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Research Article

On the induced connection on sections of Toeplitz operators

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ABSTRACT. The purpose of the present article is to show that an upper bound of the induced connection on sections of Toeplitz operators is bounded by a function of the Hankel and of the Toeplitz operators on a weighted Hilbert Bergman space on a bounded domain of a complete Kähler manifold.

Keywords: Bundle of Bergman space, Chern connection, Hankel and Toeplitz operators, Kähler manifold.

2020 Mathematics Subject Classification: 30H20, 32L05, 32A19.

1. INTRODUCTION

We state that an upper bound of the induced connection on sections of Toeplitz operators is bounded by a function of the Hankel and Toeplitz operators on a weighted Hilbert Bergman space on a bounded domain of a Kähler manifold such the weighted function is a smooth function, see below Theorem 3.1. The particular case when a weighted Hilbert Bergman space is defined on a bounded domain of $\mathbb{C}^n$, Engliš and Zhang has showed that the induced connection on sections of Toeplitz operators is equal to a function of the Hankel and Toeplitz operators, see Lemma 3.3 in [5] and see below. As it it is known that a Toeplitz operator is given in terms of a Bergman kernel. Nonetheless, the expression of this kernel cannot be provided explicitly in a Kähler manifold, e.g. Ma and Marinescu studied the asymptotic behaviour of the generalized Bergman kernels on symplectic manifolds [12, 13]. Zelditch and Schlichenmaier have obtained an asymptotic expansion of the Bergman kernel on the diagonal (resp. in a neighborhood of the diagonal) of the Cartesian product of the unit circle principal bundle in the dual of a positive holomorphic line bundle over a compact Kähler manifold [15] (resp. [10, Theorem 5.6]). Then, Hezari, Kelleher, Seto, and Xu state the existence of the asymptotic expansion of the Bergman kernel in the Böchner coordinates [8, Theorem 1.1]. M. Engliš has determined an asymptotic expansion of a weighted Bergman kernel on a pseudoconvex domain in $\mathbb{C}^n$ [3, Theorem 1]. Therefore, to avoid the use of an asymptotic expression of a weighted Bergman kernel, we cover our bounded Kähler domain by geodesic balls which are biholomorphic to Euclidean complex balls and where the Bergman kernel can be determined explicitly for a suitable weighted function.

Let M^n be an n-dimensional complete Kähler manifold which is defined by the following three tensors. The first one is a Riemannian metric g on M^n, that is, a positive definite symmetric bilinear form on TM^n, the tangent bundle of M^n, the second one is ω, the sympletic Kähler form an antisymmetric bilinear form on TM^n, and the third one is J, an almost complex structure on M^n, a TM^n-valued endomorphism map on TM^n such that $J^2 = -Id_{TM^n}$. These three tensors are connected by the following algebraic relations. Let $(u_1, u_2) \in T_pM^n \times T_pM^n$

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for $p \in M^n$, we have $g(Ju_1, u_2) = \omega(u_1, u_2)$ and $\omega(Ju_1, Jv_2) = \omega(u_1, u_2)$. Therefore, each of g, ω, and J is determined by the remaining two. Locally the expression of ω is given by $\omega = \frac{i}{2}\sum_{l,k=1}^{n} h_{l\overline{k}} dz_l \wedge d\overline{z_k}$, where $(h_{l\overline{k}})_{l,k}$ are the coefficients of a positive definite Hermitian matrix, and the n-tensor $\omega^{\otimes n} := \omega^n$ is related to the Riemannian volume element $d\mu_g$ of (M^n, g) as $\omega^n = n! d\mu_g$.

Let $z \in D$, a domain in $\mathbb{C}^m$, φ be a smooth positive function on $\mathcal{V} \times D$ such that $\mathcal{V}$ is a bounded domain of M^n and $\varphi_z(\cdot) = \varphi(z, \cdot)$. Let π be the projection map from $M^n \times D$ to D, we denote by L_2^z the Hilbert space $L_2(\mathcal{V}, e^{-\varphi_z}\omega^n)$ of square-integrable measurable functions on $\mathcal{V}$ with respect to the measure $e^{-\varphi_z}\omega^n$ and endowed with the Hermitian inner product $\langle \cdot, \cdot \rangle_{L_2^z}$ defined as

$$\langle u_z, v_z \rangle_{L_2^z} = \int_{\mathcal{V}} u_z(\xi)\overline{v_z}(\xi) e^{-\varphi_z(\xi)}\omega^n, \text{ for } (u_z = u(z, \cdot), v_z = v(z, \cdot)) \in [L_2^z(\mathcal{V}, e^{-\varphi_z}\omega^n)]^2,$$

and we denote by E_z the subspace of L_2^z of holomorphic functions with compact support on $\mathcal{V}$. Let P be the Bergman projection onto E_z on L_2^z, that is, $P(L_2^z) = E_z$. The Toeplitz operator with symbol $f_z = f(z, \cdot)$, a bounded function on $\mathcal{V}$, is defined by $T_{f_z} = PM_{f_z}$ where M_{f_z} is the multiplication operator by f_z.

The Hankel operator is defined from E_z to E_z by $H_{f_z} = (I - P)M_{f_z}$, where I represents the identity operator. Let u_z be a smooth function on $\mathcal{V}$ and $H^*_{\overline{u}_z}$ be the adjoint operator of $H_{\overline{u}_z}$. Thus by using a straightforward calculus, we get

$$T_{\overline{f_z} u_z} - T_{u_z} T_{\overline{f_z}} = H^*_{\overline{u}_z} H_{\overline{f_z}}. \tag{1.1}$$

Let (E, h^E) be the holomorphic Hermitian vector bundle over D such that its fibers are the family $(E_z)_{z\in D}$ such that the functions in E_z are holomorphic on $\mathcal{V}_z = \pi^{-1}(z)$, where π is the projection map from $M^n \times D$ to D, and a smooth section u of E close to $z_0 \in D$ is defined as a function $u = u(z, \xi) \in C^\infty(\pi^{-1}(U_{z_0}))$ such that U_{z_0} is a neighborhood of z_0, and we denote by $\Gamma(E)$ the set of smooth sections of E, we recall that h^E is the Hermitian metric on E and comprised by the smooth family $(\langle \cdot, \cdot \rangle_{E_z})_{z\in D}$ of sesquilinear maps $\langle \cdot, \cdot \rangle_{E_z} : E_z \times E_z \to \mathbb{C}$. Let (L_2, h^{L_2}) be the Hermitian vector bundle over D whose fibers are the family $(L_2^z)_{z\in D}$. We recall that a connection on $\Gamma(E)$ is defined as a linear operator $\nabla : X \to \nabla_X$ for any X, a tangent vector field on D, such that for any smooth function $\mathfrak{h}$ on D and $\mathfrak{g} \in \Gamma(E)$, we have $\nabla_X(\mathfrak{h}\mathfrak{g}) = \mathfrak{h}\nabla_X \mathfrak{g} + (X\mathfrak{h})\mathfrak{g}$ and $\nabla_{\mathfrak{h}X}\mathfrak{g} = \mathfrak{h}\nabla_X \mathfrak{g}$. While the fibers E_z are infinite-dimensional Hilbert spaces, the formalism of the Chern connection (see [9, Proposition 4.2.14]), remains true for E. To be precise, let $T^{(1,0)}D$ be the holomorphic sub-bundle of the complex tangent bundle TD, that is, $J = i$ on $T^{(1,0)}D$, also we call $T^{(1,0)}D$ the $(1,0)$ tangent vector field on D spanned by $\left(\frac{\partial}{\partial z_l}\right)_{1\le l\le n}$, we say that ∇ is a Chern connection on E if $\nabla_Z^{0,1} = \overline{\partial}_Z$ for $Z \in T^{(1,0)}D$, and ∇ is compatible with the Hermitian metric, i.e. for all $(u, v) \in [\Gamma(E)]^2$, we have

$$d\langle u_z, v_z \rangle_{E_z}(Z) = \langle \nabla_Z u_z, v_z \rangle_{E_z} + \langle u_z, \nabla_Z v_z \rangle_{E_z}, \tag{1.2}$$

where d is the exterior derivative which decomposes into the sum of the complex exterior derivative of type $(1,0)$, denoted by ∂_Z, and its conjugate $\overline{\partial}_Z = \partial_{\overline{Z}}$, the complex exterior derivative of type $(0,1)$ and where ∂_Z (resp. $\overline{\partial}_Z$) stands for the directional derivative in the direction of $Z \in T^{(1,0)}D$ (resp. $\overline{Z} \in T^{(1,0)}D$) such that $T^{(1,0)}D$ is the $(0,1)$ tangent vector field on D spanned by $\left(\frac{\partial}{\partial \overline{z_l}}\right)_{1\le l\le n}$. We recall that for a smooth function f on D, we have $\partial_Z(f) = Z(f)$, the ordinary derivative of f in the direction of Z, and $\nabla_Z^{0,1}$ is the $(0,1)$ part of the decomposition of $\nabla_Z := \nabla_Z^{1,0} + \nabla_Z^{0,1}$.

1.1. **The structure of the paper.** In the second section, we provide the expression of the $(1,0)$-part of the Chern connections on L_2 and E, respectively and their associated curvatures. Also, we state the definition of the multiplicity of a covering of a manifold and a result on the multiplicity of this covering. In the third section, we state our main result on an upper bound of the induced connection on sections of Toeplitz operators.

2. Miscellaneous lemmas and definitions

The following lemma provides the expression of the Chern connection on L_2 and on E, respectively.

Lemma 2.1. *Let $z \in D$ and $Z \in T^{(1,0)}D$. Then the $(1,0)$-part of the Chern connections on the bundles L_2 and on E are given locally by*

$$\nabla_Z^{L_2} = \partial_Z - \partial_Z \varphi_z \tag{2.3}$$

$$\nabla_Z^{E} = P\partial_Z P - T_{\partial_Z \varphi_z}. \tag{2.4}$$

Proof. We prove (2.3), let $(u_z, v_z) \in [L_2^z(\mathcal{V}, e^{-\varphi_z}\omega^n)]^2$, then we have

$$\begin{aligned} d\langle u_z, v_z\rangle_{L_2^z} &= \int_{\mathcal{V}} d\left(u_z(\xi)\overline{v_z}(\xi)e^{-\varphi_z(\xi)}\right)\omega^n \\ &= \int_{\mathcal{V}} d(u_z(\xi))\overline{v_z}\xi)e^{-\varphi_z(\xi)}\omega^n + \int_{\mathcal{V}} u_z(\xi)d(\overline{v_z}(\xi))e^{-\varphi_z(\xi)}\omega^n \\ &- \int_{\mathcal{V}} u_z(\xi)\overline{v_z}(\xi)d(\varphi_z(\xi))e^{-\varphi_z(\xi)}\omega^n \\ &= \int_{\mathcal{V}} (\partial_Z u_z(\xi) - \partial_Z\varphi_z(\xi)u_z(\xi) + \overline{\partial_Z} u_z(\xi))\overline{v_z}(\xi)e^{-\varphi_z(\xi)}\omega^n \\ &+ \int_{\mathcal{V}} u_z(\xi)\overline{(\partial_Z v_z(\xi) - \partial_Z\varphi_z(\xi)v_z(\xi) + \overline{\partial_Z} v_z(\xi))}e^{-\varphi_z(\xi)}\omega^n. \end{aligned}$$

Then the fact that ∇ is compatible with the Hermitian metric, namely,

$$d\langle u_z, v_z\rangle_{L_2^z}(Z) = \langle \nabla_Z u_z, v_z\rangle_{L_2^z} + \langle u_z, \nabla_Z v_z\rangle_{L_2^z},$$

and the fact that $\nabla_Z^{0,1} = \overline{\partial}_Z$, we obtain

$$\begin{aligned} \nabla_Z u_z &= \partial_Z u_z - \partial_Z\varphi_z u_z + \overline{\partial_Z} u_z \\ &= \nabla_Z^{L_2} u_z + \nabla_Z^{0,1} u_z. \end{aligned}$$

So the (1,0)-part of the Chern connection on L_2 is provided by

$$\nabla_Z^{L_2} = \partial_Z - \partial_Z\varphi_z.$$

Apropos the proof of (2.4), it is deduced from (2.3) and the Chern connection on E is given by

$$\nabla_Z^{E} = P\nabla^{L_2} = P\partial_Z P - T_{\partial_Z\varphi_z}. \tag{2.5}$$

□

Let $\mathcal{R}(L_2)(Z, \overline{Z})$ be the curvature of Chern connection $\nabla_Z^{L_2} + \overline{\partial}_Z$ defined as $\mathcal{R}(L_2)(Z, \overline{Z}) = (\nabla_Z^{L_2} + \overline{\partial}_Z)^2$.

Lemma 2.2. *Let $Z \in T^{1,0}D$ and $z \in D$. Then, locally we have $\mathcal{R}(L_2)(Z, \overline{Z}) = \partial_Z\overline{\partial}_Z\varphi_z$.*

Proof. It is now that the $(2,0)$-component (resp. $(0,2)$-component) of the curvature of a Chern connection vanishes [14, p.26 and Lemma 2.4], thus $\mathcal{R}^{(2,0)}(L_2)(Z,\overline{Z}) = [\nabla_Z^{L_2}]^2 = 0$ and $\mathcal{R}^{(0,2)}(L_2)(Z,\overline{Z}) = \overline{\partial}_Z^2 = 0$. So, we obtain

$$\begin{aligned}\mathcal{R}(L_2)(Z,\overline{Z}) =&(\nabla_Z^{L_2} + \overline{\partial}_Z)^2\\ =&(\nabla_Z^{L_2})^2 + \nabla_Z^{L_2}\overline{\partial}_Z + \overline{\partial}_Z\nabla_Z^{L_2} + \overline{\partial}_Z^2\\ =&\nabla_Z^{L_2}\overline{\partial}_Z + \overline{\partial}_Z\nabla_Z^{L_2}.\end{aligned}$$

Then, by using the expression of $\nabla_Z^{L_2}$ (Lemma 2.1), the fact that $\partial_Z\overline{\partial}_Z + \overline{\partial}_Z\partial_Z = 0$ [14, Lemma 2.4], and considering $f \in \Gamma(L_2)$, we get

$$\begin{aligned}\mathcal{R}(L_2)(Z,\overline{Z})f =&(\nabla_Z^{L_2}\overline{\partial}_Z + \overline{\partial}_Z\nabla_Z^{L_2})f\\ =&(\partial_Z - \partial_Z\varphi_z)\overline{\partial}_Z f + \overline{\partial}_Z(\partial_Z f - f\partial_Z\varphi_z)\\ =&(\partial_Z\overline{\partial}_Z + \overline{\partial}_Z\partial_Z)f - \partial_Z\varphi_z \wedge \overline{\partial}_Z f - \overline{\partial}_Z f \wedge \partial_Z\varphi_z + (\partial_Z\overline{\partial}_Z\varphi_z)f\\ =&(\partial_Z\overline{\partial}_Z\varphi_z)f.\end{aligned}$$

□

Consequently, if φ_z is a plurisubharmonic function on D, i.e. its complex Hessian is positive, then the curvature of the Chern connection of $\nabla_Z^{L_2} + \overline{\partial}_Z$ is positive. Concerning the curvature of the Chern connection $\nabla_Z^E + \overline{\partial}_Z$ has been conducted by Berndtsson [2] and retrieved in [5]. Precisely, by adopting our notation, we have

$$\mathcal{R}(E)(Z,\overline{Z}) = T_{\partial_Z\overline{\partial}_Z\varphi_z} - H^*_{\partial_Z\varphi_z}H_{\partial_Z\varphi_z}.$$

It is known that a connection for a vector bundle induces a new connection for other vector bundles. Therefore, for our case, we consider the new vector bundle $End(E)$, the space of endomorphisms on E, and its associated induced connection on sections of Toeplitz operators $T_{f_z} \in End(E)$ is defined by

$$\nabla_Z^{Ind(E)}T_{f_z} = [\nabla_Z^E, T_{f_z}] := \nabla_Z^E T_{f_z} - T_{f_z}\nabla_Z^E.$$

In the sequel, we need to consider M^n such that the $\mathcal{V}$-valued exponential map, noted by $\exp_{\mathfrak{p}}$, is holomorphic on the tangent space $T_{\mathfrak{p}}\mathcal{V}$ at each point $\mathfrak{p} \in \mathcal{V}$, e.g., when M^n is biholomorphic to Euclidean space. Then, by using a direct calculus, the absolute value of the Jacobean associated to $\exp_{\mathfrak{p}}$ is equal to one. Whence, through the inverse mapping theorem, $\exp_{\mathfrak{p}}$ is biholomorphic from an open neighborhood $U_0 \subset T_{\mathfrak{p}}\mathcal{V}$ of the origin, to its image $\mathcal{U}_{\mathfrak{p}}$—called *normal neighborhood*. Then, we have the following local biholomorphic chart called *normal coordinate chart* $(\mathcal{U}_{\mathfrak{p}}, \mathcal{X}_{\mathfrak{p}}^{-1})$ such that $\mathcal{X}_{\mathfrak{p}}^{-1} := \mathcal{E}_{\mathfrak{p}}^{-1} \circ \exp_{\mathfrak{p}}^{-1} : \mathcal{U}_{\mathfrak{p}} \overset{\exp_{\mathfrak{p}}^{-1}}{\mapsto} U_0 \overset{\mathcal{E}_{\mathfrak{p}}^{-1}}{\mapsto} \mathcal{X}_{\mathfrak{p}}^{-1}(\mathcal{U}_{\mathfrak{p}}) \subset \mathbb{C}^n$ with $\mathcal{E}_{\mathfrak{p}}$ is a $T_{\mathfrak{p}}\mathcal{V}$-valued isometric map on $\mathbb{C}^n$. This isometry sends the standard orthonormal basis of $\mathbb{C}^n$ to an orthonormal basis of $T_{\mathfrak{p}}\mathcal{V}$. Then, for our study, let us consider the normal coordinate chart $\left(\mathcal{B}(\mathfrak{p},\delta), \mathcal{X}^{-1}_{\mathfrak{p}\restriction_{\mathcal{B}(\mathfrak{p},\delta)}}\right)$ such that $\mathcal{X}^{-1}_{\mathfrak{p}\restriction_{\mathcal{B}(\mathfrak{p},\delta)}}(\mathcal{B}(\mathfrak{p},\delta)) = \mathbb{B}(0,\delta) \subset U_0$, where $\mathbb{B}(0,\delta)$ stands for the complex ball of center the origin 0 in $\mathbb{C}^n$ and of radius δ and $\mathcal{B}(\mathfrak{p},\delta)$ is the geodesic normal ball of center $\mathfrak{p} \in \mathcal{V}$ and of radius δ. We define by $\mathfrak{i}(\mathfrak{p},\mathcal{V}) > 0$ the injectivity radius at $\mathfrak{p}$, i.e. the large value of δ such that $\mathcal{X}_{\mathfrak{p}}^{-1}$ is a biholomorphic map on $\mathcal{B}(\mathfrak{p},\delta)$. By definition $\mathfrak{i}_{\mathcal{V}} := \inf_{\mathfrak{p}\in\mathcal{V}} \mathfrak{i}(\mathfrak{p},\mathcal{V})$ is the injectivity radius of $\mathcal{V}$. Therefore, for each $\mathfrak{p} = \mathcal{X}(0) \in \mathcal{V}$, the map $\mathcal{X}^{-1}_{\mathfrak{p}\restriction_{\mathcal{B}(\mathfrak{p},\delta)}} : \mathcal{B}(\mathfrak{p},\delta) \longmapsto \mathbb{B}(0,\delta)$ is biholomorphic with $\delta \in (0,\mathfrak{i}_{\mathcal{V}})$. Apropos of the properties of the exponential maps and the injectivity radius, we can look to, e.g., [1, Chapter 1 §3 and §4] or [6]. The following definition deals with the multiplicity of covering for $\mathcal{M}$, a manifold with bounded geometry, i.e. $\mathfrak{i}_{\mathcal{M}} > 0$ and all covariants curvature tensor are bounded.

Definition 2.1. *Let $\mathcal{M}$ be a manifold with bounded geometry and $\bigcup_{j\in J\subset\mathbb{N}} U_j$ be a covering of $\mathcal{M}$ such that $(U_j)_{j\in J}$ is a family of open sets U_j. The multiplicity of this covering is the maximum possible number N_0 of different $(j_1, j_2, \ldots, j_{N_0}) \in J^{N_0}$ with $\bigcap_{l=1}^{N_0} U_{j_l} \neq \emptyset$.*

By using Zorn Lemma and on the Gromov's paper [7], the authors V. Kondratiev and M. Shubin [11, Lemma 2.5] found an upper bound of N_0 corresponding to $(\mathcal{M}, g)$, a Riemannian Manifold of bounded geometry, that is, the corresponding injectivity radius is strictly positive and all covariants curvature tensor are bounded. More precisely, they showed:

Lemma 2.3. *Let $r \in (0, \mathfrak{i}_{\mathcal{M}})$, then there exist a covering of $(\mathcal{M}, g)$, a Riemannian Manifold of bounded geometry, by geodesic balls $\mathcal{B}_g(x_j, r)$ with the multiplicity of this covering*

$$N_0 \leq \frac{\sup\limits_{\mathfrak{x}\in\mathcal{M}} \left(vol(\mathcal{B}_g(\mathfrak{x}, 2r))\right)}{\inf\limits_{\mathfrak{x}\in\mathcal{M}} \left(vol(\mathcal{B}_g(\mathfrak{x}, \frac{r}{2}))\right)},$$

$vol(\cdot)$ is the volume w.r.t. the Riemann metrics g.

The fact that $\mathcal{V}$ is a bounded domain then has the structure of bounded geometry and we can cover $\mathcal{V}$ by geodesic normal balls of radius δ such that $2\delta \leq i_{\mathcal{O}}$, and the volume of each geodesic ball is less than the Euclidean ball volume. Whence, Lemma 2.3 yields $N_0 \leq 4^n$.

Lemma 2.4. *Let $\mathfrak{z} \in \mathbb{B}(0, \delta)$ such that $\delta \leq \mathfrak{i}_{\mathcal{V}}$ and $\mathfrak{z}$ be close to zero. Then $\det g(\mathfrak{z}) \leq 2^n n!$ and $\omega^n \leq n!^2 2^n d\mathfrak{z}$.*

Proof. As we know, the map $\mathcal{X}^{-1}_{\mathfrak{p}\restriction_{\mathcal{B}(\mathfrak{p},\delta)}} : \mathcal{B}(\mathfrak{p}, \delta) \longmapsto \mathbb{B}(0, \delta)$ is biholomorphic with $\delta \in (0, \mathfrak{i}_{\mathcal{V}})$. Therefore, we have that the coefficients $g_{jk}(\mathfrak{p})$ are equal to $\boldsymbol{\delta}_{jk}$ for $(j, k) \in \{1, \ldots, n\}^{2n}$ and the derivatives of g_{jk} at $\mathfrak{p}$ are equal to zero [1, Chapter 1, Definition 1.24]. We recall that $\mathfrak{p} = \mathcal{X}(0)$, thus its coordinates are equal to zero and the Taylor series for each smooth function g_{jk} at $\mathfrak{p} \in \mathcal{B}(\mathfrak{p}, \delta)$ provides $g_{jk}(\mathfrak{z}) = \boldsymbol{\delta}_{jk} + O(|\mathfrak{z}|^2)$ when $\mathfrak{z}$ is close to zero, i.e. $|g_{jk}(\mathfrak{z})| \leq \boldsymbol{\delta}_{jk} + C_{jk}\varepsilon_{jk}$ with $\varepsilon_{jk} \to 0^+$ and C_{jk} is a positive constant. In local coordinates, we have the famous equality $d\mu_g = \sqrt{\det g(\mathfrak{z})} d\mathfrak{z}$, where $d\mathfrak{z}$ is the Lebesgue measure in $\mathbb{C}^n$. Whence, from the definition of the determinant, we have

$$\begin{aligned}
\det g(\mathfrak{z}) &\leq \sum_{\sigma\in\mathcal{S}_n} |g_{1\sigma(1)}(\mathfrak{z})||g_{2\sigma(2)}(\mathfrak{z})| \ldots |g_{n\sigma(n)}(\mathfrak{z})| \\
&\leq \sum_{\sigma\in\mathcal{S}_n} \prod_{j=1}^{n} (\boldsymbol{\delta}_{j\sigma(j)} + C_{j\sigma(j)}\varepsilon_{j\sigma(j)}), \\
&\leq 2^n.n! \qquad (2.6)
\end{aligned}$$

$\mathcal{S}_n$ stands for the set of all symmetric permutations on $\{1, \ldots, n\}$ such that its cardinal is equal to $n!$. Regarding inequality (2.6), we have picked $\varepsilon_{j\sigma(j)}$ such that $C_{j\sigma(j)}\varepsilon_{j\sigma(j)} \leq 1$ for all $(j, \sigma) \in \{1, 2, \ldots, n\} \times \mathcal{S}_n$. So from the expression of ω and inequality (2.6), we have

$$\omega^n = n! d\mu_g = n! \sqrt{\det g(\mathfrak{z})} d\mathfrak{z} \leq n!^2 2^n d\mathfrak{z}.$$

□

3. MAIN THEOREM

The following main result states that $\nabla_Z^{Ind(E)} T_{f_z}$ is bounded by a function given in terms of the Hankel and Toeplitz operators and for the proof we use the techniques employed for the proof of Lemma 3.3 in [5].

Theorem 3.1. *Let $Z \in T^{(1,0)}D$ and $z \in D$. Then, we have*

$$\nabla_Z^{Ind(E)} T_{f_z} \leq n!^3 2^{3n} \left(T_{\partial_Z f_z} - H^*_{\overline{f_z}} H_{\partial_Z \varphi_z} \right).$$

Proof. Let us consider $\mathfrak{g} \in E_z$ which does depend on $z \in D$ and we have $[P\partial_Z P, T_{f_z}]g = \partial_Z T_{f_z} g$. Therefore, by employing (2.5), we have

$$\begin{aligned} \nabla_Z^{Ind(E)} T_{f_z} =& [\nabla_Z^E, T_{f_z}] \\ =& [P\partial_Z P - T_{\partial_Z \varphi_z}, T_{f_z}] \\ =& [P\partial_Z P, T_{f_z}] + [T_{f_z}, T_{\partial_Z \varphi_z}] \\ =& \partial_Z T_{f_z} + [T_{f_z}, T_{\partial_Z \varphi_z}]. \end{aligned} \tag{3.7}$$

Therefore, without loss in generality, below, we perform the calculation of $\langle \partial_Z T_{f_z} \mathfrak{g}, \mathfrak{h} \rangle_{E_z}$ for $(\mathfrak{g}, \mathfrak{h}) \in E_z \times E_z$ and do not rely on $z \in D$. We stress that to do directly the calculus of this latter inner product on a bounded Kähler manifold generally is a little bite hard. Therefore, to avoid this standoff, we need to employ a suitable partition of unity. Whence, the fact that the integration on a manifold does not depend on the choice of a partition of unity locally finite, let us define the following one $(\phi_k)_{k \geq 1}$ subordinated to the covering $(\mathcal{B}_\delta^{(k)})_{k \in \mathcal{J}}$ of $\mathcal{V}$, where $\mathcal{J}$ is a finite subset of $\mathbb{N}$ and $\mathcal{B}_\delta^{(k)} = \mathcal{B}(\mathfrak{p}_k, \delta)$ as follows. Let $\widetilde{\psi_k} \in C_0^\infty(\mathbb{B}_k(0, \delta))$ and be smooth positive function with compact support in $\mathbb{B}_k(0, \delta) = \mathbb{B}_\delta^{(k)}$ and bounded by one. Now, let us transfer the function $\widetilde{\psi_k}$ to $\mathcal{B}_\delta^{(k)}$, as follows. Let $\psi_k(\mathfrak{x}) = \widetilde{\psi_k}(\mathcal{X}_{\mathfrak{p}_k}^{-1}(\mathfrak{x}))$ for $(\mathfrak{x}, k) \in \mathcal{B}_\delta^{(k)} \times \mathbb{N}$. Consequently, we define the smooth function ϕ_k as follows $\phi_k(\mathfrak{x}) = \dfrac{\psi_k(\mathfrak{x})}{\sum_{l \geq 1} \psi_l(\mathfrak{x})}$. Our elected partition of unity is locally finite. Therefore, let $\mathfrak{h} \in L_2^z$ with compact support in $\mathcal{V}$ and covered by geodesic balls with finite multiplicity, so the support of $\mathfrak{h}\phi_k$ belongs to the set $\left\{ (\mathcal{B}_\delta^{(k)})_{1 \leq k \leq 2^{2n}}, \text{ such that } \bigcap_{k=1}^{N_0} \mathcal{B}_\delta^{(k)} \neq \emptyset \right\}$. The fact that $\mathcal{B}_\delta^{(k)}$ is biholomorphic to $\mathbb{B}_\delta^{(k)}$ through the exponential map, and by using Lemma 2.4 at $\mathcal{B}_\delta^{(k)}$, let us show that T_{f_z}, the Toeplitz operator with symbol f_z, on $L_{2,h}(\mathcal{B}_\delta^{(k)}, e^{-\varphi_z}\omega^n)$ is less, up to a multiplicative constant, to $T_{f_z \circ \mathcal{X}_{p_k}}$, the Toeplitz operator with symbol $f_z \circ \mathcal{X}_{p_k}$, on $L_{2,h}(\mathbb{B}_\delta^{(k)}, e^{-\varphi_z \circ \mathcal{X}_{p_k}} du)$. Thus for $\mathfrak{g}$ a holomorphic function in $\mathcal{B}_\delta^{(k)}$, we have

$$\begin{aligned} T_{f_z}\mathfrak{g}(\xi) =& \int_{\mathcal{B}_\delta^{(k)}} f_z(\zeta) K_z(\xi, \zeta) \mathfrak{g}(\zeta) e^{-\varphi_z(\zeta)} \omega^n \\ =& n! \int_{\mathbb{B}_\delta^{(k)}} f_z(\mathcal{X}_{p_k}(u)) K_z(\mathcal{X}_{p_k}(\mathfrak{z}), \mathcal{X}_{p_k}(u)) \mathfrak{g}(\mathcal{X}_{p_k}(u)) e^{-\varphi_z(\mathcal{X}_{p_k}(u))} d\mu_g \\ \leq& n!^2 2^n \int_{\mathbb{B}_\delta^{(k)}} f_z(\mathcal{X}_{p_k}(u)) K_z(\mathcal{X}_{p_k}(\mathfrak{z}), \mathcal{X}_{p_k}(u)) \mathfrak{g}(\mathcal{X}_{p_k}(u)) e^{-\varphi_z(\mathcal{X}_{p_k}(u))} du \\ \leq& n!^2 2^n \int_{\mathbb{B}_\delta^{(k)}} f_z \circ \mathcal{X}_{p_k}(u)) K_z(\mathcal{X}_{p_k}(\mathfrak{z}), \mathcal{X}_{p_k}(u)) \mathfrak{g} \circ \mathcal{X}_{p_k}(u)) e^{-\varphi_z \circ \mathcal{X}_{p_k}(u))} du \end{aligned}$$

$$=n!^2 2^n T_{f_z\circ\mathcal{X}_{p_k}}(\mathfrak{g}\circ\mathcal{X}_{p_k})(\mathfrak{z}), \tag{3.8}$$

where $K_z(\xi,\zeta)$ is the reproducing kernel for the the Hilbert space $L_{2,h}(\mathcal{B}_\delta^{(k)}, e^{-\varphi_z}\omega^n)$ and $(K_z\circ\psi)(\mathfrak{z},\mathfrak{u}) := K_z(\mathcal{X}_{p_k}(\mathfrak{z}),\mathcal{X}_{p_k}(\mathfrak{u}))$ is the reproducing kernel for $L_{2,h}(\mathbb{B}_\delta^{(k)}, e^{-\varphi_z\circ\mathcal{X}_{p_k}}du)$, e.g., by inspiring from [5, Example 2.4], let $e^{-\varphi_z\circ\mathcal{X}_{p_k}(u)} = (1-|z|^2-|u|^2)^\alpha$ for $\alpha>-1$ and $z\in\mathbb{D}$, the unit complex disk, we consider $\mathbb{B}_z = \{\xi\in\mathbb{C}^n : |\xi|^2\le 1-|z|^2\}$. Then E_z has the following reproducing kernel

$$K_z(\mathcal{X}_{p_k}(\mathfrak{z}),\mathcal{X}_{p_k}(u)) = \frac{(1-|z|^2)^{1+\alpha}}{(1-|z|^2-\mathfrak{z}\overline{\mathfrak{u}})^{1+\alpha+n}},$$

and consequently, we have

$$K_z(\xi,\zeta) = \frac{(1-|z|^2)^{1+\alpha}}{(1-|z|^2-\mathcal{X}_{p_k}^{-1}(\xi)\overline{\mathcal{X}_{p_k}^{-1}(\zeta)})^{1+\alpha+n}}.$$

Also, we have

$$\begin{aligned}\int_{\mathcal{B}_\delta^{(k)}} \partial_Z T_{f_z}\mathfrak{g}(\xi)\overline{\mathfrak{h}}(\xi)e^{-\varphi_z(\xi)}\omega^n =& n!\int_{\mathbb{B}_\delta^{(k)}} \partial_Z T_{f_z(\mathcal{X}_{p_k})}\mathfrak{g}(\mathcal{X}_{p_k}(\mathfrak{z}))\overline{\mathfrak{h}}(\mathcal{X}_{p_k}(\mathfrak{z}))e^{-\varphi_z(\mathcal{X}_{p_k}(\mathfrak{z}))}d\mu_g\\ \le& n!^2 2^n\int_{\mathbb{B}_\delta^{(k)}} \partial_Z T_{f_z(\mathcal{X}_{p_k})}\mathfrak{g}(\mathcal{X}_{p_k}(\mathfrak{z}))\overline{\mathfrak{h}}(\mathcal{X}_{p_k}(\mathfrak{z}))e^{-\varphi_z(\mathcal{X}_{p_k}(\mathfrak{z}))}d\mathfrak{z}.\end{aligned} \tag{3.9}$$

Whence, by using the integration on each chart $\left(\mathcal{B}_\delta^{(k)}, \mathcal{X}_{\mathfrak{p}_k}^{-1}\right)$ and the above partition of unity locally finite, Lemma 2.4 and inequality (3.9) in each geodesic normal ball $\mathcal{B}_\delta^{(k)}$, and the fact that $N_0\le 2^{2n}$, we have

$$\begin{aligned}&\langle\partial_Z T_{f_z}\mathfrak{g},\mathfrak{h}\rangle_{E_z}\\ =&\int_{\mathcal{V}} \partial_Z T_{f_z}\mathfrak{g}(\xi)\overline{\mathfrak{h}}(\xi)e^{-\varphi_z(\xi)}\omega^n\\ \le&\sum_{k=1}^{N_0}\int_{\mathcal{B}_\delta^{(k)}} \partial_Z T_{f_z}\mathfrak{g}(\xi)\overline{\mathfrak{h}}(\xi)\phi_k(\xi)e^{-\varphi_z(\xi)}\omega^n\\ \le& n!^2 2^n\sum_{k=1}^{N_0}\int_{\mathbb{B}_\delta^{(k)}} \partial_Z T_{f_z(\mathcal{X}_{p_k})}\mathfrak{g}(\mathcal{X}_{p_k}(\mathfrak{z}))\overline{\mathfrak{h}}(\mathcal{X}_{p_k}(\mathfrak{z}))e^{-\varphi_z(\mathcal{X}_{p_k}(\mathfrak{z}))}d\mathfrak{z}\\ =& n!^2 2^n\sum_{k=1}^{N_0}\int_{\mathbb{B}_\delta^{(k)}}\left(\int_{\mathbb{B}_\delta^{(k)}}\partial_Z\left(f_z(\mathcal{X}_{p_k}(u))\right)K_z(\mathcal{X}_{p_k}(\mathfrak{z}),\mathcal{X}_{p_k}(u))e^{-\varphi_z(\mathcal{X}_{p_k}(u))}\mathfrak{g}(\mathcal{X}_{p_k}(u))\right)du\right)\\ &\times\overline{\mathfrak{h}}(\mathcal{X}_{p_k}(\mathfrak{z}))e^{-\varphi_z(\mathcal{X}_{p_k}(\mathfrak{z}))}d\mathfrak{z}\\ =& n!^2 2^n\sum_{k=1}^{N_0}\int_{\mathbb{B}_\delta^{(k)}\times\mathbb{B}_\delta^{(k)}}\partial_Z\Big(f_z(\mathcal{X}_{p_k}(u))K_z(\mathcal{X}_{p_k}(\mathfrak{z}),\mathcal{X}_{p_k}(u))e^{-\varphi_z(u)}\\ &\quad\times\mathfrak{g}(\mathcal{X}_{p_k}(u))\overline{\mathfrak{h}}(\mathcal{X}_{p_k}(\mathfrak{z}))e^{-\varphi_z(\mathcal{X}_{p_k}(\mathfrak{z}))}\Big)\,du d\mathfrak{z}\\ -& n!^2 2^n\sum_{k=1}^{N_0}\int_{\mathbb{B}_\delta^{(k)}\times\mathbb{B}_\delta^{(k)}} f_z(\mathcal{X}_{p_k}(u))K_z(\mathcal{X}_{p_k}(\mathfrak{z}),\mathcal{X}_{p_k}(u))e^{-\varphi_z(\mathcal{X}_{p_k}(u))}\\ &\times\mathfrak{g}(\mathcal{X}_{p_k}(u))\partial_Z(\overline{\mathfrak{h}}(\mathcal{X}_{p_k}(\mathfrak{z}))e^{-\varphi_z(\mathcal{X}_{p_k}(\mathfrak{z}))})du d\mathfrak{z}.\end{aligned}$$

Hence,

$$\begin{aligned}
\langle \partial_Z T_{f_z}\mathfrak{g},\mathfrak{h}\rangle_{E_z} \leq & n!^2 2^n \sum_{k=1}^{n} \partial_Z \langle T_{f_z\circ\mathcal{X}_{p_k}}(\mathfrak{g}\circ\mathcal{X}_{p_k}),\mathfrak{h}\circ\mathcal{X}_{p_k}\rangle_{L_2(\mathbb{B}_k,e^{-\varphi_z\circ\mathcal{X}_{p_k}}d\mathfrak{z})} \\
&+ n!^2 2^n \sum_{k=1}^{N_0} \int_{\mathbb{B}_\delta^{(k)}\times\mathbb{B}_\delta^{(k)}} f_z(\mathcal{X}_{p_k}(u))K_z(\mathcal{X}_{p_k}(\mathfrak{z}),\mathcal{X}_{p_k}(u))e^{-\varphi_z(\mathcal{X}_{p_k}(u))}\mathfrak{g}(\mathcal{X}_{p_k}(u)) \\
&\times\overline{\mathfrak{h}}(\mathcal{X}_{p_k}(\mathfrak{z}))\partial_Z\varphi_z(\mathcal{X}_{p_k}(\mathfrak{z}))e^{-\varphi_z(\mathcal{X}_{p_k}(\mathfrak{z}))}du d\mathfrak{z} \\
=& n!^2 2^n \sum_{k=1}^{N_0} \int_{\mathbb{B}_\delta^{(k)}} \partial_Z\left(f_z(\mathcal{X}_{p_k}(\mathfrak{z}))e^{-\varphi_z(\mathcal{X}_{p_k}(\mathfrak{z}))}\right)(\mathfrak{g}\overline{\mathfrak{h}})(\mathcal{X}_{p_k}(\mathfrak{z}))d\mathfrak{z} \\
&+ n!^2 2^n \sum_{k=1}^{n} \int_{\mathbb{B}_\delta^{(k)}} (T_{f_z\circ\mathcal{X}_{p_k}}\mathfrak{g}\circ\mathcal{X}_{p_k})(\partial_Z\varphi_z(\mathcal{X}_{p_k}(\mathfrak{z}))\overline{\mathfrak{h}}(\mathcal{X}_{p_k}(\mathfrak{z}))e^{-\varphi_z(\mathcal{X}_{p_k}(\mathfrak{z}))}d\mathfrak{z} \\
=& n!^2 2^n \sum_{k=1}^{N_0} \int_{\mathcal{B}_\delta^{(k)}} \partial_Z(f_z(\xi)e^{-\varphi_z(\xi)})(\mathfrak{g}\overline{\mathfrak{h}})(\xi)\omega^n \\
&+ n!^2 2^n \sum_{k=1}^{N_0} \int_{\mathcal{B}_\delta^{(k)}} (T_{f_z}\mathfrak{g})(\partial_Z\varphi_z(\xi)\overline{\mathfrak{h}}(\xi))e^{-\varphi_z(\xi)}\omega^n \\
\leq& n!^2 2^{3n}\langle (T_{e^{\varphi_z}\partial_Z(f_z e^{-\varphi_z})} + T_{\partial_Z\varphi_z}T_{f_z})\mathfrak{g},\mathfrak{h}\rangle_{E_z} \\
=& n!^2 2^{3n}\langle (T_{\partial_Z f_z - f_z\partial_Z\varphi_z} + T_{\partial_Z\varphi_z}T_{f_z})\mathfrak{g},\mathfrak{h}\rangle_{E_z}.
\end{aligned}$$

Whence, we have

$$\partial_Z T_{f_z} \leq n!^3 2^{3n}(T_{\partial_Z f_z - f_z\partial_Z\varphi_z} + T_{\partial_Z\varphi_z}T_{f_z}). \tag{3.10}$$

Let us recall equality (1.1).

$$T_{\overline{f_z}u_z} - T_{u_z}T_{\overline{f_z}} = H^*_{\overline{u}_z}H_{\overline{f_z}}. \tag{3.11}$$

Therefore, by using (3.11) with $u_z = \partial_Z\varphi_z$ and (3.10), inequality (3.7) becomes:

$$\nabla_Z^{Ind(E)}T_{f_z} \leq n!^3 2^{3n}(T_{\partial_Z f_z} - H^*_{\overline{f}_z}H_{\partial_Z\varphi_z}). \tag{3.12}$$

□

Remark 3.1. *For the particular case when $\mathcal{V}$ is a bounded subspace of $\mathbb{C}^n$, Engliš and Zhang state that $\nabla_Z^{Ind(E)}T_{f_z} = T_{\partial_Z f_z} - H^*_{\overline{f_z}}H_{\partial_Z\varphi_z}$, see* [5, Lemma 3.3]. *Thus, we are asking whether it is also possible to look for on a lower bound of $\nabla_Z^{Ind(E)}T_{f_z}$ in (3.12). Likewise, it is also possible to provide a upper bounded of $\partial_{\overline{Z}}^{Ind(E)}$, the induced connection associated to the $(0,1)$-part of the Chern connection for $\overline{Z} \in T^{(0,1)}D$, the anti-holomorphic sub-bundle of the complex tangent bundle TD, i.e. $J = -i$ on $T^{(0,1)}D$.*

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