alpha \langle \alpha \langle x, e \rangle e, e \rangle e = \alpha^2 \langle x, e \rangle e,$$

using $\langle e, e \rangle = \|e\|_2^2 = 1$. Thus the coincidence equation $f(x) = g(x)$ reads

$$\alpha^2 \langle x, e \rangle e = \alpha \langle x, e \rangle e \iff \alpha(\alpha - 1) \langle x, e \rangle e = 0 \iff \langle x, e \rangle = 0$$

(since $0 < \alpha < 1$ implies $\alpha(\alpha - 1) \neq 0$). Since coincidence points must lie in the common domain C of f and g , the coincidence set of (f, g) is

$$\text{Coin}(f, g) = C \cap \left\{ x \in L^2([0, 1]) : \int_0^1 x(s) ds = 0 \right\} = B_R \cap e^\perp,$$

the intersection of the ball $C = B_R$ with the hyperplane $e^\perp = \{x : \langle x, e \rangle = 0\}$ – an infinite-dimensional, closed, bounded, convex subset of C , in sharp contrast with the earlier fixed point problem $g(x) = x$, whose only solution was $\text{Fix}(g) = \{0\}$. This does not mean the coincidence problem is easier to solve than the fixed point problem: by the previous remark, $\text{Coin}(f, g) \neq \emptyset$ exactly because $\text{Fix}(g) = \{0\} \neq \emptyset$. What the computation shows is that the coincidence set can be vastly larger than the fixed point set: every zero-mean function $x \in C$ satisfies $g(x) = 0 \in \text{Fix}(g)$, and is therefore a coincidence point of (f, g) , regardless of the value of x away from its mean. We now compute the barycentric approximation explicitly. Take $v_0 = e$. Since $g^k(e) = \alpha^k e$ for $k \geq 1$ (shown earlier), (H7) gives

$$z_j = g^{j+1}(e) = \alpha^{j+1} e, \quad j = 0, \dots, n-1,$$

and by (H9),

$$(H10) \quad B_n^{(2)} = \frac{1}{n} \sum_{j=0}^{n-1} \alpha^{j+1} e = \frac{\alpha(1 - \alpha^n)}{n(1 - \alpha)} e.$$

Since $g(x) = \alpha \langle x, e \rangle e$, we may take

$$x_n := \frac{1 - \alpha^n}{n(1 - \alpha)} e,$$

for which $g(x_n) = \alpha \langle x_n, e \rangle e = \frac{\alpha(1 - \alpha^n)}{n(1 - \alpha)} e = B_n^{(2)}$, as required. Then

$$g(g(x_n)) = g(B_n^{(2)}) = \alpha \left\langle \frac{\alpha(1 - \alpha^n)}{n(1 - \alpha)} e, e \right\rangle e = \frac{\alpha^2(1 - \alpha^n)}{n(1 - \alpha)} e,$$

so that

$$g(x_n) - g(g(x_n)) = \frac{\alpha(1 - \alpha^n)}{n(1 - \alpha)} e - \frac{\alpha^2(1 - \alpha^n)}{n(1 - \alpha)} e = \frac{\alpha(1 - \alpha^n)}{n} e,$$

and

$$(H11) \quad \|g(x_n) - g(g(x_n))\|_2 = \frac{\alpha(1 - \alpha^n)}{n} \leq \frac{\alpha}{n}.$$

As in the earlier fixed point example, the residual in (H11) decays like $\mathcal{O}(n^{-1})$, faster than the general $\mathcal{O}(n^{-1/2})$ bound from Theorem 4.5. Moreover $x_n = \frac{1-\alpha^n}{n(1-\alpha)}e \rightarrow 0$ strongly as $n \rightarrow \infty$, and the limit $x^* = 0$ does lie in the coincidence set $B_R \cap e^\perp$ identified above, so the barycentric sequence (x_n) converges to an actual coincidence point of (f, g) -though, as the computation above shows, it is only one of infinitely many.

Remark 4.12. *This example shows concretely how the coincidence-point formulation (H6) genuinely differs from the fixed point equation $g(x) = x$: although solvability of the two problems is equivalent here (both hold exactly because $0 \in \text{Fix}(g)$), their solution sets differ enormously in size and structure; a single point versus an infinite-dimensional convex subset of C ; even though the residual estimate governing the approximation, (H11), has exactly the same algebraic form as in the fixed point case, up to the extra factor of α contributed by the additional application of g .*

5. CONCLUSION

We have presented two complementary explicit approximation schemes for coincidence points in our setting in [3], together with an application to a class of Hammerstein integral equations.

- Cesàro averaging of our approximate coincidence sequence (u_n) , and gives an $\mathcal{O}((\log n)/n)$ scheme requiring f, g affine and weak-to-weak continuous, with no convexity assumption on $g(C)$ (Theorem 3.2).
- The induced map $A: g(C) \rightarrow f(C)$ of Proposition 2.1 converts the coincidence problem for (f, g) into a fixed point problem for A , the conceptual bridge on which both approximation schemes rest.
- A quantitative p -barycenter asymptotic regularity theorem for nonexpansive self-maps of a closed, bounded, convex set (Theorem 3.3) transfers, via the induced map, to an $\mathcal{O}(n^{-1/p})$ coincidence-approximation scheme requiring no affinity, but requiring $g(C)$ closed, bounded, and convex (Corollary 3.2); the two schemes are compared directly in Section 3.
- For a Hammerstein operator g generated by a Lipschitz nonlinearity, the canonical pair $(f, g) = (g \circ g, g)$ is automatically g -nonexpansive with $f(C) \subseteq g(C)$; specializing to an affine nonlinearity secures the range geometry needed for existence (via Theorem 2.1) and for the barycentric approximation (Section 4). Thus the quantitative Hammerstein application is not asserted for general nonlinear Lipschitz Nemytskii terms under the stated hypotheses alone. In the present application, the affine-nonlinearity assumption ensures the closedness, boundedness, and convexity of $g(C)$ required by the existence and approximation theorems.
- The rank-one example gives an explicit $\mathcal{O}(n^{-1})$ rate, faster than the general $\mathcal{O}(n^{-1/2})$ Hilbert-space estimate, and exhibits a coincidence set, $B_R \cap e^\perp$, that is infinite-dimensional despite the fixed point set of g being the single point $\{0\}$.

Both schemes complement our existence theorem in earlier work by providing quantitative approximation procedures, and the induced-map construction unifies them: the same reduction to a fixed point problem for a single nonexpansive map underlies both the Cesàro and the barycentric approximation.

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