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Research Article

Imaginary geometries

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ABSTRACT. In this paper, we axiomatize the geometries obtained from the long root subgroup geometries by taking as new lines the so-called imaginary lines. A generic such line is the union of the orbits of the centers of the two root groups corresponding to two opposite long roots, which share at least two points. This extends characterizations of Cuypers and Hall on copolar spaces, who treated the quadrangular case. Here, we treat the remaining case, the hexagonal one. Our results hold over any field of size at least 5 and characteristic different from 2.

Keywords: Long root geometry, root groups, spherical buildings, Moufang buildings.

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1. INTRODUCTION

1.1. **General context and motivation.** Buildings, sometimes also called Tits-buildings, were introduced by Jacques Tits [26] and give a geometric interpretation of semi-simple groups of algebraic origin (semi-simple algebraic groups, classical groups, groups of mixed type, (twisted) Chevalley groups). These buildings are, at first glance, complicated combinatorial structures; however, the properties of spherical buildings can be made more accessible using associated point-line geometries. The most commonly used point-line geometries that can be associated to a spherical building Δ of type (W, S) are the so called *Lie incidence geometries* [6]. For every nonempty subset $J \subseteq S$, there is a canonical procedure that yields a Lie incidence geometry with point set the set of J-simplices of Δ. Classical examples are given by the (Grassmannians of) projective and polar spaces, which are associated to buildings of type A_n, and B_n or D_n, respectively.

For every irreducible Moufang building Δ (of rank at least two, not an octagon or a Moufang quadrangle of type F_4), there is some (not necessarily unique, see Remark 2.3) subset $J \subseteq S$ for which there is a natural correspondence between the points of the associated Lie incidence geometry and the long root subgroups of Δ. This geometry is referred to as the *long root (subgroup) geometry* of Δ and either forms a polar space–in which case we call it *quadrangular*–or contains a lot of non-thick generalized hexagons with thick lines–in which case we call it *hexagonal*.

Long root geometries have been studied from different angles. From an algebraic point of view, they were studied in the context of *Timmesfelds theory of abstract root subgroups* [25], which axiomatizes the behaviour of (centers of) the root subgroups of long roots in spherical buildings. Moreover, these long root geometries appear as the so called extremal geometries ([5, 9]) of certain *Lie algebras*, and more recently, provide important classes of examples of *Tits quadrangles and Tits hexagons* ([18, 19]), which are higher rank generalizations of Moufang polygons.

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From an incidence geometric point of view, there are two main approaches. Firstly, the long root geometries are studied in the context of other Lie incidence geometries, and are hence classified as so called *parapolar spaces* that satisfy certain extra regularity conditions ([10, 23]). Secondly, they appear as the most important examples of *root filtration spaces* ([3, 4]), which are point-line geometries equipped with five relations between points that must satisfy a list of axioms (none of which involves any groups). These two incidence-geometric approaches, while very powerful, have the disadvantage that they capture a broader class of incidence geometries (namely, the *root shadow spaces*, see [4]), whose point set not necessarily coincides with the long root subgroups of a spherical building.

We propose and axiomatize an alternative point-line geometry associated to Δ. This point-line geometry, which we call the *imaginary geometry* of Δ, takes as point set the same point set as the long root geometry of Δ (i.e. the centers of the long root subgroups of Δ). Its lines, which we call *imaginary* lines, are induced by the rank one groups generated by two opposite long root subgroups. When the long root geometry is quadrangular, this imaginary geometry has been studied and axiomatized before ([7, 11], see also Section 3.2). In this paper, we focus on the hexagonal imaginary geometries.

In such hexagonal imaginary geometry, the set of imaginary lines through a point can be given the structure of a *Freudenthal triple system*, and as such, these geometries have been studied (implicitly) throughout the literature (for example in [16]). We complement this algebraic approach by providing an axiomatization of the hexagonal imaginary geometries. The protagonists of this incidence-geometric point of view are the imaginary geometries of buildings of type A_2, which are called A_2*-planes* and should be considered as the imaginary counterparts of non-thick generalized hexagons with thick lines. The main theorem roughly states that the hexagonal imaginary geometries are characterized by these A_2-planes and the local interactions that they have with other points of the geometry. As was shown in [16], Freudenthal triple systems (when generalized to arbitrary characteristic) can behave very different over fields of characteristic 2. As a consequence, the imaginary geometries suffer from the same disease, and for the axiomatization, we will restrict ourselves to hexagonal imaginary geometries defined over fields of characteristic not two.

In [18], it is shown that every Moufang building of rank one of so called *polar type* arises as the fixed point structure of a Galois involution of some imaginary geometry. The imaginary lines of this imaginary geometry induce imaginary lines of the Moufang set, and the geometry obtained like this is exactly the *Tits web* of the Moufang set, as was introduced in [27]. The imaginary geometries defined here should be seen as higher rank counterparts of these Tits webs.

Finally, we mention a further motivation. The rank one analogues of the Tits polygons mentioned above are the *Tits sets*, introduced in [17]. The abelian Tits sets have recently been classified by the first author in her PhD thesis [12] under a mild (and natural) additional condition, and they arise from higher rank spherical buildings by considering the vertices of a so-called *Jordan type* (the middle node in the A_n diagram, the extreme nodes in the other classical diagrams, and the node labeled 7 in the E_7 diagram). The next natural class to consider is the class of Tits sets corresponding to the long root geometries, and for this class, it is expected that the characterization in the present paper will be very useful.

1.2. **Formulation of the main results.** The purpose of this paper is twofold. First of all, we introduce imaginary geometries and investigate their behaviour. Secondly, we propose and prove an incidence geometric axiomatization of hexagonal imaginary geometries. For notation and definitions, we refer to Section 2.

Definition 1.1. *Let Δ be an irreducible spherical Moufang building of rank at least two, and let $\mathscr{E}$ be a (conjugacy) class of centers of long root subgroups of Δ. For A, B two opposite elements of $\mathscr{E}$, define the imaginary line through A and B to be the set $\{A\} \cup B^A$. We define $\mathrm{Im}(\Delta, \mathscr{E})$ to be the point-line geometry with as point set $\mathscr{E}$ and as line set the set of all imaginary lines. This space is called hexagonal when there exist $A, B \in \mathscr{E}$ with $[A, B] \in \mathscr{E}$, and quadrangular if no such A, B exist. Any point-line geometry obtained like this is called an imaginary geometry.*

The only irreducible spherical Moufang buildings of rank at least two that do not posses a class of long root subgroups are the octagons and the Moufang quadrangle of type F_4, so in particular, we attach an imaginary geometry to all irreducible spherical Moufang buildings of rank at least two that are not of those types. Moreover, whenever such a building Δ is either simply laced or defined over a field of characteristic not two or three, the set $\mathscr{E}$ is uniquely determined. In this case, we denote $\mathrm{Im}(\Delta) := \mathrm{Im}(\Delta, \mathscr{E})$.

The imaginary geometry $\mathrm{Im}(\Delta, \mathscr{E})$ fully determines Δ and $\mathscr{E}$, implying that studying $\mathrm{Im}(\Delta, \mathscr{E})$ is equivalent to studying Δ. There is a unique Lie incidence geometry of Δ, called the *long root geometry*, whose point set coincides with $\mathscr{E}$.

We will focus on *hexagonal imaginary geometries*. It turns out that these are exactly the imaginary geometries which contain imaginary geometries $\mathrm{Im}(\mathsf{A}_2(k))$ for some skew field k. which we will call A_2-planes. We now provide an explicit construction of these A_2-planes, along with the most essential definitions to understand the main theorem below (referring forward for details).

Definition 1.2. *Let k be a skew field. Consider the projective plane $\mathbb{P}(k^3)$, and denote its point and line set with $\mathscr{P}_\tau$ and $\mathscr{L}_\tau$, respectively. The geometry $\mathrm{Im}(\mathsf{A}_2(k))$ is the point-line geometry $(\mathscr{E}, \mathscr{I})$ with*

$$\mathscr{E} := \{(p, l) \mid p \in \mathscr{P}_\tau, l \in \mathscr{L}_\tau, p \in l\},$$
$$\mathscr{I} := \{[q, m] \mid q \in \mathscr{P}_\tau, m \in \mathscr{L}_\tau, q \notin m\}, \text{ where}$$
$$[q, m] := \{(p, pq) \mid p \in \mathscr{P}, p \in m\} \text{ for all } q \in \mathscr{P}_\tau, m \in \mathscr{L}_\tau \text{ with } q \notin m.$$

This geometry $\mathrm{Im}(\mathsf{A}_2(k))$ is called the A_2-plane over k. The corresponding long root geometry is the point-line geometry $(\mathscr{E}, \mathscr{L})$ with $\mathscr{L} := \{T_p \mid p \in \mathscr{P}_\tau\} \cup \{T_l \mid l \in \mathscr{L}_\tau\}$, where

$$T_p := \{(p, m) \mid m \in \mathscr{L}_\tau, m \ni p\}, \text{ for all } p \in \mathscr{P}_\tau,$$
$$T_l := \{(q, l) \mid q \in \mathscr{P}_\tau, q \in l\} \text{ for all } l \in \mathscr{L}_\tau.$$

One should note that the long root geometry $(\mathscr{E}, \mathscr{L})$ is a non-thick generalized hexagon with thick lines. We will prove that elements of $\mathscr{L}$ are fully determined by the geometry $\mathrm{Im}(\mathsf{A}_2(k))$. We can hence refer to these elements as transversals of $\mathrm{Im}(\mathsf{A}_2(k))$.

We refer forward to Section 3.4 to see that an A_2-plane contains many dual affine planes. It hence makes sense to consider the following definitions.

Definition 1.3. *Let A be an A_2-plane over a field k. A conical subset of A is a subset of A that intersects any dual affine plane of A in a conic (Definition 2.6). Such a conical subset is called a conical subspace when it intersects every transversal of A in 0, 1 or all of its points.*

Notation 1. *Let Y be an imaginary geometry (or a geometry axiomatizing it, as in the Main Theorem) and let p be a point of Y, then $p^{\not\equiv}$ denotes the set of points in Y noncollinear to p.*

Proposition 1.1. *Let $Y = \mathrm{Im}(\Delta, \mathscr{E})$ be a hexagonal imaginary geometry. Assume that every line of Y contains at least four points (or equivalently, Δ is not defined over $\mathbb{F}_2$). Moreover, if Δ is of type A_n, assume that $\Delta = \mathsf{A}_n(k)$ for some field k. Then Y is a connected partial linear space. Moreover, the following properties hold:*

(Im_1) *Let l and m be two intersecting lines, and p a point on l.*
(1) If $|p^{\not\equiv} \cap m| = 1$, any point of $m \setminus \{l \cap m\}$ is noncollinear to exactly one point of l.
(2) If $|p^{\not\equiv} \cap m| = 2$, the lines l and m generate an A_2-plane over some field.
The situation in (ii) occurs at least once.

(Im_2) *For any A_2-plane A, and any point p, the set $p^{\not\equiv} \cap A$ forms a conical subspace of A and contains three mutually collinear points, not on a common line.*

(Im_3) *For any points p, q, if $p^{\not\equiv} = q^{\not\equiv}$, then $p = q$.*

It turns out that, at least when no A_2-plane is defined over a field of characteristic two, the axioms (Im_1), (Im_2) and (Im_3) suffice to characterize all imaginary geometries.

Main Theorem. *Let Y be a connected partial linear space that satisfies*(Im_1), (Im_2) *and* (Im_3). *Assume that no A_2-plane of Y is defined over a field of characteristic 2 or over $\mathbb{F}_3$. Then Y is the hexagonal imaginary geometry of a spherical building, defined over a field k with $|k| \geq 5$ and $\mathrm{char}(k) \neq 2$, or an infinite rank analagon of such a hexagonal imaginary geometry (as defined in* Remark 3.5*).*

1.3. **Outline of paper.** In Section 3, we introduce the notion of an imaginary geometry and give several examples. In Section 4, we focus on the properties of hexagonal imaginary geometries, in particular, we prove that such a geometry satisfies axioms (Im_1), (Im_2) and (Im_3). In Section 5, we discuss several different classes of conical subspaces of A_2-planes. Sections 6 to 8 comprise the proof of the main Theorem.

The idea of the proof of the main theorem is the following: we start with the partial linear space $Y = (\mathscr{E}, \mathscr{I})$, and use the presence of A_2-planes in Y to define four possible relations between two distinct points: linelike, symplectic, special and collinear, and to define a set of transversals $\mathscr{L}$. The goal is to show that the point-line geometry $(\mathscr{E}, \mathscr{L})$, equipped with the relations defined above, forms a root filtration space. A priori however, it is not at all clear whether these relations are disjoint, or whether the point-line geometry $(\mathscr{E}, \mathscr{L})$ is a partial linear space.

The first difficulty we tackle, is proving that the four defined relations are disjoint. This is done in Section 6. Next, we focus on the relation between two special points p and q: in Section 7, we prove that the behaviour of any point linelike to both p and q is fully determined by the behaviour of p and q, Axiom (Im_3) will then ensure that there is a unique such a point. The next difficulty is to find a way to distinguish between linelike and symplectic points, which is done in Section 8.1, and again heavily relies on Axiom (Im_3). At this point, one has all the tools to prove that the four relations indeed define a filtration on $\mathscr{E}$, which is done in Section 8.2. A subtle, yet tedious detail is that one should still prove that $(\mathscr{E}, \mathscr{L})$ is a partial linear space; this is done in Section 8.3. Once we have that $(\mathscr{E}, \mathscr{L})$ is non-degenerate root filtration space, we can apply Theorem 2.1 to obtain that it is a hexagonal root shadow space. In Section 8.4, we then conclude that $(\mathscr{E}, \mathscr{L})$ is a long root geometry, and that Y is the corresponding imaginary geometry.

2. Preliminaries

In this section, we discuss four different classes of incidence structures. Schematically, and ignoring the peculiarity that some root shadow spaces (namely those of infinite rank) are not Lie incidence geometries, see Remark 2.2, these classes can be depicted as follows:

Point-line geometries $\supset$ Lie incidence geometries $\supset$ Root shadow spaces $\supset$ Long root geometries.

Throughout the section, we will use some basic definitions regarding buildings, for which we refer to [22].

2.1. **Point-line geometries.** The most general incidence structures studied in incidence geometry are point-line geometries. We recall some basic definitions, which can all be found in [24].

Definition 2.4. *A point-line geometry is a pair $(\mathscr{P}, \mathscr{L})$ consisting of a nonempty set $\mathscr{P}$, and a nonempty set $\mathscr{L}$ of subsets of $\mathscr{P}$. The elements of $\mathscr{P}$ are called points, those of $\mathscr{L}$ are called lines. We say that two points are* collinear *when they are contained in a common line, and say that they are* noncollinear *when they are not. Let $X = (\mathscr{P}, \mathscr{L})$ be a point-line geometry.*

1. *A subspace S of X is a subset of $\mathscr{P}$ for which every line that contains at least two points of S, is contained in S. A subspace that consists of mutually collinear points is called a singular subspace. A subspace that intersects every line in at least a point, is called a hyperplane.*
2. *For any set $P \subseteq \mathscr{P}$, the subspace generated by P is defined to be the intersection of all subspaces that contain P. A subset generated by three mutually collinear points, not on a common line, is called a plane.*
3. *The point-line incidence graph of X is the bipartite graph that has vertex set $\mathscr{P} \cup \mathscr{L}$ and edge set $\{(p, l) \mid p \in \mathscr{P}, l \in \mathscr{L}, p \in l\})$. We denote this graph with Γ^X.*
4. *The geometry X is called (co)connected when (the complement of) Γ^X is a connected graph.*
5. *The distance between points x, y is defined to be the half the distance between x and y in Γ^X. In particular, x and y are collinear if and only if they are at distance one.*
6. *The geometry X is called a partial linear space when every two collinear points p and q are contained in exactly one line (which we then denote with pq), and where moreover every line contains at least three points.*
7. *We define $\mathrm{Aut}(X)$ to be the group $\{\sigma \in \mathrm{Sym}(\mathscr{P}) \mid \mathscr{L}^\sigma = \mathscr{L}\}$.*

We give some examples of partial linear spaces that will be useful later on.

Example 2.1. *Let V be a (left) vector space of dimension $n \geq 3$ over some skew field k. The projective space $\mathbb{P}(V)$ is the partial linear space that has as points the 1-dimensional subspaces of V. The lines are induced by the 2-dimensional subspaces of V. When $n = 3$, this is called the* projective plane defined over k.

Example 2.2. *Let V be a (left) vector space (possibly of infinte dimension) over some skew field k. The dual V^* of V is a (right) vector space over k. Let W^* be a subspace of V^* such that $\{v \in V \mid \phi(v) = 0,\ \forall \phi \in W^*\} = \{\vec{o}\}$. If V is finite dimensional, the only possibility for W^* is V^*. Denote $\mathbb{P} := \mathbb{P}(V)$ and $\mathbb{H} = \mathbb{P}(W^*)$. Note that $\mathbb{H}$ is a set of hyperplanes of $\mathbb{P}$ such that $\mathbb{H}$ forms a subspace of the dual of $\mathbb{P}$ and no point of $\mathbb{P}$ is contained in all elements of $\mathbb{H}$.*

The partial linear space $\mathscr{E}(\mathbb{P}, \mathbb{H})$ is defined as follows. The point set is the set $\{(p, H) \mid (p, H) \in \mathbb{P} \times \mathbb{H},\ p \in H\}$. The lineset consists of two types: subsets of the form $\{(p, H) \mid p \in l\}$ where l is a line of $\mathbb{P}$ that is contained in H, and subsets of the form $\{(p, H) \mid H \supset K\}$ where K is a codimension-2 subspace of $\mathbb{P}$ that contains p for which there are at least two elements of $\mathbb{H}$ containing it.

A projective space defined over a field k and of dimension n is denoted by $\mathbb{P}(k^n)$.

Definition 2.5. *A polar space is a point-line geometry in which every point is collinear to one or all points of a line. It is called nondegenerate when no point is collinear to all other points. We say that a polar space has rank $n \in \mathbb{N}$ when every chain $M_1 \subset M_2 \subset \dots$ of nonempty singular subspaces has length at most n (where the length of a chain is defined to be the number of subspaces contained in it), and when there moreover exists such a chain of length n. When no such n exists, we say that the polar space has infinite rank.*

We gather two examples of such polar spaces.

Example 2.3. *Let V be a vector space defined over some field k, let $q : V \to k$ be a quadratic form with associated symmetric bilinear form $f : V \times V \to k$, where $f(v, w) = q(v + w) - q(v) - q(w)$. A* *subspace S of V is called* isotropic *when $q(s) = 0$ for all s of S. Assume that q contains some isotropic 2-space and that q is nondegenerate, that is, $\{v \in V \mid q(v) = f(v, w) = 0, \forall w \in V\} = \{\vec{o}\}$. The point-line geometry with as points the isotropic 1-spaces of V and as lines the isotropic 2-spaces of V is a nondegenerate polar space. Any polar space that can be realized like this is called an orthogonal polar space.*

Example 2.4. *Let V be a $2n$-dimensional vector space ($n \geq 2$) with basis $\{e_i\}_{1\leq i\leq 2n}$, and let f be the alternating bilinear form on V given by $f(x, y) = x_1y_2 - y_1x_2 + \cdots + x_{2n-1}y_{2n} - y_{2n-1}x_{2n}$ for $x = \sum x_ie_i$ and $y = \sum y_ie_i$. A subspace S of V is called isotropic when $f(v, w) = 0$ for all $v, w \in S$. Note that every 1-space of V is isotropic. The point-line geometry with as points the isotropic 1-spaces of V and as lines the isotropic 2-spaces of V is a nondegenerate polar space of rank n. A polar space that can be realized like this is called a symplectic polar space.*

All polar spaces of rank at least 3 (including those of infinite rank) have been classified, and besides the orthogonal and symplectic ones, there are the polar spaces defined using a pseudo-quadratic form with an associated Hermitian form, and also the so-called nonembeddable ones. We will not need the latter two classes. For more background and the proof of this classification we refer to [26].

We finish this subsection with a few more definitions regarding (conics of) dual affine planes, as they will play a crucial role in what follows.

Definition 2.6.

(1) *A partial linear space $\tau = (\mathscr{P}, \mathscr{L})$ is called a dual affine plane if noncollinearity, denoted by $\not\equiv$, induces an equivalence relation on $\mathscr{P}$ and moreover, for ∞ a (new) symbol not in $\mathscr{P}$, the following point-line geometry is a projective plane:*

$$\tau_\infty := (\mathscr{P} \cup \{\infty\}, \mathscr{L} \cup \{T \cup \{\infty\} \mid T \text{ equivalence class of } \not\equiv\}).$$

If $\tau_\infty = \mathbb{P}(k^3)$ for some skew field k, we say that τ is defined over k.

(2) *Suppose that $\tau = (\mathscr{P}, \mathscr{L})$ is a dual affine plane defined over a field k. A subset $\mathscr{C}$ of $\mathscr{P}$ is called a conic of τ when either $\mathscr{C}$ or $\mathscr{C} \cup \{\infty\}$ is a conic of τ_∞. In the latter case, we say that it is a conic through the missing point of τ.*

(3) *Also when $\tau = (\mathscr{P}, \mathscr{L})$ is not defined over a field, we have the notion of a degenerate conic of τ, which is a set $\mathscr{C}$ of $\mathscr{P}$ such that either $\mathscr{C}$ or $\mathscr{C} \cup \{\infty\}$ is empty, a point, a line, the union of two lines or the whole of τ_∞. By convention, we say that both the empty set and the plane τ are degenerate conics of τ (both through the missing point of τ).*

2.2. **Lie incidence geometries.** In this section, we recall the definition of Lie incidence geometries, which should be seen as Grassmannians of spherical buildings, and as such, are generalizations of the projective and polar spaces discussed above. They were introduced in [6] as Lie incidence systems.

Definition 2.7. *Let (W, S) be a finite irreducible Coxeter system of rank at least 2, and Δ a thick building of type (W, S). For any $J \subseteq S$, we define a point-line geometry $(\mathscr{P}_J, \mathscr{L}_J)$:*

$$\mathscr{P}_J := \{J\text{-simplices of } \Delta\}$$
$$\mathscr{L}_J := \{j\text{-lines of } \Delta \mid j \in J\}.$$

For $j \in J$, a j-line is defined to be a set of the form $\{K \mid K \text{ is } J\text{-residue incident with } F\}$, with F a simplex of type $S \setminus \{j\}$. Any geometry that arrises like this is called a Lie incidence geometry. If Δ has type X_n, we say that $(\mathscr{P}_J, \mathscr{L}_J)$ is the Lie incidence geometry of type $\mathsf{X}_{n,J}$ related to Δ.

This definition is quite abstract, so we provide some examples related to classical buildings.

Example 2.5. *Let k be a skew field and let Δ be the building $\mathsf{A}_n(k)$, for $n \geq 2$.*

(1) *The Lie incidence geometry of type $\mathsf{A}_{n,1}$ related to Δ is the projective space $\mathbb{P} := \mathbb{P}(k^{n+1})$.*
(2) *The Lie incidence geometry of type $\mathsf{A}_{n,\{1,n\}}$ related to Δ is the point-line geometry $\mathscr{E}(\mathbb{P}, \mathbb{H})$ as defined is Example 2.2 with $\mathbb{H}$ the set of all hyperplanes of $\mathbb{P}$. Note that this geometry has two types of lines, the 1-lines and the n-lines, which is of course due to the fact that J has size 2.*

Example 2.6. *Let Δ be a building of type X_n with X_n either $\mathsf{B}_n (n \geq 3)$ or $\mathsf{D}_n (n \geq 4)$.*

(1) *The Lie incidence geometry of type $\mathsf{X}_{n,1}$ related to Δ is a polar space Γ.*
(2) *The Lie incidence geometry of type $\mathsf{X}_{n,2}$, $n \geq 3$, related to Δ is the line Grassmannian of this polar space Γ. This* line Grassmannian *of any polar space Γ (possibly of infinite rank) is defined as follows: its points are the lines of Γ, two points L and M are collinear when the corresponding lines in Γ intersect in a point p of Γ and at the same time span a singular plane π of Γ. In this case, the line LM is the set of lines in Γ through p in π.*

Remark 2.1. *If we refer to a Lie incidence geometry of a building Δ, then Δ is not necessarily assumed to be Moufang (see the addendum of [26]). As a consequence, any thick generalized quadrangle and hexagon is a Lie incidence geometry of type $\mathsf{B}_{2,1}$ and $\mathsf{G}_{2,1}$, respectively.*

2.3. **Root shadow spaces and root filtration spaces.** Some Lie incidence geometries behave nicer than others. One particularly nice class of Lie incidence geometries are the root shadow spaces, which are discussed in detail in [4].

Definition 2.8. *Let X_n, $n \geq 2$, be an irreducible crystallographic Coxeter diagram. There is at least one Dynkin diagram Y_n that has X_n as underlying Coxeter diagram. The extended (or affine) diagram of Y_n is obtained by adding one extra node to Y_n corresponding to the highest root. Let J be the set of nodes in Y_n connected to this additional node. Any Lie incidence geometry of type $\mathsf{X}_{n,J}$ is called a root shadow geometry. These are exactly the Lie incidence geometries of the following types (where $n \geq 2$ unless stated otherwise):*

$$\mathsf{A}_{n,\{1,n\}}, \mathsf{B}_{n,1}, \mathsf{B}_{n,2}, \mathsf{D}_{n,2} \text{ (for } n \geq 4), \mathsf{F}_{4,1}, \mathsf{F}_{4,4}, \mathsf{G}_{2,1}, \mathsf{G}_{2,2}, \mathsf{E}_{6,2}, \mathsf{E}_{7,1}, \mathsf{E}_{8,8}.$$

Remark 2.2. *There are three more classes of geometries which are not Lie incidence geometries, but still behave very similarly to the geometries above. We will also refer to them as root shadow geometries (of infinite rank). They are the following:*

(1) *Polar spaces of infinite rank. These geometries behave similarly to Lie incidence geometries of type $\mathsf{B}_{n,1}$ for $n \geq 2$ and $\mathsf{D}_{n,1}$ for $n \geq 4$.*
(2) *Line Grassmannians of polar spaces of infinite rank. These geometries behave similarly to Lie incidence geometries of type $\mathsf{B}_{n,2}$ for $n \geq 3$ and $\mathsf{D}_{n,2}$ for $n \geq 4$.*
(3) *A geometry $\mathscr{E}(\mathbb{P}, \mathbb{H})$ as defined in Example 2.2 with $\mathbb{P}$ an infinite dimensional projective space. These geometries behave similarly to the Lie incidence geometries of type $\mathsf{A}_{n,\{1,n\}}$.*

Definition 2.9. *A root shadow space is called quadrangular when it is a polar space (including infinite rank!), and hexagonal when it is not.*

There are several frameworks that axiomatize the hexagonal root shadow spaces. We will make use of the notion of root filtration spaces.

Definition 2.10. *A partial linear space $X = (\mathscr{E}, \mathscr{L})$ is a root filtration space with filtration $\mathscr{E}_i$, $-2 \leq i \leq 2$ if the sets $\mathscr{E}_i \subseteq \mathscr{E} \times \mathscr{E}$ with $-2 \leq i \leq 2$, provide a partition of $\mathscr{E} \times \mathscr{E}$ into five symmetric relations satisfying the following for all $x, y, z \in \mathscr{E}$:*

(Rf_1) *The relation $\mathscr{E}_{-2}$ is equality.*
(Rf_2) *The relation $\mathscr{E}_{-1}$ is collinearity of distinct points.*

(Rf_3) *For each $(x,y) \in \mathscr{E}_1$, there exists a unique point, denoted with $[x,y]$, such that*
$$\mathscr{E}_i(x) \cap \mathscr{E}_j(y) \subseteq \mathscr{E}_{\leq i+j}([x,y]).$$
(Rf_4) *If $(x,y) \in \mathscr{E}_2$, then $\mathscr{E}_{\leq 0}(x) \cap \mathscr{E}_{\leq -1}(y) = \emptyset$.*
(Rf_5) *The subsets $\mathscr{E}_{\leq i}$ are subspaces of Γ, for $-2 \leq i \leq 2$.*
(Rf_6) *The subset $\mathscr{E}_{\leq 1}$ is a geometric hyperplane of Γ.*

The root filtration space X is called nondegenerate when:

(Rf_7) *The set $\mathscr{E}_2$ is nonempty.*
(Rf_8) *The space X is connected.*

Here we have denoted $\mathscr{E}_{\leq i} = \bigcup_{j=-2}^{i} \mathscr{E}_j$ and $\mathscr{E}_{(\leq)i}(x) := \{y \in \mathscr{E} \mid (x,y) \in \mathscr{E}_{(\leq)i}\}$.

Theorem 2.1 ([4] and [13])**.** *Every nondegenerate root filtration space is a hexagonal root shadow space. Conversely, for every hexagonal root shadow space X (possibly of infinite rank), there is a unique filtration such that it forms a root filtration space. The filtration can be defined as follows:*

$(x,y) \in \mathscr{E}_{-2} \iff x = y.$

$(x,y) \in \mathscr{E}_{-1} \iff$ *x and y are collinear in X.*

$(x,y) \in \mathscr{E}_{0} \iff$ *x and y are at distance 2 in X and have at least 2 common neighbours; in this case we say that x and y are **symplectic**.*

$(x,y) \in \mathscr{E}_{1} \iff$ *x and y are at distance 2 in X and have exactly 1 common neighbour : $[x,y]$; in this case we say that x and y are **special**.*

$(x,y) \in \mathscr{E}_{2} \iff$ *x and y are at distance 3 in X. in this case we say that x and y are **opposite**.*

2.4. **Long roots, abstract root subgroups and long root geometries.** For (almost) every irreducible, spherical, thick Moufang building Δ of rank at least two, there is one root shadow space related to Δ for which its points coincide with the root subgroups of the long roots of Δ. In this subsection, we recall the definition of these long roots, the special role of their root groups and their connection to root shadow geometries. An excellent reference for background on root groups and buildings is [1]. For the long root (subgroup) geometries themselves, see [25].

Notation 2. *In this subsection, Δ denotes a thick, irreducible Moufang building of rank at least two. For a root (also called a half-apartment) α of Δ, the group U_α denotes the root group of α. Moreover, set $G^+ := \langle U_\alpha \mid \alpha \text{ root of } \Delta\rangle$.*

We first recall which roots of Δ are called long roots. More details can be found in [15].

Definition 2.11. *Let Σ be an apartment of Δ, and let α, β be two roots of Σ. We define the angle θ between α and β as follows. If $\alpha = \beta$, set $\theta(\alpha,\beta) := 0$. If $\alpha = -\beta$, set $\theta(\alpha,\beta) := \pi$. Suppose that $\alpha \neq \pm\beta$. Let T be a rank 2 residue of Δ such that both $\alpha \cap T$ and $\alpha \cap T$ are roots of T. If T is a n-gon, and $\alpha \cap \beta \cap T$ contains p chambers, then we define $\theta(\alpha,\beta) := \frac{(n-p)}{n}\pi$. One can check that such T always exists, and that $\theta(\alpha,\beta)$ is independent of the choice of T.*

Definition 2.12. *A root α of Δ is called long when for every apartment Σ containing α and every root β of Σ one of the following holds:*

(1) $\theta(\alpha,\beta) > \pi/3$ or $\alpha = \beta$,
(2) $\theta(\alpha,\beta) = \pi/3$, the group U_α is abelian, $[U_\alpha, U_\beta] = U_\gamma$ with γ the unique root of Σ at angle $\pi/3$ with both α and β,

(3) $\theta(\alpha,\beta) \leq \pi/2$, $Z(U_\alpha) \neq 1$ *and* $[Z(U_\alpha), U_\beta] = 1$.

A G^+*-orbit of long roots in* Δ *called a class of long roots. A root group of a long root is called a long root subgroup.*

Proposition 2.2 (Theorem 3.8 of [15])**.** *If* Δ *is not an octagon or a Moufang quadrangle of type* F_4*, it contains a class of long roots.*

Classes of long roots of Δ are particularly interesting because the centers of their root groups form a set of abstract root subgroups, which were studied by Timmesfeld in [25].

Definition 2.13. *A rank one group is a group generated by two nontrivial nilpotent subgroups* A *and* B *such that for each* $a \in A^*$*, there exists an element* $b \in B^*$ *with* $A^b = B^a$ *and vice versa.*

An example of a rank one group is given by $\mathsf{PSL}_2(k)$, with k a field. This group is generated by the upper and lower triangular matrices with 1s on the diagonal.

Definition 2.14. *Let* G *be a group, with* $\mathscr{E}$ *a conjugacy class of abelian subgroups of* G *such that* $G = \langle \mathscr{E} \rangle$*. The set* $\mathscr{E}$ *is called a class of abstract root subgroups of* G *when for each* $A, B \in \mathscr{E}$*, exactly one of the following occurs:*

($\mathscr{E}_{\leq 0}$) *The groups* A *and* B *commute.*
($\mathscr{E}_1$) *The group* $[A, B]$ *belongs to* $\mathscr{E}$ *and equals* $[A, b] = [a, B]$ *for all* a *in* A *and* b *in* B*.*
($\mathscr{E}_2$) *The group* $\langle A, B \rangle$ *is a rank one group. In this case,* A *and* B *are called opposite.*

If all possibilities above occur, we call $\mathscr{E}$ *nondegenerate. If possibilities (1) and (3) occur, but not (2), we call* $\mathscr{E}$ *a class of abstract transvection groups. If for all opposite elements* A, B *of* $\mathscr{E}$*, the rank one group* $\langle A, B \rangle \cong \mathsf{PSL}_2(k)$ *for some fixed field* k*, then* $\mathscr{E}$ *is called a class of* k*-root subgroups of* $\mathscr{E}$ *(or a class of* k*-transvection groups).*

Proposition 2.3 ([25])**.** *Let* M *be a class of long roots of* Δ*, and set* $\mathscr{E} := \{Z(U_\alpha) \mid \alpha \in M\}$*. One of the following holds:*

(1) The set $\mathscr{E}$ *is a nondegenerate class of abstract root subgroups of* G^+*. Define the line set* $\mathscr{L}$ *to be the set of all subsets of* $\mathscr{E}$ *of cardinality at least* 3 *that are of the form*

$$\{C \mid C \leq AB\} \text{ for } A, B \in \mathscr{E} \text{ with } [A, B] = 1.$$

The point-line geometry $(\mathscr{E}, \mathscr{L})$ *forms a hexagonal root shadow space related to* Δ*.*

(2) The set $\mathscr{E}$ *is a class of abstract transvection groups of* G^+*. For* A *in* $\mathscr{E}$*, set* $C_{\mathscr{E}}(A) := \{B \in \mathscr{E} \mid [A, B] = 1\}$*. Moreover, define the line set* $\mathscr{L}$ *to be the subsets of* $\mathscr{E}$ *of the form*

$$\{C \mid C \leq Z(\langle C_{\mathscr{E}}(A) \cap C_{\mathscr{E}}(B) \rangle)\} \text{ for } A, B \in \mathscr{E} \text{ with } [A, B] = 1.$$

The point-line geometry $(\mathscr{E}, \mathscr{L})$ *forms a quadrangular root shadow space related to* Δ*.*

A root shadow space that can be obtained like this is called a long root geometry (related to Δ*).*

We give a quick overview of the long root geometries obtained in Proposition 2.3.

Example 2.7. *If* Δ *is simply laced (or equivalently, of type* $\mathsf{A}_n, \mathsf{D}_n$ *for* $n \geq 4$ *or* E_n *for* $6 \leq n \leq 8$*), the set of all of its roots forms a* G^+ *orbit, implying that there is exactly one class of long roots. At the same time, there is only one root shadow space related to* Δ*, which is always hexagonal. Its point set coincides with the set of all root subgroups of* Δ*.*

Example 2.8. *If* Δ *is not simply laced, the group* G^+ *has two orbits on the roots of* Δ*, and there are two root shadow spaces related to* Δ*.*

(1) If Δ *has type* B_n *for* $n \geq 3$*, the Lie incidence geometry of type* $\mathsf{B}_{n,1}$ *related to* Δ *is a polar space* Γ*.*

(a) If Γ is an orthogonal polar space, the line Grassmannian of Γ is a hexagonal long root geometry.
(b) If Γ is not orthogonal, then the polar space Γ itself is a quadrangular long root geometry.

(2) If Δ has type F_4, any root shadow space for which the convex closure of symplectic points forms an orthogonal polar space, is an hexagonal long root geometry.

(3) If Δ has type B_2 or G_2, one can easily read off from the commutator relations in [29] which roots are long. Unless Δ is a Moufang quadrangle of type F_4, we can always find at least one long root geometry related to Δ, which is quadrangular or hexagonal depending on whether Δ is of type B_2 or G_2.

Remark 2.3. *When the building Δ is defined over a bad characteristic (which is 2 if Δ is a building of type B_n or F_4, and 3 if Δ is a building of type G_2), it could be that Δ has two classes of long root subgroups, and hence has two distinct long root geometries related to it.*

Definition 2.15. *As in Remark 2.2, there are some classes of geometries of infinite rank which are not Lie incidence geometries, but behave similarly to the long root geometries defined in Proposition 2.3. These geometries are the following:*

(1) Non-orthogonal polar spaces of infinite rank.
(2) Line Grassmannians of orthogonal polar spaces of infinite rank.
(3) The geometries $\mathscr{E}(\mathbb{P},\mathbb{H})$ with $\mathbb{P}$ an infinite dimensional projective space (see Example 2.2).

We will refer to them as long root geometries (of infinite rank).

Remark 2.4. *Let $X = (\mathscr{E},\mathscr{L})$ be a hexagonal long root geometry (possibly of infinite rank), then for any point $x \in \mathscr{E}$, we define the group*

$$Z_x := \{\theta \in Aut(X) \,|\, y^\theta = y, \forall\, y \in \mathscr{E}_{\leq 0}(x) \text{ and } y^\theta \in y[x,y], \forall\, y \in \mathscr{E}_1(x)\}.$$

The set $\{Z_x \,|\, x \in \mathscr{E}\}$ is a class of abstract root subgroups of $\langle Z_x \,|\, x \in \mathscr{E}\rangle$. We refer to this set as the canonical class of root subgroups related to X. If some points x and y are collinear or symplectic, then $[Z_x, Z_y] = 0$. If they are special, then $[Z_x, Z_y] = Z_{[x,y]}$. In the latter case, the group Z_x acts sharply transitively on the points of the line $y[x,y]$ different from $[x,y]$. If they are opposite, then $\langle Z_x, Z_y\rangle$ is a rank one group.

Proposition 2.4. *Let $X = (\mathscr{E},\mathscr{L})$ be an hexagonal long root geometry with $\{Z_x \mid x \in \mathscr{E}\}$ its canonical class of root subgroups. Let x and y be two opposite points of X, and let p and q be two special points of X for which $[p,q] = x$. Denote with S the smallest subspace of X for which the following hold:*

(1) it contains the points p, q and y.
(2) for every two special points p' and q' of S, $[p',q'] \in S$.

Then S is a long root geometry of type $\mathsf{A}_{2,\{1,2\}}$, which is defined over a skew field k as soon as X itself is not of type $\mathsf{A}_{2,\{1,2\}}$. If X is not of type $\mathscr{E}(\mathbb{P},\mathbb{H})$, then k is automatically a field. No two points of S are symplectic, and points in the long root geometry S are opposite (collinear, special) if and only if they are opposite (collinear, special) in X.

Proof. This is proved for all cases throughout [14]. Alternatively, one can also argue as in the proof of Proposition 4.6 (Im_2), see Section 4.2. Finally, one can translate the assertion to the root subgroup language and then use Section V.2 of [25]. □

We finish this subsection by mentioning a common property of hexagonal long root geometries.

Lemma 2.1. *Let X be a hexagonal root long root geometry (possibly of infinite rank), and let y_1 and y_2 be symplectic points, both opposite some point x. There is a point w that is opposite x and collinear to*

y_1 and y_2. If X is not of type $\mathscr{E}(\mathbb{P},\mathbb{H})$ or of type $\mathsf{B}_{3,2}$, then there also exists a point u that is special to x and collinear to y_1 and y_2.

Proof. Properties (a) and (b) of Section 3 of [4] imply that y_1 and y_2 are contained in a subspace Γ isomorphic to a polar space containing a (unique) point z symplectic to x. Under the given assumptions, Γ, which is an orthogonal polar space, contains a point w collinear to both y_1 and y_2, but not to z, and a point u collinear to all of y_1, y_2 and z. The assertion now follows from Conditions (Rf_4) and (Rf_6). □

3. Imaginary geometries

In this section, we define the main objects of this paper: the imaginary geometries. These geometries have the same point set as long root geometries, but have a different set of lines. Depending on whether the corresponding long root geometry is quadrangular or hexagonal, the imaginary geometry behaves very differently.

3.1. **Definition of imaginary geometries.** We start by defining imaginary geometries related to spherical buildings.

Definition 3.16. *Let Δ be a thick, irreducible, spherical Moufang building of rank at least two, and let $\mathscr{E}$ be a class of centers of long root subgroups of Δ. If A, B in $\mathscr{E}$ are opposite, it follows from* [25, Lemma 2.1] *that*

$$\{C \in \mathscr{E} \mid C \in \langle A, B\rangle\} = A^{\langle A,B\rangle} = \{A\} \cup B^A = \{B\} \cup A^B.$$

We refer to this set as the imaginary line through A and B. Define $\mathrm{Im}(\Delta, \mathscr{E})$ to be the point-line geometry with as point set $\mathscr{E}$ and as line set the set of all imaginary lines. If Δ has type X_n, we say that the imaginary geometry $\mathrm{Im}(\Delta, \mathscr{E})$ is also of type X_n

As in Definition 2.15, there are some geometries that are not associated to spherical buildings, but still behave very similarly to the geometries $\mathrm{Im}(\Delta, \mathscr{E})$. We will therefore work with the following more general definition.

Definition 3.17. *Let $X = (\mathscr{E}, \mathscr{L})$ be a long root geometry (possibly of infinite rank) with canonical set of abstract root subgroups $\{Z_x \mid x \in \mathscr{E}\}$. For any two opposite points x, y of $\mathscr{E}$, the group $\langle Z_x, Z_y\rangle$ is an abstract root subgroup. It follows from* [25, Lemma 2.1] *that*

$$\{z \in \mathscr{E} \mid Z_z \leq \langle Z_x, Z_y\rangle\} = x^{\langle Z_x, Z_y\rangle} = y^{\langle Z_x, Z_y\rangle} = \{x\} \cup y^{Z_x} = \{y\} \cup x^{Z_y}.$$

We denote this set with xy and call it the imaginary line defined by x and y.

Definition 3.18. *A point-line geometry Y is called an imaginary geometry if there is a long root geometry X such that the point set of X coincides with the point set of Y and the lines are the imaginary lines of X. If this is the case, we will say that Y is the imaginary geometry of X. We call Y hexagonal (quadrangular) when X is hexagonal (quadrangular).*

If Y is the imaginary geometry of the long root geometry X, it could a priori be that there is another long root geometry X', not isomorphic to X, such that Y is also the imaginary geometry of X', in particular, Y could even both be quadrangular and hexagonal. This is of course not the case, as we will prove in Proposition 4.6.

Notation 3. *In an imaginary geometry Y (or more general, in an incidence structure that axiomatizes such an imaginary geometry), we denote collinearity with $\equiv$, and noncollinearity with $\not\equiv$. Moreover, for any point p, we denote*

$$p^{\equiv} = \{q \mid q \text{ point of } Y \text{ with } q \equiv p\} \text{ and } p^{\not\equiv} = \{q \mid q \text{ point of } Y \text{ with } q \not\equiv p\}.$$

For any sets S_1, S_2 of points, we denote

$$S_1^{\not\equiv} = \bigcap_{s \in S_1} s^{\not\equiv} \text{ and } S_1 \not\equiv S_2 \text{ if } S_2 \subseteq S_1^{\not\equiv}.$$

3.2. **Quadrangular imaginary geometries.** Imaginary geometries of quadrangular long root geometries have been studied and axiomatized before, for example in [7], [8] and [11]. In the former two, imaginary lines are called hyperbolic lines, and the imaginary geometry is referred to as *the hyperbolic geometry of polar spaces*. The reason why we call it "imaginary" is that we reserve this name for objects that contain points at distance 3 in the original geometry, while "hyperbolic" refers to objects at distance 2 in the original geometry (this is conform the terminology in Chapter 6 of [28]). In this subsection, we shortly discuss one example of a quadrangular imaginary geometry, and state the axiomatization theorem of quadrangular imaginary geometries, as obtained in [8].

Construction 1. *Let Y be a quadrangular long root geometry. Two points x and y of X are opposite when they are not collinear, in which case they are at distance 2 in X. Suppose this is the case, then the imaginary line xy coincides with $(\{x, y\}^{\equiv})^{\equiv}$.*

Proof. If X is a polar space of rank at least 3, this follows from [7, Section 4]. If X has rank 2, it is a Moufang quadrangle, not of type F_4, and the result immediately follows from the commutator relations in the appropriate, but various chapters of [29]. □

Example 3.9. *Let X be a symplectic polar space of rank n. The point set of X coincides with the point set of the projective space $\mathbb{P}(k^{2n+1})$. Two points p_1 and p_2 of X are opposite when they are not collinear in X, in this case, the imaginary line through p_1 and p_2 is the (non-isotropic) line p_1p_2 of $\mathbb{P}(k^{2n+1})$.*

Lemma 3.2 ([7]). *The following properties hold in a quadrangular imaginary geometry Y:*

(1) *For any line l and point p, the point p is collinear to all, all but one or no points of l.*
(2) *A plane is either a dual affine plane or a linear plane (that is, a plane in which any two points are collinear).*
(3) *There is a unique quadrangular long root geometry X for which Y is the imaginary geometry of X.*

Proposition 3.5 ([7]). *Let $Y = (\mathscr{E}, \mathscr{I})$ be a connected and coconnected partial linear space in which the axioms below hold (where we denote (non)collinearity as in Notation 3):*

(1) *If l is a line and p is a point with $|p^{\not\equiv} \cap l| = 1$, then p and l generate a dual affine plane.*
(2) *If π is a subspace of Y isomorphic to a dual affine plane, containing a point q. If $|q^{\not\equiv} \cap p^{\not\equiv} \cap \pi| \geq 2$ for some point p, then $q^{\not\equiv} \cap \pi \subseteq p^{\not\equiv}$.*
(3) *If p and q are points with $p^{\equiv} \subseteq q^{\equiv}$, then $p = q$.*
(4) *Every line contains at least four points.*

Then $X = (\mathscr{E}, \mathscr{L})$ with $\mathscr{L}$ the set of subsets of $\mathscr{E}$ given by

$$pq := \{r \mid p^{\not\equiv} \cap q^{\not\equiv} \subseteq r^{\not\equiv}\} \text{ for } p \text{ and } q \text{ elements of } \mathscr{E} \text{ with } p \not\equiv q$$

is a quadrangular long root geometry. Moreover, if every line $l \in \mathscr{I}$ coincides with $(l^{\not\equiv})^{\not\equiv}$, the set $\mathscr{I}$ is the set of imaginary lines of X, and Y is the imaginary geometry of X.

3.3. **Hexagonal imaginary geometries.** We provide a construction of imaginary lines in a hexagonal long root geometry, and apply this construction to give some explicit examples of hexagonal imaginary geometries, using the corresponding long root geometries.

Construction 2. *Let $X = (\mathscr{E}, \mathscr{L})$ be a hexagonal long root geometry containing opposite points x and y. Let p_x and q_x be points collinear to x and special to y, such that p_x and q_x are also special. Denote $p_y := [p_x, y]$ and $q_y := [q_x, y]$. The imaginary line xy coincides with*

$$\{[p, q] \mid p \in p_x p_y, q \in q_x q_y \text{ with } p \text{ special to } q\}.$$

Proof. It follows from basic properties of root shadow spaces [4] that every point of $p_x p_y$ is special to a unique point of $q_x q_y$, while being opposite to all other points of that line. Moreover, we find that p_x and q_x are special, with $x = [p_x, q_x]$ and that p_y and q_y are special, with $y = [p_y, q_y]$.

Let $\{Z_x \mid x \in \mathscr{E}\}$ be the canonical class of root subgroups related to X, defined in Remark 2.4. Using Definition 3.18, we find that $xy = \{x\} \cup y^{Z_x} = \{[p_x, q_x]\} \cup \{[p_y^z, q_y^z] \mid z \in Z_x\}$. The proof now follows from the fact that, as noted in Remark 2.4, the group Z_x acts transitively on the points of the line $p_x p_y$ different from p_x. □

Example 3.10. *Let X be the hexagonal long root geometry $\mathscr{E}(\mathbb{P}, \mathbb{H})$, with $\mathbb{P}$ and $\mathbb{H}$ as in Example 2.2. Two points (p_1, H_1) and (p_2, H_2) are opposite in X when $p_1 \notin H_2$ and $p_2 \notin H_1$. In this case, the imaginary line through (p_1, H_1) and (p_2, H_2) is given by the set*

$$\{(q, \langle q, H_1 \cap H_2\rangle) \mid q \text{ point on } p_1p_2\} = \{(H \cap p_1p_2, H) \mid H \text{ hyperplane through } H_1 \cap H_2\}.$$

If Y is an imaginary geometry of X, we will say that Y is of type $\mathscr{E}(\mathbb{P}, \mathbb{H})$. When $\mathbb{P}$ is a projective plane, this example is discussed in more detail in Section 3.4.

Example 3.11. *Let Γ be an orthogonal polar space (possibly of infinite rank), and let X be the hexagonal long root geometry related to Γ (that is, the line Grassmannian of Γ). Two lines l and m of Γ are opposite points of X when in Γ every point of l is collinear to a unique point of m and vice versa. Let k_1 and k_2 be two lines of Γ that intersect both l and m. Then the imaginary line lm of X is the set of lines of Γ that intersect both k_1 and k_2. This set is independent of the choice of k_1 and k_2.*

Remark 3.5. *An hexagonal imaginary geometry of infinite rank is one of the following:*

(1) an imaginary geometry of type $\mathscr{E}(\mathbb{P}, \mathbb{H})$ with $\mathbb{P}$ an infinite-dimensional projective space.
(2) an imaginary geometry of a line Grassmannian of an orthogonal polar space of infinite rank.

3.4. **Imaginary geometries of type A_2.** We finish this section by zooming in on one particular example of hexagonal imaginary geometries, namely those related to a building of type A_2, as this will be the main building block of the axiomatic hexagonal imaginary geometries.

Notation 4. *In this subsection, Δ denotes a thick Moufang building of type A_2 defined over some field k. As mentioned in Example 2.5, the Lie incidence geometry of type $\mathsf{A}_{2,1}$ related to Δ is a projective plane $\tau = \mathbb{P}(k^3)$, whose point and line set we denote with $\mathscr{P}_\tau$ and $\mathscr{L}_\tau$. Throughout the subsection, we assume that $|k| \geq 3$.*

We first describe the long root geometry and the imaginary geometry of Δ. Note that these are exactly the geometries obtained in Example 2.2 and Example 3.10 with $\mathbb{P} = \tau$.

Example 3.12. *The long root geometry related to τ is the point-line geometry $X = (\mathscr{E}, \mathscr{L})$ with*

$$\mathscr{E} := \{(p, l) \in \mathscr{P}_\tau \times \mathscr{L}_\tau \mid p \in l\},$$
$$\mathscr{L} := \{T_p \mid p \in \mathscr{P}_\tau\} \cup \{T_l \mid l \in \mathscr{L}_\tau\},$$

where for any point p of τ and line l of τ, the sets T_p and T_l are defined as follows:

$$T_p := \{(p, m) \mid m \in \mathscr{L}_\tau, m \ni p\} \text{ and } T_l := \{(q, l) \mid q \in \mathscr{P}_\tau, q \in l\}.$$

As already noted, X is a non-thick generalized hexagon with thick lines.

Example 3.13. *The imaginary geometry related to Δ is the point-line geometry $Y = (\mathscr{E}, \mathscr{I})$, where $\mathscr{E}$ is the point set of X defined in Example 3.12 and*

$$\mathscr{I} := \{[q,m] \mid q \in \mathscr{P}_\tau, m \in \mathscr{L}_\tau, q \notin m\} \text{ with } [q,m] := \{(p,pq) \mid p \in m\} = \{(l \cap m, l) \mid q \in l\}.$$

Notation 5. *In the rest of this subsection, we will work with both X and Y from Example 3.12 and Example 3.13, which have the same point set but a different set of lines. We refer to lines of X as transversals of X, and lines of Y as lines. Two distinct points are called collinear when they are contained in a common line and noncollinear when they are not. We will make use of Notation 3 for Y. If two points are contained in a common transversal of X, they are called linelike in X.*

In Theorem 2.1, we saw that there are five possible relations between points of a long root geometry. In X however, no two points are symplectic, so this amounts to four different relations between points. We describe them explicitly in the following lemma.

Lemma 3.3. *Let (p,l) and (q,m) be two points of X. Exactly one of the following occurs:*

- $(\mathscr{E}_{-2})$ *$p = q$ and $l = m$. The points are equal.*
- $(\mathscr{E}_{-1})$ *$p = q$ or $l = m$, but not both. There exists a unique transversal (namely T_p or T_l respectively) that contains both (p,l) and (q,m). The points are linelike in X.*
- $(\mathscr{E}_1)$ *$p \in m$ or $q \in l$, but not both. There exists a unique point, namely (p,m) or (q,l) respectively, which is linelike in X to both (p,l) and (q,m). We denote this point with $[(p,l),(q,m)]$. The points are special in X.*
- $(\mathscr{E}_2)$ *$p \notin m$ and $q \notin l$. There is a unique line that contains (p,l) and (q,m), namely $[l \cap m, pq]$. The points are collinear (in Y).*

When two distinct points in Y are noncollinear, they can hence either be linelike or special in X. We come back to that in Lemma 3.6.

As pointed out in Lemma 3.2, every quadrangular imaginary geometry contains a lot of dual affine planes. We show that this is also the case for the hexagonal imaginary geometry Y.

Definition 3.19. *For $p \in \mathscr{P}_\tau$ and $l \in \mathscr{L}_\tau$, we define the following subsets of Y:*

$$\pi_p := \{(q,m) \in \mathscr{E} \mid q \neq p \text{ and } p \in m\}.$$
$$\pi_l := \{(q,m) \in \mathscr{E} \mid m \neq l \text{ and } q \in l\}.$$

Lemma 3.4. *For $p \in \mathscr{P}_\tau$ and $l \in \mathscr{L}_\tau$, the subsets π_p and π_l form subspaces of Y and are dual affine planes.*

Proof. We prove this for π_p, the proof for π_l then follows immediately by dualizing. Each line of Y which contains two points of π_p is of the form $[p,l]$, and is hence fully contained in π_p, which implies that π_p is a subspace of Y. The points of τ different from p, together with the lines of τ not through p form a dual affine plane. The map

$$\pi_p \to \tau \setminus \{p\} : (q,m) \mapsto q$$

is clearly an isomorphism between π_p and this dual affine plane. □

The next lemma determines the planes of Y. The proof is an easy verification in τ and is omitted.

Lemma 3.5. *Let $[p,l]$ and $[q,m]$ be two lines in Y that intersect in the point (r,n). Exactly one of the following cases occurs:*

(1) *$p = q$ and $l = m$. In this case, (p,l) and (q,m) are equal.*
(2) *$p = q$ or $l = m$, but not both. The two lines generate the dual affine plane π_p (or π_l). Any point of $[p,l] \setminus \{(r,n)\}$ is noncollinear to a unique point of $[q,m]$ and vice versa.*

(3) $p \neq q$ *and* $l \neq m$. *The lines generate* A. *Any point of* $[p, l] \setminus \{(r, n)\}$ *is noncollinear with exactly two points of* $[q, m]$ *and vice versa.*

In particular, every plane of Y *is either* Y *itself, or is one of the dual affine planes described in* Definition 3.19.

Remark 3.6. *If* k *would be equal to* $\mathbb{F}_2$*, then the subspace in* Y *generated by* $[p, l]$ *and* $[q, m]$ *with* $p \notin m$ *and* $q \notin l$ *would just be* $[p, l] \cup [q, m]$*, which is not the whole of* A.

We can use Lemma 3.5 to distinguish whether two noncollinear points in Y are linelike or special in X. This is done in the next lemma. The proof of this lemma is again just a verification, and is hence omitted.

Lemma 3.6. *Let* p *and* q *be two distinct noncollinear points* p *and* q *in* Y*. The following hold:*

(1) *The points* p *and* q *are linelike in* X *if and only if there is a dual affine plane of* Y *that contains both* p *and* q*. In this case, the transversal in* X *through* p *and* q *is given by* $(\{p, q\}^{\not\equiv})^{\not\equiv}$.
(2) *The points* p *and* q *are special in* X *when there is no dual affine plane of* Y *that contains both* p *and* q*. In this case,* $[p, q]$ *is the unique point in* Y *linelike to both* p *and* q.

Remark 3.7. *Lemma 3.6 implies that the imaginary geometry* Y *determines whether two distinct noncolllinear points are linelike or special in* X*. Moreover, the transversals of* X *are determined by the lines of* Y*. We can hence say that two points are linelike (or special) in* Y *and speak of transversals of* Y.

Definition 3.20. *For a dual affine plane* π *of* Y*, a transversal of* Y *is called a* transversal of π *if it contains at least two points of* π*. Define* $\bar{\pi}$ *to be the union of all transversals of* π*, and define* $T_\pi := \bar{\pi} \setminus \pi$*. We will refer to* $\bar{\pi}$ *as the* transversal closure of π.

Notation 6. *For a transversal* T *of* Y*, the set* $T^{\not\equiv}$ *is the union of all transversals of* Y *that intersect* T *in a point. The set* $T^{\not\equiv} \setminus T$ *is a dual affine plane, which we denote with* π_T.

Remark 3.8. *Using Definitions 6 and 3.20, one deduces the following natural correspondence between dual affine planes of* A *and transversals of* A*: a transversal* T *corresponds to the dual affine plane* π_T *and a dual affine plane* π *corresponds to the transversal* T_π*. Note that for a point* p *and a line* l *of* τ*, the transversals* T_p *and* T_l *correspond to dual affine planes* π_p *and* π_l.

We finish this subsection with one more lemma, which will be useful later on. The proof is once again an easy verification, and is hence omitted.

Lemma 3.7. *Let* q *be a point and* π *be a dual affine plane of* Y*, with* $q \notin \bar{\pi}$.

(1) *There is exactly one point* p *of* π *linelike with* q.
(2) $q^{\not\equiv} \cap \bar{\pi} = T \cup l$*, with* T *the transversal of* π *through* p *and* l *a line of* π *through* p.
(3) *Every line through* q *intersects* $\bar{\pi}$ *in exactly one point.*

Remark 3.9. *By picking a coordinate system for* τ*, we obtain (projective) coordinates for the points and lines of* τ *and hence coordinates for the points of* A*, which are incident point-line pairs of* τ*. As such, we obtain a map* σ *from* A *to the projective space* $\mathbb{P}(k^8)$*:*

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \right\} \longmapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} = \begin{pmatrix} x_1a_1 & x_1a_2 & x_1a_3 \\ x_2a_1 & x_2a_2 & x_2a_3 \\ x_3a_1 & x_3a_2 & x_3a_3 \end{pmatrix}.$$

Note that the image of this map consists exactly of the matrices of rank 1 *and trace* 0*. The points of* A *are hence all contained in a hyperplane of* $\mathbb{P}(k^8)$*, and are even the intersection of the Segre variety (formed by all matrices of rank* 1*) with this hyperplane.*

We can consider the images under σ *of transversals, lines and dual affine planes of* Y.

- *Let T be a transversal of A, then $\sigma(T)$ is a line in $\mathbb{P}(k^8)$.*
- *Let L be a line of A, then $\langle \sigma(L) \rangle$ is a plane of $\mathbb{P}(k^8)$. We have that $\sigma(L) = \sigma(A) \cap \langle \sigma(L) \rangle$ is a conic in this plane.*
- *Let π be a dual affine plane of A, then $\langle \sigma(\pi) \rangle$ is a 4-dimensional subspace of $\mathbb{P}(k^8)$. We have that $\sigma(A) \cap \langle \sigma(\pi) \rangle = \sigma(\bar{\pi})$.*

Now let Q be a hyperplane of $\mathbb{P}(k^8)$ and set $\mathscr{Q} := Q \cap \sigma(A)$. Then we can consider the intersection of $\mathscr{Q}$ with the objects above:

- *Let T be a transversal of A, then $\sigma(T) \cap \mathscr{Q}$ is either a point or the whole of $\sigma(T)$.*
- *Let L be a line of A, then $\sigma(L) \cap \mathscr{Q}$ is either empty, exactly one point, exactly two points or is the whole of $\sigma(L)$.*
- *Let π be a dual affine plane of A, then $\sigma(\pi) \cap \mathscr{Q}$ is either the whole of $\sigma(\pi)$ or it is of the form $\sigma(\mathscr{C})$ with $\mathscr{C}$ some conic of π through the missing point of π.*

Remark 3.10. *The representation of X and Y using a Segre variety as in Remark 3.9 is precisely the polarized embedding arising from the adjoint module of the Lie algebra $\mathfrak{sl}_2(k)$ as described in [2], as is readily verified.*

4. Properties of hexagonal imaginary geometries

We discuss some properties of hexagonal imaginary geometries, where we focus on the properties occurring in the Main Theorem. In particular, we will prove the following proposition.

Proposition 4.6. *Let Y be an imaginary geometry where lines contain at least four points. Then Y is a connected partial linear space that satisfies (Im_1), (Im_2) and (Im_3). Moreover, there exists a unique long root geometry X such that Y is the imaginary geometry of X.*

Notation 7. *In this section, the point-line geometry $Y = (\mathscr{E}, \mathscr{I})$ is a hexagonal imaginary geometry related to some hexagonal long root geometry $X = (\mathscr{E}, \mathscr{L})$, where X is possibly of infinite rank. We denote with $\{Z_x \mid x \in \mathscr{E}\}$ the canonical class of root subgroups of X, as defined in Remark 2.4.*

The point-line geometries X and Y have the same point set, but a different line set. As in Notation 5, we refer to the lines of X as transversals, and lines of Y as lines. If two distinct points are contained in a common line, we say that they are collinear and if they are not contained in a common line, we say that they are noncollinear. We use the notations $\equiv$ and $\not\equiv$ introduced in Notation 3. When points are noncollinear, it follows from Theorem 2.1 that there are three options: they can be linelike in X, symplectic in X or special in X, in the latter case, there is a unique point that is linelike in X to both p and q, we denote this point with $[p, q]$.

Throughout the subsection, we assume that transversals (and hence also lines) contain at least four points.

Lemma 4.8. *The imaginary geometry Y is a partial linear space.*

Proof. Suppose that x and y are collinear points in Y, and that l is any line that contains x and y. We aim to prove that $l = x^{\langle Z_x, Z_y \rangle}$. The imaginary line l is of the form $p^{\langle Z_p, Z_q \rangle}$ with p and q points of $\mathscr{E}$ that are collinear in X, which implies that $x = p^{u_1}$ and $y = p^{u_2}$, for $u_1, u_2 \in \langle Z_p, Z_q \rangle$. It then follows from [25, II.2.1] that $\langle Z_x, Z_y \rangle = \langle Z_p, Z_q \rangle$, which indeed implies that $l = x^{\langle Z_x, Z_y \rangle}$. $\square$

4.1. **The A_2-planes contained in hexagonal imaginary geometries.** By Proposition 2.4, the long root geometry X contains subspaces that are long root geometries of type A_2. In this subsection, we investigate corresponding subspaces of the imaginary geometry Y that are imaginary geometries of type A_2 (which are the A_2-planes of Definition 4.21).

Definition 4.21. *A subspace of Y that is isomorphic to an imaginary geometry of type A_2 (defined over a skew field k) is called an A_2-plane (defined over k).*

Remark 4.11. *Suppose that A is an A_2-plane of Y. Then two points p, q of Y contained in A are collinear in A if and only if they are collinear in Y. If these two points are noncollinear in A, it follows by Remark 3.7 that they are linelike in A (in which case there is a transversal of A containing them) or special in A. A priori, it is not clear that the two points are linelike (special) in A if and only if they are linelike (special) in X. In the next lemma, we construct A_2-planes of Y where this is the case.*

Lemma 4.9. *Let y, p, q be points of Y such that p and q are special in X, and y is collinear to p, q and $x := [p, q]$. The plane A in Y generated by p, q and r is an A_2-plane. For each pair of points p' and q' of A, the following hold:*

(1) *The points p' and q' are linelike (special, opposite) in X if and only if they are linelike (special, opposite) in A.*
(2) *If p' and q' are linelike, the transversal in X that contains p' and q' coincides with the transversal in A that contains p' and q'.*
(3) *The points p' and q' are not symplectic in X.*

If X is not of type $A_{2,\{1,2\}}$, the A_2-plane A is defined over a skew field k. If X is not of type $\mathscr{E}(\mathbb{P}, \mathbb{H})$, k is a field.

Proof. Let A be the subspace of X that contains p, q, x and y, and is closed under taking special paths, obtained in Proposition 2.4. Denote with $\mathscr{L}_A$ the set of transversals of X contained in A. Then $X_A = (\mathscr{P}_A, \mathscr{L}_A)$ is a long root geometry of type $A_{2,\{1,2\}}$. Moreover, two points are linelike (special, opposite) in X_A if and only if they are linelike (special, opposite) in X.

Let p and q be two points of $\mathscr{P}_A$ that are collinear in Y. Then these points must be opposite in X_A, so we can construct the imaginary line in X_A through p and q using Construction 2. Since lines of X_A are transversals of X, this is of course the imaginary line of X through p and q, which is by definition the line pq of Y. Let $\mathscr{I}_A$ be the set of lines of Y that contain two points of $\mathscr{P}_A$. Then every element of $\mathscr{I}_A$ is hence completely contained in $\mathscr{P}_A$, and coincides with an imaginary line of X_A. This translates to the fact that $\mathscr{P}_A$ is a subspace of Y. The point-line geometry $A = (\mathscr{P}_A, \mathscr{I}_A)$ is isomorphic to the imaginary geometry of X_A, meaning that it is an A_2-plane. It is clear that A is defined over a (skew) field if and only if X_A is. This completes the proof. □

Lemma 4.10. *Let l and m be two lines of Y intersecting in some point q. If some point of $l \setminus \{q\}$ is noncollinear to exactly i points of m ($i \in \mathbb{N}$), then any point of $l \setminus \{q\}$ is noncollinear to exactly i points of m.*

Proof. Denote $q = l \cap m$. Let $\hat{p}$ be any point of $l \setminus \{q\}$. The group Z_q acts transitively on the points of $l \setminus \{q\}$, so there exists some $u \in Z_q$ with $\hat{p} = p^u$. The group Z_q stabilizes the line m, so

$$i = |\{p^{\not\equiv} \cap m\}| = |\{p^{\not\equiv} \cap m\}^u| = |\{\hat{p}^{\not\equiv} \cap m\}|.$$

□

Remark 4.12. *In the statement (and proof) of Lemma 4.10, we can of course replace noncollinear with linelike, symplectic or special in X.*

Lemma 4.11. *Let l and m be two intersecting lines of Y. Exactly one of the following occurs:*

(1) *Every point of l is collinear to every point of m.*
(2) *Every point of $l \setminus \{l \cap m\}$ is linelike (symplectic, special) in X to exactly one point of $m \setminus \{l \cap m\}$, and collinear to all other points of m, and vice versa.*

(3) *Every point of $l \setminus \{l \cap m\}$ is special in X to exactly two points of $m \setminus \{l \cap m\}$ and collinear to all other points of m, and vice versa. The subspace in Y generated by l and m is an A_2-plane that has the properties listed in* Lemma 4.9.

Proof. Denote $q = l \cap m$, and let p be any point of $l \setminus \{q\}$. If p is collinear to all points of m, the claim follows from Lemma 4.10. Suppose that p is noncollinear to some point r of m. In X, the points p and r are either linelike, symplectic or special. If they are special, there exists a unique point $[p, r]$ linelike to both. We make a case distinction.

(1) Suppose that p is linelike or symplectic to r. The group Z_r fixes p and it acts transitively on the points of $m \setminus \{r\}$. The point p is collinear to $q \in m$, and is hence collinear to all points in $q^{Z_r} = m \setminus \{r\}$. This implies that p is linelike (or symplectic) to a unique point of m (namely r), and that it is collinear to all other points of m. Using Lemma 4.10, we find that this is the case for every point of $l \setminus \{q\}$. By reversing the roles of r and p, we also find that every point of $m \setminus \{q\}$ is linelike (or symplectic) to a unique point of l, and collinear to all other points of l.

(2) Suppose that p is special to r and that q is noncollinear to $s := [p, r]$. The point q is special to s and collinear to all other points of the transversal sp. The group Z_r fixes s, stabilizes the transversal sp, and acts transitively on $m \setminus \{r\}$. This implies that every point of $q^{Z_r} = m \setminus \{r\}$ is special to s and collinear to all other points of the transversal sp, in particular to p. The point p is hence noncollinear to a unique point of l, namely r. Using Lemma 4.10, we find that every point of $l \setminus \{q\}$ is noncollinear to a unique point of m. By reversing the roles of r and p, we also find that every point of $m \setminus \{q\}$ is noncollinear to a unique point of l.

(3) Suppose that p is special to r and that q is collinear to $s := [p, r]$. We can apply Lemma 4.9 to obtain that q, p and r (and hence also l and m) generate an A_2-plane in Y. It follows from Lemma 3.5 that every point of $l \setminus \{q\}$ is special to exactly two points of $m \setminus \{q\}$ and vice versa.

□

Lemma 4.12. *Every A_2-plane of Y has the properties listed in* Lemma 4.9.

Proof. Let A is an A_2-plane of Y. It follows from Lemma 3.5 that A is generated by two lines l and m where every point of $l \setminus \{l \cap m\}$ is noncollinear to two points of $m \setminus \{l \cap m\}$ and vice versa. It then follows from Lemma 4.11 that A indeed has the properties of Lemma 4.9. □

The next corollary should be compared to Lemma 3.6.

Corollary 4.1. *Let p and q be distinct noncollinear points. The following hold:*

(1) *The points p and q are linelike in X if and only if there is an A_2-plane A of Y that contains p and q such that p and q are linelike in A. In this case, the transversal in X that contains p and q coincides with the transversal in A that contains p and q.*

(2) *The points p and q are symplectic in X if and only if there does not exist any A_2-plane of Y that contains both p and q.*

(3) *The points p and q are special in X if and only if there is an A_2-plane A of Y such that p and q are special in A.*

Proof. This follows directly from Lemmas 4.9 and 4.11. □

Remark 4.13. *Corollary 4.1 implies that the hexagonal imaginary geometry Y determines whether two distinct noncollinear points are linelike, symplectic, or special in X. Moreover, the transversals in X are determined by the lines of Y. We can hence say that two points are linkelike (symplectic or special) (in Y) and speak of transversals (of Y).*

4.2. **Conclusion.** By now, we have gathered enough information to conclude the proof of Proposition 4.6

Proof of Proposition 4.6. It follows from Lemma 4.8 that Y is a partial linear space and from [3, Lemma 5] that Y is connected. Moreover, it follows from Corollary 4.1 that X is the unique long root geometry for which Y is the imaginary geometry of X. Then Axiom (Im_1) follows from Lemma 4.11.

We now show Axiom (Im_2). By letting the Lie algebra $\mathfrak{sl}_2(k)$ corresponding to a rank one group generated by two opposite long root groups act in its adjoint representation on the Lie algebra corresponding to X, we deduce, using Remark 3.10, that, in the embedding of X corresponding to the adjoint module (as in [2]), an A_2-plane is embedded as in Remark 3.9. Then Axiom (Im_2) follows from (Rf_6) and Remark 3.9.

Finally, we prove Axiom (Im_3). To that end, let p and q be two distinct points. We prove that $p^{\equiv} \neq q^{\equiv}$. If p and q are symplectic, this follows from [4, Lemma 8]. If p and q are linelike, special or opposite, then p and q are contained in some A_2-plane A, and it is clear that $p^{\equiv} \cap A \neq q^{\equiv} \cap A$. $\square$

5. Conical subspaces of imaginary geometries of type A_2

5.1. **Definitions and notations.** As the title of this section suggests, we will discuss conical subspaces of hexagonal imaginary geometries of type A_2.

Notation 8. *In this section, k is a skew field, with $|k| \geq 4$. We denote with A the hexagonal imaginary geometry related to the building $\mathsf{A}_2(k)$, and with $\tau = (\mathscr{P}_\tau, \mathscr{L}_\tau)$ the projective plane $\mathbb{P}(k^3)$. Note that τ is the Lie incidence geometry of type $\mathsf{A}_{2,1}$ of $\mathsf{A}_2(k)$.*

Definition 5.22.

(1) A conical subset of A is a subset of A which intersects every dual affine plane of A in a (possibly empty) conic, as defined in Definition 2.6. It is called fully degenerate when all these conics are degenerate.

(2) A conical subspace of A is a conical subset $\mathscr{C}$ for which every transversal of A that contains two points of $\mathscr{C}$, is automatically contained in $\mathscr{C}$. If moreover every transversal of A contains a point of $\mathscr{C}$, the subset $\mathscr{C}$ is called a conical hyperplane of A.

In general, a conical subspace is not a subspace of A, we use this terminology because it is a subspace of the long root geometry of A.

Remark 5.14. *It is clear that every conical hyperplane of an imaginary geometry of type A_2 is automatically a conical subspace, and it is easy to check that it indeed contains three mutually collinear points.*

Lemma 5.13. *A conical subset of A intersects any line or transversal of A in all or at most two of its points.*

Proof. Every line of A is contained in a dual affine plane of A. Moreover, for any transversal T, and any point $p \in T$, there is a dual affine plane of A that contains $T \setminus \{p\}$. $\square$

Notation 9. *Let $\mathscr{Q}$ be a conical subset of A, let π be a dual affine plane of A and let T be a transversal of π. If $\mathscr{Q} \cap \pi = T \setminus \{T \cap T_\pi\}$, we simplify notation by writing $\mathscr{Q} \cap \pi = T$.*

5.2. **Fully degenerate conical subsets.** In this subsection, we discuss fully degenerate conical subsets of A. We first describe a class of examples.

Lemma 5.14. *Let π_1 and π_2 be two (possibly coinciding) dual affine planes of A. Then $\bar{\pi}_1 \cup \bar{\pi}_2$ is a fully degenerate conical subset of A.*

Proof. Set $\mathscr{Q} := \bar{\pi}_1 \cup \bar{\pi}_2$. Let π be any dual affine plane of A. If $\pi = \pi_1$ or $\pi = \pi_2$, then $\pi \cap \mathscr{Q} = \pi$, which is indeed a degenerate conic of π. We can hence assume that $\pi \neq \pi_1, \pi_2$. Then

$$\pi \cap \mathscr{Q} = (\pi \cap \bar{\pi}_1) \cup (\pi \cap \bar{\pi}_2).$$

The intersection $\pi \cap \pi_i$ is either a line or a transversal ($i = 1, 2$). The intersection $\pi \cap \mathscr{Q}$ is hence either a line, a transversal, the union of two lines, the union of two transversals or the union of a line and a transversal. All these structures are degenerate conics of π. □

Some choices of π_1 and π_2 yield conical hyperplanes, others do not even yield a conical subspace. Below, we list all different possibilities.

Example 5.14.

(1) *Let p be a point of τ and l a line of τ. Then $\mathscr{Q} := \bar{\pi}_p \cup \bar{\pi}_l$ forms a conical hyperplane. If $p \in l$, then this set $\mathscr{Q}$ is exactly the set of points in A which are noncollinear with the point (p, l) of A.*

(2) *Let p be a point of τ, then $\mathscr{Q} := \bar{\pi}_p$ forms a conical subspace of A. For any line l of τ that does not contain p, the transversal T_l intersects $\mathscr{Q}$ trivially, the set $\mathscr{Q}$ is hence not a conical hyperplane of A. Dually, for a line l of τ, the set $\bar{\pi}_l$ also forms a conical subspace of A which is not a conical hyperplane of A.*

(3) *Let p and q be two points of τ, then $\mathscr{Q} := \bar{\pi}_p \cup \bar{\pi}_q$ forms a conical subset. Let r be a point of τ not on pq, then the transversal T_r intersects $\mathscr{Q}$ in exactly two points, namely (r, rp) and (r, rq). The set $\mathscr{Q}$ hence does not form a conical subspace.*

It turns out that, as soon as a fully degenerate conical subset contains *enough* points, it is either the whole of A or it is as in Lemma 5.14. In order to prove this, we first gather some easy lemmas, which we mention here without proof.

Lemma 5.15. *Let $\mathscr{Q}$ be a fully degenerate conical subset which is not a conical subspace. Then either $\mathscr{Q}$ is a conical subspace of A, or there is a dual affine plane π of A which intersects $\mathscr{Q}$ in the union of two lines.*

Lemma 5.16. *Let π_1 and π_2 be two distinct dual affine planes in A. Any conical subset that contains π_1 and π_2 either equals $\bar{\pi}_1 \cup \bar{\pi}_2$ or A itself.*

We are now ready to prove the previously mentioned result.

Proposition 5.7. *A fully degenerate conical subset of A that contains three mutually collinear points, not on common line, is either the whole of A, or it is of the form $\bar{\pi}_1 \cup \bar{\pi}_2$ with π_1 and π_2 two (possibly coinciding) dual affine planes of A.*

Proof. Let $\mathscr{Q}$ be a conical subset of A which contains at least three mutually collinear points, not on a common line. For any dual affine plane π of A, the intersection $\pi \cap \mathscr{Q}$ is empty, a point, a line, a transversal, the union of two lines, the union of two transversals, the union of a line and a transversal or the whole of π. We will make a case distinction.

Case 1: *The set $\mathscr{Q}$ does not form a conical subspace.*

By Lemma 5.15, there exists a dual affine plane π of A which intersects $\mathscr{Q}$ in the union of two lines. Without loss of generality, we may assume that π is of the form π_p, with p some point of τ. The intersection $\mathscr{Q} \cap \pi_p$ is then the union $[p, l] \cup [p, m]$ with l and m some lines of τ not through p. Let q be the intersection in τ of l and m.

We first prove that we find a point r on $pq \setminus \{p, q\}$ for which $T_r \cap \mathscr{Q} = \emptyset$. To that end, let n be any line through p in τ. Since $\pi_p \cap \mathscr{Q}$ is the union of two lines, we have that $T_n \nsubseteq \mathscr{Q}$. By Lemma 5.13, $T_n \cap \mathscr{Q}$ contains at most two points. If n does not contain q, this line n intersects l and m in distinct points q_l and q_m, implying that $T_n \cap \mathscr{Q} = \{(q_l, n), (q_m, n)\}$ and hence that $(p, n) \notin \mathscr{Q}$. As a result, the only point of T_p that can be contained in $\mathscr{Q}$, is (p, pq), and hence $\pi_{pq} \cap \mathscr{Q}$ does not contain any line. We can use this to determine $\pi_{pq} \cap \mathscr{Q}$: it is either at most one point (s, k) with $s \in pq$ and $s \in k \neq pq$, or is the transversal T_q. Taking r on $pq \setminus \{p, q, s\}$, we find that $T_r \cap \mathscr{Q} = \emptyset$.

Next, we prove that $\pi_l, \pi_m \subseteq \mathscr{Q}$. To that end, take x_l on $l \setminus \{q\}$ and set $x_m := rx_l \cap m$. The set $\mathscr{Q}$ contains (x_l, px_l) and (x_m, px_m) and moreover has empty intersection with T_r. Considering $\pi_{x_l x_m} \cap \mathscr{Q}$ we hence find that T_{x_l} and T_{x_m} are contained in $\mathscr{Q}$. Since x_l is any point on l different from q, we find that $\pi_l \setminus T_q \subseteq \mathscr{Q}$, which indeed implies that $\pi_l \subseteq \mathscr{Q}$. Similarly, we find that $\pi_m \subseteq \mathscr{Q}$. It now follows from Lemma 5.16 that $\mathscr{Q} = \bar{\pi}_l \cup \bar{\pi}_m$. This concludes Case 1. From now on, we assume that $\mathscr{Q}$ forms a conical subspace, that is, each transversal of A that is not contained in $\mathscr{Q}$ intersects $\mathscr{Q}$ in at most one point.

Case 2: *There is a dual affine plane that intersects $\mathscr{Q}$ in the union of a transversal and a line.*
Without loss of generality, we may assume that this plane is of the form π_p with p some point of τ. Then $\pi_p \cap \mathscr{Q} = T_l \cup [p, m]$ for lines l and m in τ with $p \in l$ and $p \notin m$. Set $q := l \cap m$.

First suppose that there exists some point r_l of $l \setminus \{q, p\}$ for which $T_{r_l} \nsubseteq \mathscr{Q}$. Let r_m be any point of m different from q. The line $[p, r_l r_m]$ intersects $\mathscr{Q}$ in exactly two points, namely (r_l, pr_l) and (r_m, pr_m). As a result, the plane $\pi_{r_l r_m}$ intersects $\mathscr{Q}$ in either the union of two lines or the union of a line and a transversal. Since we assume that $\mathscr{Q}$ is a conical subspace, the former cannot happen. Moreover, we assumed that $T_{r_l} \nsubseteq \mathscr{Q}$. We hence find that $\pi_{r_l r_m} \cap \mathscr{Q}$ is the union of a line through (r_l, pr_l), (which equals $[s, r_l r_m]$, for some point s of τ) and a transversal through (r_m, pr_m) (which equals T_{r_m}). The point r_m was an arbitrary point of $m \setminus q$, so we find that $\pi_m \setminus T_q \subseteq \mathscr{Q}$, which implies that $\pi_m \cap \mathscr{Q} = \pi_m$. Moreover, the plane $\pi_{r_l r_m}$ plays the same role as π_p, where T_{pr_m} plays the role of T_{r_l}, and $[s, r_l r_m]$ that of $[p, m]$. So with the same reasoning as above, we find that $\pi_s \subseteq \mathscr{Q}$. By Lemma 5.16, we can conclude that $\mathscr{Q} = \bar{\pi}_m \cup \bar{\pi}_s$.

Next, suppose that $T_{r_l} \subseteq \mathscr{Q}$ for every point r_l of $l \setminus \{q, p\}$. Then $\pi_l \setminus \{T_p, T_q\} \subseteq \mathscr{Q}$, which, given that $|k| \geq 4$, implies that $\pi_l \cap \mathscr{Q} = \pi_l$. Moreover, $T_q \cup [p, m] \subseteq \pi_m \cap \mathscr{Q}$, so either $\pi_m \subseteq \mathscr{Q}$, or $\pi_m \cap \mathscr{Q} = T_q \cup [p, m]$. In the latter case, π_m plays the same role as π_p, so we can apply the same arguments on π_m instead of π_p. We find that either $\pi_p \subseteq \mathscr{Q}$ (which cannot happen by the assumption on π_p) or $\pi_q \subseteq \mathscr{Q}$. By Lemma 5.16, we hence find that either $\mathscr{Q} = \bar{\pi}_l \cup \bar{\pi}_m$ (which actually cannot happen since we assumed $\mathscr{Q}$ to be a conical subspace) or $\mathscr{Q} = \bar{\pi}_l \cup \bar{\pi}_p$. This concludes Case 2.

Case 3: *There is a dual affine plane that intersects $\mathscr{Q}$ in the union of two transversals.*
Without loss of generality, we may assume that this plane is of the form π_p with p some point of τ. We have that $\mathscr{Q} \cap \pi_p = T_l \cup T_m$ for l and m some lines of τ through p. We may assume that Case 2 does not occur and show that this leads to a contradiction. Let n be any line of τ not through p. Then $[p, n]$ is contained in π_p and hence contains exactly two points of $\mathscr{Q}$. Keeping in mind that $\pi_n \cap \mathscr{Q}$ cannot be the union of two lines or the union of a line and a transversal, the intersection $\pi_n \cap \mathscr{Q}$ is the union of two transversals (namely $T_{n \cap m}$ and $T_{n \cap l}$). Varying n, we see that $\pi_l \cup \pi_m \subseteq \mathscr{Q}$, a contradiction as before.

Case 4: *Every line of A that contains two points of $\mathscr{Q}$, is contained in $\mathscr{Q}$.*
By assumption, we find three pairwise collinear points x_1, x_2, x_3 in $\mathscr{Q}$ not on a line. As pointed out in Lemma 3.5, every plane of A is either the whole of A or is a dual affine plane. Suppose that $\mathscr{Q}$ is not A, then x_1, x_2 and x_3 must lie in some dual affine plane π of A. Since every line of A that contains two points of $\mathscr{Q}$ is contained in $\mathscr{Q}$, the plane π, which is generated by the

points x_1, x_2 and x_3, is contained in $\mathscr{Q}$. By Lemma 5.13, the transversal T_π is also contained in $\mathscr{Q}$. If $\mathscr{Q}$ contains another point q of A, it will follow from Lemma 3.7 that $\mathscr{Q} = A$. □

We use Proposition 5.7 to obtain another classification, which will be very useful in the proof of the Main Theorem.

Definition 5.23. *We say that $\mathscr{Q}$ is a conical subset with vertex q when $\mathscr{Q}$ is a conical subset containing the point q for which every line through q in A is either contained in $\mathscr{Q}$ or intersects $\mathscr{Q}$ in a unique point, namely q.*

Lemma 5.17. *Let $\mathscr{Q}$ be a fully degenerate conical subset with vertex q that contains three mutually collinear points, not on a line. Then the set $\mathscr{Q}$ is one of the following:*

(1) the whole set A.
(2) a set of the form $\bar{\pi}_1 \cup \bar{\pi}_2$ with π_1 and π_2 dual affine planes in A such that $q \in \bar{\pi}_1 \cap \bar{\pi}_2$.

Proof. By Proposition 5.7, the set $\mathscr{Q}$ either equals A, or is of the form $\bar{\pi}_1 \cup \bar{\pi}_2$ with π_1 and π_2 two dual affine planes of A. In the former case, there is nothing more to prove, we may hence assume that we are in the latter case. There is some line l through q for which $l \nsubseteq \mathscr{Q}$. Suppose that $q \notin \bar{\pi}_1$, then the line l intersects $\bar{\pi}_1$ in some point different from q, which is contained in $\mathscr{Q}$. This implies that $l \subseteq \mathscr{Q}$, a contradiction. □

We finish this section by giving a condition that ensures that a conical subset is fully degenerate, and by gathering an easy observation on these fully degenerate conical subsets.

Lemma 5.18. *A conical subset of A that contains a dual affine plane of A is a fully degenerate conical subset of A.*

Proof. Let $\mathscr{Q}$ be a conical subset of A that contains some dual affine plane π. Any transversal of π intersects π in all but one of its points, and is, by Lemma 5.13, contained in $\mathscr{Q}$. This implies that $\bar{\pi} \subseteq \mathscr{Q}$. Let π' be any other dual affine plane of A, then the intersection $\pi' \cap \mathscr{Q}$ contains $\pi' \cap \bar{\pi}$, which is either a line or a transversal of π'. This implies that $\mathscr{Q} \cap \pi'$ is indeed a degenerate conic of π', and concludes the proof. □

Lemma 5.19. *Let π_1 and π_2 be two (possibly coinciding) dual affine planes of A. Then for any point $q \in \mathscr{Q} := \bar{\pi}_1 \cup \bar{\pi}_2$, there is at least one transversal of A through q that is contained in $\mathscr{Q}$.*

5.3. **Conical subspaces are often conical hyperplanes.** This subsection is devoted to proving the following Proposition.

Proposition 5.8. *Let k be a field with $\mathrm{char}(k) \neq 2$ and $|k| \geq 5$. A conical subspace of A which contains three mutually collinear points, not on a common line, is either the transversal closure of a dual affine plane or is a conical hyperplane of A.*

Notation 10. *From now on, let k be a field with $\mathrm{char}(k) \neq 2$ and $|k| \geq 5$. Let $\mathscr{Q}$ be a conical subspace of A which contains at least three mutually collinear points, not on a common line.*

We first gather information regarding the possible intersections of $\mathscr{Q}$ with dual affine planes of A.

Lemma 5.20. *Let π be a dual affine plane of A. Then one of the following occurs:*

(1) The set $\pi \cap \mathscr{Q}$ is empty, in this case, $T_\pi \cap \mathscr{Q}$ is either empty, one point, or T_π.
(2) The set $\pi \cap \mathscr{Q}$ is a point p. In this case, $T_\pi \cap \mathscr{Q}$ is either empty or a point q. In the latter case, p and q are not linelike.
(3) The set $\pi \cap \mathscr{Q}$ is a line. In this case, $T_\pi \cap \mathscr{Q}$ is empty.
(4) The set $\pi \cap \mathscr{Q}$ is a transversal T of π. In this case, $T_\pi \cap \mathscr{Q}$ is either $T_\pi \cap T$ or T_π.

(5) The set $\pi \cap \mathscr{Q}$ is the union of a line and a transversal T. In this case, $T_\pi \cap \mathscr{Q} = T \cap \mathscr{Q}$.
(6) The set $\pi \cap \mathscr{Q}$ is the union of two transversals of π. In this case, $T_\pi \cap \mathscr{Q} = T_\pi$.
(7) The set $\pi \cap \mathscr{Q} = \pi$. In this case, $T_\pi \cap \mathscr{Q} = T_\pi$.
(8) The set $\pi \cap \mathscr{Q}$ is a nondegenerate conic $\mathscr{C}$ of π through the missing point of π. In this case, $T_\pi \cap \mathscr{Q}$ is either empty or equals $T_\pi \cap T$, with T the transversal of π that corresponds to the tangent line of $\mathscr{C}$ through the missing point.

Proof. This follows immediately when combining the condition that $\pi \cap \mathscr{Q}$ is a conic and that $\mathscr{Q}$ contains 0, 1 or all points of any transversal of $\bar{\pi}$. □

Remark 5.15. *If* $\mathsf{char}(k)$ *were equal to* 2 *(which we do not allow here), there would be one extra possibility in Lemma 5.20, namely where $\pi \cap \mathscr{Q}$ is a nondegenerate conic, while the nucleus of this conic is the missing point of π.*

Lemma 5.21. *Let π be a dual affine plane of A such that T_π contains a point of $\mathscr{Q}$, but some transversal T of π does not contains any point of $\mathscr{Q}$. Then $\pi \cap \mathscr{Q}$ is empty, a point or a transversal. In particular, every line in π intersects $\mathscr{Q}$ in at most one point.*

Proof. By assumption, the set $T_\pi \cap \mathscr{Q}$ is a unique point. Using Lemma 5.20, we can hence see that it suffices to prove that $\pi \cap \mathscr{Q}$ is neither the union of a line and a transversal, nor a nondegenerate conic through the missing point of π. First suppose that $\pi \cap \mathscr{Q}$ contains a line l. Every transversal of π intersects l, contradicting the fact that there is a transversal of π that contains no point of $\mathscr{Q}$. Next, suppose that $\pi \cap \mathscr{Q}$ is a nondegenerate conic $\mathscr{C}$. Let T be any transversal of π. Either T contains a point of $\mathscr{C}$, in which case $T \cap \mathscr{Q} \neq \emptyset$, or T corresponds to the tangent line of $\mathscr{C}$ through the missing point, in which case $T_\pi \cap T \in \mathscr{Q}$. Again a contradiction. □

Lemma 5.22. *If no line of A intersects $\mathscr{Q}$ in exactly two points, the set $\mathscr{Q}$ is the transversal closure of one dual affine plane or equals A.*

Proof. One easily checks that $\mathscr{Q}$ is a fully degenerate conical subspace of A. The claim then follows from Proposition 5.7 and Example 5.14. □

We will need the following rather technical lemma.

Lemma 5.23. *Suppose that there exists some line l of τ such that for each, but at most one, point q of τ on l the following assertion holds:*

"For all lines m of τ not through q, the line $[q, m]$ of A intersects $\mathscr{Q}$ in at most one point."

Then $\mathscr{Q}$ is the transversal closure of one dual affine plane.

Proof. Let l be as in the lemma, let s be a point of l and suppose that the assertion holds for all points of l different from s. We claim that no line $[p, m]$ of A intersects $\mathscr{Q}$ in exactly two points. To that end, first let m be any line of τ different from l and set $r := m \cap l$. Then for any point p on $l \setminus \{s, r\}$, the line $[p, m]$ of π_m contains, by assumption, at most one point of $\mathscr{Q}$. If $\pi_m \cap \mathscr{Q}$ was a nondegenerate conic, then, since we assume $|k| \geq 5$ and $\mathrm{char}(k) \neq 2$, there would be at least two lines in π_m through (r, l) which would intersect $\mathscr{Q}$ in exactly two points, a contradiction. The set $\pi_m \cap \mathscr{Q}$ is a degenerate conic, and there is at most one line through (r, l) in π_m that intersects this conic in more than one point. We find that $\pi_m \cap \mathscr{Q}$ is either empty, a point, a line, or a transversal. In particular, there is no line $[p, m]$ that intersects $\mathscr{Q}$ in exactly two points. Next, let p be a point of τ not on l. Using the fact that no line $[p, m]$ with $m \neq l$ and $p \notin m$ intersects $\mathscr{Q}$ in at most one point, we find that $\pi_p \cap \mathscr{Q}$ is empty, a point, a line, or a transversal. In particular, we find that $[p, l]$ does not intersect $\mathscr{Q}$ in exactly two points. This proves the claim. The lemma now follows immediately from Lemma 5.22. □

Lemma 5.24. *Let π be a dual affine plane of A for which $T_\pi \cap \mathscr{Q} = \emptyset$ and $|\pi \cap \mathscr{Q}| \geq 2$. Then the set $\mathscr{Q}$ is the transversal closure of one dual affine plane.*

Proof. Without loss of generality, we may assume that π is of the form π_l with l some line of τ. Using Lemma 5.20, one sees that $\pi \cap \mathscr{Q}$ is either a line or a nondegenerate conic of π_l. In any case, there exists at most one transversal T of τ for which $T \cap \mathscr{Q} = \emptyset$. Let $p \in l$ be the point of τ for which $T = T_p$, and let q be any other point of l. Then $|T_q \cap \mathscr{Q}| = 1$. Moreover, T_l is a transversal of π_q disjoint from $\mathscr{Q}$. Applying Lemma 5.21 to π_q, we find that every line of A of the from $[q, n]$, with n a line of τ not through q, intersects $\mathscr{Q}$ in at most one point. The assertion now follows from Lemma 5.23. □

We have now gathered all ingredients needed to finish the proof of Proposition 5.8.

Proof of Proposition 5.8. Assume for a contradiction that $\mathscr{Q}$ is neither a conical hyperplane, nor the transversal closure of one dual affine plane. By assumption, there exists some transversal T of A that intersects $\mathscr{Q}$ trivially. Without loss of generality, we may assume that T is of the form T_l for some line l of τ. It follows from Lemma 5.24 that $\pi_l \cap \mathscr{Q}$ contains at most one point, which in particular implies that $T_q \cap \mathscr{Q} = \emptyset$ for each, but at most one, point q of l. Let q be such a point, then we can apply Lemma 5.24 to π_q, and obtain that $\pi_q \cap \mathscr{Q}$ is at most one point, and hence that every line in A of the form $[q, m]$ (with m a line of τ not through q) intersects $\mathscr{Q}$ in at most one point. We can now apply Lemma 5.23 to the line l of τ, and obtain a contradiction. □

All examples of conical subsets that we have seen so far are fully degenerate. It is however good to keep in mind that there are other examples.

Example 5.15. *Let p be a point and l a line of τ, and consider a projectivity*

$$\phi : \{\textit{Points on } l\} \to \{\textit{Lines through } p\}.$$

Then the following set forms a conical hyperplane of A:

$$\mathscr{Q}(p, l, \phi) := \{[q, \phi(q)] \mid q \in l \textit{ and } q \notin \phi(q)\} \cup \{T_q \cup T_{\phi(q)} \mid q \in l \textit{ and } q \in \phi(q)\}.$$

6. Defining five distinct point relations

Notation 11. *In this section, Y denotes a connected partial linear space that satisfies Axioms* (Im_1) *and* (Im_2). *We assume that no A_2-plane of Y is defined over $\mathbb{F}_3$ or over a field of characteristic 2. We make use of Notation 3.*

In this section, we will define five point relations on Y, and prove that these relations are disjoint. Along the way, we prove that every point p of Y is noncollinear to a conical hyperplane of any A_2-plane of Y, which is a stronger version of Axiom (Im_2).

6.1. Some initial observations.

We start by gathering some initial observations on Y.

Lemma 6.25. *Let l and m be two intersecting lines such that some point of l is noncollinear to exactly one point of m. Then noncollinearity induces a bijection between $l \setminus \{l \cap m\}$ and $m \setminus \{l \cap m\}$.*

Proof. Let p be the intersection point of l and m. By Axiom $(\mathrm{Im}_1)(i)$, any point of $m \setminus \{p\}$ is noncollinear to a unique point of $l \setminus \{p\}$. We can however apply this same axiom again while interchanging the roles of l and m. We then indeed obtain that noncollinearity induces a bijection between $l \setminus \{p\}$ and $m \setminus \{p\}$. □

Lemma 6.26. *Let l and m be two intersecting lines such that some point of l is noncollinear to exactly two points of m. Then each point of $l \setminus \{l \cap m\}$ is noncollinear to exactly two points of $m \setminus \{l \cap m\}$ and vice versa.*

Proof. If this is the case, then, by $(\mathrm{Im}_1)(ii)$, the lines l and m generate an A_2-plane defined over a field. By Lemma 3.5, the claim is true in every A_2-plane. □

Lemma 6.27. *Let p be a point and l a line. If l is contained in some A_2-plane, then p is collinear to no or all but at most 2 points of l.*

Proof. Let A be an A_2-plane that contains l. By Axiom (Im_2), the point p is noncollinear to a conical subspace of A. The claim now follows from Lemma 5.13. □

Remark 6.16. *Axiom (Im_1) stipulates that Y contains an A_2-plane. A priori however, we do not know whether every line is contained in an A_2-plane.*

Lemma 6.28. *The space Y contains dual affine planes.*

Proof. By Axiom (Im_1), the space Y contains an A_2-plane A. As explained in Lemma 3.4, this plane A contains several dual affine planes. □

Next, we introduce some definitions and notations, which are of course inspired on the observations we made in Section 4.

Definition 6.24.

(1) *A dual affine plane π of Y that is contained in some A_2-plane A of Y, is called a linelike plane. In general, not every dual affine plane of Y is a linelike plane.*
(2) *Let A be an A_2-plane of Y. Using Remark 3.8, we define the following.*
   (a) *For a transversal T of A, we denote with π_T^A the dual affine plane π of A that corresponds to T.*
   (b) *For a dual affine plane π of A, we denote with T_π^A the transversal in A corresponding to π. We define $\bar{\pi}^A := \pi \cup T_\pi^A$, and call this the* transversal closure of π in A.
(3) *A transversal of Y is defined to be any subset $T \subset X$ for which there exists an A_2-plane $A \supset T$ such that T is a transversal of A.*
(4) *Let π be a linelike plane and T be a transversal of Y. We say that that T* is a transversal of π *when there exists an A_2-plane A that contains both π and T in which T is a transversal of π.*

Remark 6.17. *A priori, two transversals of Y can intersect in an arbitrary number of points. If the linelike plane π is contained in two distinct A_2-planes A_0 and A_1 of Y, and q is a point of π, it could, in principle, even happen that the transversal T_1 of π in A_1 does not fully coincide with the transversal T_2 of π in A_2. Of course, we do have that $T_1 \cap \pi = T_2 \cap \pi = q^{\not\equiv} \cap \pi$. This implies that it could for example happen that $T_\pi^{A_0} \neq T_\pi^{A_1}$.*

Remark 6.18. *We repeat Notation 9. Let p be a point and π be a linelike plane. If $p^{\not\equiv} \cap \pi = T \cap \pi$ for some transversal T of π, we simply write $p^{\not\equiv} \cap \pi = T$, and say that $p^{\not\equiv} \cap \pi$ is a transversal T of π.*

Lemma 6.29. *Let p be a point and let A be an A_2-plane. The set $p^{\not\equiv} \cap A$ is either a conical hyperplane of A, or is of the form $\bar{\pi}^A$ for some dual affine plane π of A.*

Proof. By Axiom (Im_2) the set $p^{\not\equiv} \cap A$ is a conical subspace of A which contains at least three mutually collinear points, not on a common line. By Proposition 5.8, such a subset is either a conical hyperplane or the transversal closure of a dual affine plane of A. □

Corollary 6.2. *Let p be a point and T a transversal. If $|p^{\not\equiv} \cap T| \geq 2$, then $p \not\equiv T$.*

Proof. Let A be an A_2-plane that contains T. By Lemma 6.29, the point p is noncollinear to a conical subspace of A. By definition, a conical subspace of A intersects a transversal of A in zero, one or all of its points. □

6.2. **Relations between a point and a linelike plane.** In this subsection, we will investigate sets $p^{\not\equiv} \cap \pi$ with p a point and π a linelike plane. We start with a very elementary lemma, which is based on Lemma 5.20.

Lemma 6.30. *Let p be a point and π be a linelike plane. For any A_2-plane A containing π, exactly one of the following holds:*

(1) *The set $p^{\not\equiv} \cap \pi$ is empty. In this case, $T_\pi^A \subseteq p^{\not\equiv}$.*
(2) *The set $p^{\not\equiv} \cap \pi$ is a line. In this case, $T_\pi^A \cap p^{\not\equiv}$ is empty.*
(3) *The set $p^{\not\equiv} \cap \pi$ is a transversal T of π in A and $T_\pi^A \cap p^{\not\equiv} = T \cap T_\pi^A$.*
(4) *The set $p^{\not\equiv} \cap \pi$ is a transversal T of π in A and $T_\pi^A \subseteq p^{\not\equiv}$.*
(5) *The set $p^{\not\equiv} \cap \pi$ is the union of two disjoint transversals of π in A. In this case, $T_\pi^A \subseteq p^{\not\equiv}$.*
(6) *The set $p^{\not\equiv} \cap \pi$ is the union of a line and a transversal T of π in A. In this case, $p^{\not\equiv} \cap T_\pi^A = T \cap T_\pi^A$.*
(7) *The set $p^{\not\equiv} \cap \pi$ is a nondegenerate conic of π through the missing point of π.*
(8) *The plane π is contained in $p^{\not\equiv}$. In this case, also T_π^A is contained in $p^{\not\equiv}$.*

If we are in case (1), (5) (6) or (7), the set $p^{\not\equiv} \cap A$ is automatically a conical hyperplane of A. If we are in case (2) or (3), the set $p^{\not\equiv} \cap A$ is of the form $\bar{\pi}_1^A$ for some dual affine plane π_1 of A.

Proof. By Lemma 6.29, we find that $\mathscr{Q} := p^{\not\equiv} \cap A$ is either of the form $\bar{\pi}_1^A$ for some dual affine plane π_1 of A, or is a conical hyperplane of A. In the former case, we can easily deduce that $\mathscr{Q}$ intersects $\bar{\pi}$ as described in $(2), (3), (4)$ or (8). In the latter case, the set $\mathscr{Q} \cap \pi$ is a conic of π, which intersects every transversal of π (and the transversal T_π^A) in one or all of its points, which implies that $\mathscr{Q}$ intersects $\bar{\pi}$ as described in $(1), (4), (5), (6), (7)$ or (8). □

Lemma 6.31. *Let l be a line containing distinct points p, q, r, and let π be a linelike plane through q but not through l. Suppose that p is collinear to all points of π. Then r is as well. For any A_2-plane A that contains π, we have that $l \not\equiv T_\pi^A$.*

Proof. Assume for a contradiction that r is noncollinear to at least one point of π. Suppose first that there is a point s in π collinear to q but noncollinear to r. Then we can consider the line $m := sq$. Since m is contained in the plane π which is contained in some A_2-plane, Lemma 6.27 implies that the point r is noncollinear to one or two points of m (one of which is s). Then Lemma 6.25 or Lemma 6.26, respectively, implies that p is also noncollinear with one or two points, respectively, of $m \subset \tau$, a contradiction. We hence conclude that r is collinear to all points of π which are collinear to q, i.e. $r^{\not\equiv} \cap \pi \subseteq T$, with T the transversal of π containing q. Lemma 6.30 then implies that $r^{\not\equiv} \cap \pi$ either equals T or is empty. The point r is collinear with $q \in T$, so r is indeed collinear to all points of π.

The point q is contained in π, so $q \not\equiv T_\pi^A$. Moreover, it follows from (1) of Lemma 6.30 that r and p are noncollinear to T_π^A. The point r however can be chosen arbitrarily on $l \setminus \{p, q\}$, so we indeed obtain that $l \not\equiv T_\pi^A$.

□

Lemma 6.32. *Let p and q be collinear points, and let π be a linelike plane through q. The following statements are equivalent:*

(1) *The set $p^{\not\equiv} \cap \pi$ is either a line or a transversal of π.*
(2) *The point p is noncollinear to a unique point of every line in π through q.*

Proof. Let p, q and π be as stated. Every line of π intersects every other line of π and every transversal of π in a unique point. So if the first claim holds, the second one holds, too. On the other hand, we know that the set $p^{\not\equiv} \cap \pi$ is one of the possibilities in Lemma 6.30. The only possibilities where p is noncollinear to a unique point of every line in π through a certain point

q collinear with p, are those where $p^{\not\equiv} \cap \pi$ is either a line, or a transversal of π. We conclude that the two claims are equivalent. $\square$

Lemma 6.33. *Let l be a line containing distinct points p, q, r, and let π be a linelike plane through q but not through l. Suppose that $p^{\not\equiv} \cap \pi$ is a line. Then $r^{\not\equiv} \cap \pi$ is a line as well, which moreover contains the point $p^{\not\equiv} \cap q^{\not\equiv} \cap \pi$.*

Proof. Let π, p, q and r be as stated. By assumption, the set $k := p^{\not\equiv} \cap \pi$ is a line. Let m be any line in π through q. Then Lemma 6.32 implies that p is noncollinear to a exactly one point of m. By Lemma 6.25, the point r is also noncollinear to exactly one point of m. The line m through q in π was arbitrary, so $r^{\not\equiv} \cap \pi$ contains exactly one point of every line in π through q. Lemma 6.32 then implies that $r^{\not\equiv} \cap \pi$ is either a line of π or a transversal of π. In either case, the set $r^{\not\equiv} \cap \pi$ intersects the line k in a point s. Suppose that s is collinear to q. As above, we find that p is noncollinear to a unique point of the line qs, so by Axiom (Im_1), the point s is noncollinear to a unique point of l, a contradiction to the fact that it is noncollinear to both p and r. We hence conclude that s is noncollinear to q. Any transversal of π through s contains q, and r is collinear to q, so we conclude that $r^{\not\equiv} \cap \pi$ is indeed a line through s. $\square$

Lemma 6.34. *Let l be a line containing distinct points p, q, r, and let π be a linelike plane through q but not through l. Suppose that $p^{\not\equiv} \cap \pi$ is a transversal of π. Then $r^{\not\equiv} \cap \pi$ is a transversal of π as well.*

Proof. The proof is very similar to that of Lemma 6.33. We start by using Lemma 6.32 to obtain that p is noncollinear to exactly one point of every line through q. Secondly, we invoke Lemma 6.25 to see that the same holds for the point r. Next, we use Lemma 6.32 again to obtain that $r^{\not\equiv} \cap \pi$ is either a line or a transversal. If it were a line however, then we could apply Lemma 6.33 with the roles of p and r interchanged to obtain that $p^{\not\equiv} \cap \pi$ would be a line, a contradiction. We can hence indeed conclude that $r^{\not\equiv} \cap \pi$ is a transversal of π. $\square$

Lemma 6.35. *Let l be a line containing distinct points p, q, r, and let π be a linelike plane through q but not through l. Suppose that $p^{\not\equiv} \cap \pi$ is the union of two disjoint transversals of π. Then $r^{\not\equiv} \cap \pi$ is the union of two disjoint transversals of π as well.*

Proof. Let π, p, q and r be as stated. We prove that l is contained in some A_2-plane and that r is noncollinear to exactly two points of every line in π through q. Let m be any such line. The point p is noncollinear to two disjoint transversals of π and hence to exactly two points of m. Lemma 6.26 implies that the point r is noncollinear to exactly two points of m. Moreover, using Axiom (Im_1), we find that l is indeed contained in the A_2-plane $\langle l, m\rangle$.

Considering the possibilities in Lemma 6.30, we see that exactly one of the following statements holds for $r^{\not\equiv} \cap \pi$:

(1) a union of two disjoint transversals of π.
(2) a nondegenerate conic $\mathscr{C}$ of π through the missing point of π such that every line of π through q intersects $\mathscr{C}$ in exactly two points.

Suppose for a contradiction that the second statement holds. Let A be an A_2-plane that contains π. By Lemma 6.30, the set $r^{\not\equiv} \cap A$ is a conical hyperplane of A, implying that r is noncollinear to some point s of T_π^A. By Lemma 6.30, both points p and q are noncollinear to T_π^A, so $s \in T_\pi^A$ is noncollinear to the points p, q, r of l. We argued in the first paragraph of this proof that l is contained in some A_2-plane. As a result, we can apply Lemma 6.27 and obtain that $s \not\equiv l$.

Denote with T_s the transversal of π in A that contains s. We claim that $q \notin T_s$. Suppose that this would be the case. Denote with π_∞ the projective plane obtained by adding one point to π, and denote this point with ∞. Define $\mathscr{C} = r^{\not\equiv} \cap \pi$. By assumption, the set $\mathscr{C}_\infty := \mathscr{C} \cup \{\infty\}$ forms a nondegenerate conic of π_∞. It follows from Corollary 6.2 that $T_s \cap r^{\not\equiv} = \{s\}$. Hence the point

q is contained in a tangent line to $\mathscr{C}_\infty$ in π_∞, namely $q\infty$. The projective plane π_∞, however, is, by Axiom (Im_1) (ii), defined over a field of characteristic different from two. This implies that q lies on exactly two tangent lines to $\mathscr{C}_\infty$ in π_∞. Translating this back to π, we find that there is a line through q in π which intersects $\mathscr{C} = r^{\not\equiv} \cap \pi$ in exactly one point, a contradiction. We conclude that $q \notin T_s$.

Take $t \in T_s \cap \pi$. The point q is not contained in T_s, and is hence collinear to t. Consider the line $m := tq$. By Lemma 6.26, applied to l and m, the point t is noncollinear to exactly two points of l, at least one of which is different from p; call this r'. The point r' is noncollinear to both t and s of T_s, implying that $r' \not\equiv T_s$. Since r' plays the same role as r, we see that $r'^{\not\equiv} \cap \pi$ is the union of two distinct transversals, one of which is T_s. Lemma 6.30 moreover implies that $r' \not\equiv T_\pi^A$. But then every point of T_π^A is noncollinear to p, r' and q. So, by Lemma 6.27, we have that $T_\pi^A \not\equiv l$. The point $r \in l$, however, is collinear to all points of $T_\pi^A \setminus \{s\}$, a contradiction. This proves that $r^{\not\equiv} \cap \pi$ is indeed the union of two disjoint transversals. □

Lemma 6.36. *Let l be a line containing distinct points p, q, r, and let π be a linelike plane through q but not through l. Suppose that $p^{\not\equiv} \cap \pi$ is the union of a line and a transversal of π. Then $r^{\not\equiv} \cap \pi$ is the union of a line and a transversal of π too, where the line contains the point $p^{\not\equiv} \cap q^{\not\equiv} \cap \pi$.*

Proof. Let π, p, q and r be as stated. Let A be an A_2-plane that contains π. By assumption, we have that $p^{\not\equiv} \cap \pi = m \cup T$, for some line m and some transversal T of π, define $x := m \cap T$. Let T_q be the transversal of π in A that contains q, and set $y := T_q \cap T_\pi^A$. Note that $y \in T \setminus \{T \cap T_\pi^A\}$. Lemma 6.30 implies that p is collinear to y.

Let s be any point of $l \setminus \{p, q\}$. We determine the possibilities for $s^{\not\equiv} \cap \pi$. For every line n in π through q different from qx, the point p is noncollinear to exactly two points of n. Axiom $(\mathrm{Im}_1)(ii)$ implies that $\langle l, n\rangle$ is an A_2-plane, which we assumed to be defined over a field of at least five elements, implying that l contains at least six points. Moreover, Lemma 6.26 implies that the point s is noncollinear to exactly two points of n. The point p is noncollinear to exactly one point of the line qx. By Lemma 6.25, the point s is noncollinear to a unique point of qx, which we denote with x_s. Taking into account the different possibilities in Lemma 6.30, we see that $s^{\not\equiv} \cap \pi$ is one of the following:

(1) The union of a line m_s and a transversal T_s of π in A, which intersect in the point x_s. In this case, s is collinear to y.
(2) A nondegenerate conic $\mathscr{C}$ through the missing point of π. The line qx intersects $\mathscr{C}$ in exactly one point, namely x_s. Every other line in π through q intersects $\mathscr{C}$ in exactly two points. Since A, and hence π, is defined over a field of characteristic not two, the point s is in this case noncollinear to y.

We have to prove that the first statement holds for the point $r \in l \setminus \{p, q\}$. Assume, for a contradiction, that this is not the case. First suppose that the second statement holds for some s of $l \setminus \{r, p, q\}$. Then the point y is noncollinear to three distinct points of l, namely q, r and s. We already noted before that l is contained in some A_2-plane, so by Lemma 6.27, the point y would be noncollinear to the whole of l, and in particular to p, a contradiction. This implies that the first statement holds for all points s of $l \setminus \{q, r\}$. Denote with T_r the transversal of π in A that contains x_r (which was defined to be the unique point on qx not collinear to r).

We claim that for any two points s_1 and s_2 of $l \setminus \{q, r\}$, the intersection $m_{s_1} \cap m_{s_2}$ is contained in T_r or T_q. Assume this was not the case. Let y_s be the unique point on qx which is noncollinear to $m_{s_1} \cap m_{s_2}$. By assumption, y_s is different from q and x_r. Lemma 6.25 implies that there is a unique point s on l noncollinear to y_s. This point s is different from q and r, which implies that $s^{\not\equiv} \cap \pi$ is the union of a line and a transversal of π, which intersect in $y_s \in qx$. The point s is hence noncollinear to the transversal of A in π that contains y_s, and in particular to $m_{s_1} \cap m_{s_2}$. But then the point $m_{s_1} \cap m_{s_2}$ is noncollinear to three points of l, namely s_1, s_2 and s. By Lemma 6.27, it is noncollinear to all points of l, in particular to q, a contradiction. This proves the claim.

We argued before that l contains at least 6 points, so in particular, we find two distinct points s_1 and s_2 of $l \setminus \{q, r, p\}$. Using the previous paragraph, one sees that the lines m_{s_1}, m_{s_2} and m intersect in one point z of π, which lies either on T_r or on T_q. In either case, the point z is noncollinear to these three points of l. By Lemma 6.27, it is noncollinear to the whole of l, in particular, to q. We conclude that z is contained in T_q. The point r however, is then noncollinear to both y and z of T_q, and by Corollary 6.2, also to $q \in T_q$, a contradiction. This concludes the proof. □

6.3. **Relations between points.** In Lemma 3.3, we distinguished four different relations between two points in an A_2-plane. We will use this to define five relations between two points of Y. These definitions are of course inspired on the observations made in Corollary 4.1.

Definition 6.25. *Let p and q be two points of Y. One (or more) of the following occurs:*

(1) *The points p and q are equal.*
(2) *There is an A_2-plane A of Y containing p and q such that p and q are linelike in A. In this case, we say that p and q are linelike.*
(3) *There is no A_2-plane of Y that contains both p and q. In this case, we say that p and q are symplectic.*
(4) *There is an A_2-plane A of Y containing p and q such that p and q are special in A, i.e. there exists a unique point in A, which we denote with $[p,q]_A$, which is linelike in A to both p and q. In this case, we say that p and q are special.*
(5) *The points p and q are collinear.*

Our first goal is to prove that the five relations defined above are disjoint. A priori, this might not be the case. We have for example not yet proven that every line is contained in some A_2-plane, so two points could at the same time be symplectic and collinear. Some of the relations are of course automatically disjoint, so we start with these.

Lemma 6.37. *Let p and q be two points which are either collinear or symplectic. Then p and q are not linelike nor special.*

Proof. If p and q are collinear, they are collinear in every plane (and hence every A_2-plane) that contains them both. This implies that they are neither linelike nor special. If p and q are symplectic, then, by definition, they are not contained in any common A_2-plane, and are hence neither linelike nor special. □

If two points p and q are noncollinear but contained in different A_2-planes, it could be that they are linelike in one of these planes, but special in the other one. In Proposition 6.9, we will prove that this cannot occur, that is, two points that are linelike cannot be special. In preparation of the proof of this proposition, we first gather a few Lemmas.

Lemma 6.38. *Let l be a line containing distinct points p, q, r and let A be an A_2-plane through q but not through l. Suppose that $p^{\not\equiv} \cap A = \bar{\pi}^A$ for some dual affine plane π of A. Denote with T the transversal of π in A noncollinear to q. Then $r^{\not\equiv} \cap A = \bar{\pi}'^A$, where π' is some dual affine plane of A different from π in which T is a transversal.*

Proof. Denote with τ the projective plane related to A. Without loss of generality, the plane π is of the form π_x for some point x of τ. The point $q \in A$ is collinear with p, and is hence not contained in $\bar{\pi}_x$. As a result, it is of the form (y, m) with y some point of τ and m a line of τ not through x. Note that the transversal T is the transversal T_{xy}.

Let m' be a line of τ through y, different from m and xy, then the plane $\pi_{m'}$ contains $q = (y, m)$, while the intersection $\pi_{m'} \cap \bar{\pi}_x$ is a line, namely $[x, m']$. The point p is hence noncollinear to exactly a line of the linelike plane $\pi_{m'}$, which contains q. By Lemma 6.33, the set $r^{\not\equiv} \cap \pi_{m'}$ is also a line of $\pi_{m'}$, which contains $p^{\not\equiv} \cap q^{\not\equiv} \cap \pi_m = (y, yx)$, and is hence of the from $[z, m']$ with z some point of τ on $yx \setminus \{y, x\}$. With Lemma 6.30, we obtain that the set $r^{\not\equiv} \cap A$ has the form $\bar{\pi}'$ with π' some dual affine plane of A. This plane π' must of course contain the line $[z, m']$ and hence equals either π_z or $\pi_{m'}$. It cannot be the latter, since we showed before that r is noncollinear to exactly a line of that plane. The transversal $T = T_{xy}$ is indeed a transversal of the plane π_z. □

Corollary 6.3. *Let l be a line containing distinct points p, q, r, and let A be an A_2-plane through q. If $p^{\not\equiv} \cap A$ is a conical hyperplane of A, then so is $r^{\not\equiv} \cap A$.*

Proof. Each point of A is noncollinear to a conical hyperplane of A. If $p \in A$, then also $r \in A$, implying that $r^{\not\equiv} \cap A$ is a conical hyperplane. If $p \notin A$, then the claim follows from Lemma 6.38. □

Lemma 6.39. *Let l be a line containing distinct points p, q, let π be a linelike plane through q, and let A be any A_2-plane through π. If $p \not\equiv T_\pi^A$, then $l \not\equiv T_\pi^A$.*

Proof. Let p, q, l, π and A be as in the statement of the lemma. Let r be any point of $l \setminus \{p, q\}$, we have to prove that $r \not\equiv T_\pi^A$.

First assume that p is noncollinear to a conical hyperplane of A. By Corollary 6.3, the point r is also noncollinear to a conical hyperplane of A. Using Lemma 6.30, we see that exactly one of the following three cases occurs:

(1) *The point p is collinear to all points of π.* By Lemma 6.31, the point r is also collinear to all points of π, and is hence noncollinear to T_π^A.
(2) *The point p is noncollinear to exactly one transversal of π in A.* By Lemma 6.34, the point r is also noncollinear to exactly one transversal of π in A. As $r^{\not\equiv} \cap A$ is a conical hyperplane, we find that r is noncollinear to T_π^A.
(3) *The point p is noncollinear to exactly two disjoint transversals of π in A.* By Lemma 6.35, the point r is also noncollinear to exactly two transversals of π in A. The set $r^{\not\equiv} \cap A$ is again a conical hyperplane of A, so also in this case r is noncollinear to all points of T_π^A.

Next, assume that $p^{\not\equiv} \cap A$ is not a conical hyperplane of A. By Lemma 6.29, the set $p^{\not\equiv} \cap A = \bar{\pi}'$ for some dual affine plane π' of A. The point p is collinear to $q \in \pi$ and noncollinear to T_π^A. This transversal is also noncollinear to q, so Lemma 6.38 implies that $r^{\not\equiv} \cap A$ indeed also contains T_π^A. This concludes the proof. □

We are now ready to prove the crucial lemma in the run up to Proposition 6.9.

Lemma 6.40. *Let p and q be linelike points. Then for every line l through q, the point p is noncollinear to either exactly one point of l, namely q, or to all points of l.*

Proof. Let l be a line through q, and suppose that p is noncollinear to some point $x \in l \setminus \{q\}$. We have to prove that p is noncollinear to l, or equivalently, that every point of l is noncollinear to p. The points p and q are linelike, so by definition, there is some A_2-plane A containing p and q such that p and q are linelike in A. Denote with T the transversal in A that contains both p and q. Let T' be the unique transversal in A that contains p but not q, and set $\pi := \pi_{T'}^A$.

If x is noncollinear to $T' = T_\pi^A$, we can use Lemma 6.39, with x in the role of p, to obtain that every point of l is noncollinear to T', and in particular to p. Assume that this is not the case, then $x^{\not\equiv} \cap T'$ contains at most one point, and hence equals $\{p\}$. By Lemma 6.30, the set

$x^{\not\equiv} \cap \pi$ has to be a nondegenerate conic $\mathscr{C}$ through the missing point of π. Denote with π_∞ the projective plane obtained by adding one point, denoted ∞, to π. The set $\mathscr{C}_\infty := \mathscr{C} \cup \{\infty\}$ is a conic in π_∞. Since $x^{\not\equiv} \cap T = \{p\}$, we find that the line $q\infty$ is the tangent line to $\mathscr{C}_\infty$ at ∞. The plane π_∞ is defined over a field of characteristic not two, so there is exactly one other tangent line m to $\mathscr{C}_\infty$ through q. In π, this means that there is exactly one line m through q which contains one point of $\mathscr{C}$, while all other lines of π through $\mathscr{C}$ contain zero or two points of $\mathscr{C}$. Let r be any point of $l \setminus \{x, q\}$. Using Lemma 6.25 and Lemma 6.26, we find that the line m contains exactly one point noncollinear to r, while all other lines through q in π contain exactly zero or two points noncollinear to r. Considering the possibilities in Lemma 6.30, we can hence conclude that $r^{\not\equiv} \cap \pi$ is one of the following:

(1) The union of a line and a transversal of π. Lemma 6.36 would then imply that $x^{\not\equiv} \cap \pi$ is also the union of a line and a transversal of π, a contradiction.

(2) A nondegenerate conic $\mathscr{C}_r$ through the missing point of π such that there is exactly one line through q in π which contains exactly one point noncollinear to r. The set $\mathscr{C}_r \cup \{\infty\}$ then forms a conic in π_∞, with $q\infty$ the tangent line at ∞. This implies in particular that $r^{\not\equiv} \cap T' = p$, and hence that r is noncollinear to p.

This concludes the proof of the lemma. □

Proposition 6.9. *Let p and q be two linelike points. Then they are not special.*

Proof. Let p and q be two linelike points, and let A be an A_2-plane that contains p and q such that p and q are special in A. In A, there exists some point x which is collinear to both p and q. By Lemma 3.5, the point p is noncollinear to exactly two points of $m = qx$, a contradiction to Lemma 6.40. □

6.4. **A point is noncollinear to a conical hyperplane of any A_2-plane.** As the title of this subsection suggests, the next goal is to prove that a point p is noncollinear to a conical hyperplane of any A_2-plane. Afterwards, we use this to prove that two collinear points cannot be symplectic. We first gather a natural in-between result.

Lemma 6.41. *Let A and A' be two A_2-planes that contain special points q_1 and q_2. Every point of A' that is collinear to both q_1 and q_2 is noncollinear to a conical hyperplane of A.*

Proof. Let p' be a point of A' that is collinear to both q_1 and q_2. By Lemma 6.29, it suffices to prove that $p'^{\not\equiv} \cap A$ is not of the form $\bar{\pi}^A$ with π a certain dual affine plane of A. Suppose for a contradiction that this is the case. For $i = 1, 2$, denote with T_i the transversal in A that contains q_i and $[q_1, q_2]_A$. If the point p' were noncollinear to $[q_1, q_2]_A$, it would follow from Lemma 5.19 that it would be noncollinear to $T_i \ni q_i$ for some i, a contradiction. So p' is collinear with $[q_1, q_2]_A$.

By Lemma 3.7, there is a unique point x of π that is linelike to $[q_1, q_2]_A$. Without loss of generality, we may assume that $x \in T_1$. Let T_x be the transversal in A through x different from T_1. By Lemma 5.19, the set $\bar{\pi}^A$ contains T_x, implying that p' is noncollinear to T_x. Next, consider the line $p'q_1$. By Lemma 3.5, there is a point r on $p'q_1 \setminus \{q_1\}$ which is noncollinear to q_2. Applying Lemma 6.38, with p' taking the role of p, q_1 that of q and T_x that of T, we find that $r^{\not\equiv} \cap A$ is the transversal closure in A of a dual affine plane of A that has T_x as a transversal. Together with the fact that r is noncollinear to q_2, this implies that $r^{\not\equiv} \cap A = \bar{\pi}^A_{T_1} \ni q_1$, a contradiction to the fact that r is collinear to q_1. □

We can now use the previous lemma to reach the goal of this section.

Proposition 6.10. *Let p be a point and let A be any A_2-plane. Then p is noncollinear to a conical hyperplane of A.*

Proof. By Lemma 6.29, it suffices to prove that $p^{\not\equiv} \cap A$ is not of the from $\overline{\pi}^A$ for some dual affine plane π of A. Suppose for a contradiction that this is the case. Let q_1 be a point of A collinear to p. By Lemma 3.7, the set $q_1^{\not\equiv} \cap \pi$ is the union of a transversal T and a line m. Let q_2 be a point of m not on T. We claim that $q_2^{\not\equiv} \cap pq_1 = \{p, q_1\}$. Indeed, let p' be any point of $pq_1 \setminus \{p, q_1\}$. Then, using Lemma 6.38 with p' in the role of r, we see that $p'^{\not\equiv} \cap A$ is the transversal closure in A of a dual affine plane π' of A, where π' contains T, but is different from π. We hence indeed find that p' is collinear q_2, which proves the claim. By Axiom (Im_1) (ii), the point q_2 and the line pq_1 then generate an A_2-plane A'. Both the A_2-planes A and A' contain the special points q_1 and q_2, while A' contains points (namely any point of $pq_1 \setminus \{p, q_1\}$) that are collinear to q_1 and q_2 but are not collinear to a conical hyperplane of A. This contradicts Lemma 6.41 and hence concludes the proof. □

Corollary 6.4. *Let p be a point and T a transversal, then p is noncollinear to at least one point of T.*

Proof. This is an immediate consequence of Proposition 6.10. □

We can use this proposition to obtain the following useful lemma.

Lemma 6.42. *Every line is contained in some A_2-plane.*

Proof. The space Y is connected, and contains at least one A_2-plane. It hence suffices to prove that every line l of Y that intersects an A_2-plane, is itself contained in an A_2-plane. So let l be a line, and A an A_2-plane that intersects l. If l is contained in A, there is nothing to prove. Suppose that l intersects A in some point p. Let q be a point of l different from p. Suppose that there is some line m through p in A for which q is noncollinear to exactly two points of m. Then Axiom $(\mathrm{Im}_1)(ii)$ implies that $\langle l, m\rangle$ is an A_2-plane, which contains l. Suppose for a contradiction that this is not the case. By Lemma 6.27, and the fact that q is collinear to p, we find that q is noncollinear to at most one point of every line m through p in A. Let T be a transversal through p in A, let r be a point of $T \setminus \{p\}$ and let T_r be the transversal in A through r but not through p. The dual affine plane $\pi^A_{T_r}$ contains p. Considering the possibilities in Lemma 6.30, and keeping in mind that no line through p in π intersects $q^{\not\equiv}$ in more than one point, we find that $q^{\not\equiv} \cap \pi$ is either empty or is a transversal. By Proposition 6.10, the set $q^{\not\equiv} \cap A$ is a conical hyperplane of A, so in each of these two cases, the transversal T_r, and in particular r is contained in $q^{\not\equiv}$. We hence find that $T \setminus \{p\} \subseteq q^{\not\equiv}$. Corollary 6.2 then, however, implies that q is also noncollinear to p, a contradiction to the fact that they both belong to the line l. □

Corollary 6.5. *Two collinear points cannot be symplectic.*

Proof. Two points are symplectic when they are not contained in a common A_2-plane. By Lemma 6.42, every pair of collinear points is contained in a common A_2-plane. □

7. Special points

Notation 12. *In this section, Y denotes a connected partial linear space that satisfies Axioms (Im_1), (Im_2) and (Im_3). We assume that no A_2-plane of Y is defined over $\mathbb{F}_3$ or over a field of characteristic two. We make use of Notation 3.*

In this section, we prove that the behaviour of two special points completely determines the behaviour of any point that is linelike to both of them. This will allow us to invoke Axiom (Im_3), and obtain that this point is uniquely determined.

7.1. **When a point is linelike or symplectic to some point of an A_2-plane.** We start by discussing what happens when a point is linelike or symplectic to some point of an A_2-plane.

Lemma 7.43. *Let p and q be linelike or symplectic points. For every line l through q, the point p is noncollinear to exactly one point of l, namely q, or to all points of l.*

Proof. Suppose that q is collinear to some point of l. It follows from Lemma 6.42 that the line l is contained in some A_2-plane. We can hence apply Lemma 6.27 and obtain that $|p^{\not\equiv} \cap l| \leq 2$. However, if p is noncollinear to exactly two points of l, it follows from Axiom (Im_2) that $\langle p, l\rangle$ is an A_2-plane. Using Lemma 3.5, we find that p and q are special in A, a contradiction to the assumption that they are linelike or symplectic. □

Lemma 7.44. *Let p be a point and T a transversal. If p is linelike or symplectic with some point q of T, then p is noncollinear to every point of T.*

Proof. There exists some dual affine plane π that contains q such that T is a transversal of π. The point p is noncollinear to q and, by Lemma 7.43, to one or all points of every line through q in π. Using Lemma 6.30, one then easily concludes that $p \not\equiv T$. □

Lemma 7.45. *Suppose that p is linelike to a point of a line l, then p is neither linelike, nor symplectic to any other point of l.*

Proof. Let q be a point of l that is linelike to p. Then there exists some A_2-plane A in which p and q are contained on some common transversal T. Let r be a point of $l \setminus \{q\}$. If r was linelike or symplectic to p, Lemma 7.44 would imply that r is noncollinear to the whole transversal T, and in particular to q, a contradiction. □

Lemma 7.46. *Let p be a point and A be an A_2-plane. If $p^{\not\equiv} \cap \pi$ is a degenerate conic for every dual affine plane π of A, and p is moreover linelike or symplectic with some point $q \in A$, then $p^{\not\equiv} \cap A$ is one of the following:*

(1) The whole set A.
(2) A set of the form $\bar\pi_1^A \cup \bar\pi_2^A$ with π_1 and π_2 dual affine planes in A such that $\pi_1 \cap \pi_2$ is a line through q.
(3) A set of the form $z^{\not\equiv} \cap A$ with z some point in A linelike with q. This equals $\bar\pi_{T_1}^A \cup \bar\pi_{T_2}^A$, with T_1 and T_2 the two transversals in A through z.

Proof. Using the terminology of Definition 5.23, we find that $p^{\not\equiv} \cap A$ is a fully degenerate conical subset with vertex q. Using Lemma 5.17, we hence find that $p^{\not\equiv} \cap A$ is either A or a subset $\bar\pi_1^A \cup \bar\pi_2^A$ with π_1 and π_2 dual affine planes in A for which $q \in \bar\pi_1^A \cap \bar\pi_2^A$. At the same time, the set $p^{\not\equiv} \cap A$ is a conical hyperplane of A. Using Example 5.14, one easily sees that this indeed implies the claim. □

7.2. **When a point is linelike or symplectic to several points of an A_2-plane.** A point can of course also be linelike or symplectic to several points of an A_2-plane. We investigate some particular cases that will be useful later on.

Lemma 7.47. *Let p be a point and let A be an A_2-plane containing linelike points q_1 and q_2. If p is linelike or symplectic to both q_1 and q_2, the set $p^{\not\equiv} \cap A$ is either the whole of A, or equals $q^{\not\equiv} \cap A$ for some point q on the transversal in A that contains q_1 and q_2.*

Proof. Let T be the transversal in A that contains both q_1 and q_2. We first prove that $p \not\equiv \pi_T^A$, or equivalently, that $p \not\equiv T'$ for every transversal T' of A that intersects T. Let T' be such a transversal, and set $q' := T' \cap T$. First suppose that $q' = q_1$ or q_2. Then the point p is linelike or symplectic to q', so it follows from Lemma 7.44, that $p \not\equiv T'$. Next, suppose that $q' \notin$

$\{q_1, q_2\}$. The plane $\pi' := \pi^A_{T'}$ contains q_1 and q_2. One can easily argue, using Lemma 6.30 and Lemma 7.43, that $p^{\not\equiv} \cap \pi'$ is either T' or π'. Since $p^{\not\equiv} \cap A$ is a conical hyperplane, it follows that $p \not\equiv T'$. This proves the claim. One can now use Lemma 5.18 to obtain that $p^{\not\equiv} \cap \pi$ is a degenerate conic for every dual affine plane π of A. The result now follows from Lemma 7.46. □

Lemma 7.48. *Let p be a point and let A be an A_2-plane containing special points q_1 and q_2. If p is linelike or symplectic to both q_1 and q_2, the set $p^{\not\equiv} \cap A$ is either the whole set A or equals $[q_1, q_2]^{\not\equiv}_A \cap A$.*

Proof. For $i = 1, 2$, let T_i be the transversal in A that contains q_i and $[q_1, q_2]_A$, and let Q_i be the transversal in A through q_i different from T_i. By Lemma 7.44, the set $p^{\not\equiv} \cap \pi^A_{T_1}$ contains T_2 and Q_1. The point p is moreover linelike or symplectic to $q_2 \in \pi^A_{T_1}$, so using Lemma 7.43, one easily argues that $p \not\equiv \pi^A_{T_1}$. Similarly, one finds that $p \not\equiv \pi^A_{T_2}$. Lemma 5.16 then implies that $p^{\not\equiv} \cap A$ is either the whole of A or the set $\bar{\pi}^A_{T_1} \cup \bar{\pi}^A_{T_2} = [q_1, q_2]^{\not\equiv}_A \cap A$. This concludes the Lemma. □

Lemma 7.49. *Let p' be a point and let A be an A_2-plane containing special points q_1 and q_2. Suppose that p' is linelike to q_1 and linelike or symplectic to q_2. Let T be a transversal that contains both p' and q_1 and let T_1 be the transversal in A that contains q_1 and $[q_1, q_2]_A$. For any point $p \in T \setminus \{p', q_1\}$, the set $p^{\not\equiv} \cap A$ is of the form $x_p^{\not\equiv} \cap A$ with x_p some point in $T_1 \setminus \{[q_1, q_2]_A\}$.*

Proof. By Lemma 7.48, the set $p'^{\not\equiv} \cap A$ is either A or $[q_1, q_2]^{\not\equiv}_A \cap A$. Suppose that we are in the former case, then Corollary 6.2 implies that p is noncollinear to all points that are noncollinear to both p' and q_1, and in particular to $q_1^{\not\equiv} \cap A$. If p were noncollinear to any other point y of A, this same corollary would imply that y would also be noncollinear to q_1, a contradiction. This implies that $p^{\not\equiv} \cap A = q_1^{\not\equiv} \cap A$. Next, assume that $p'^{\not\equiv} \cap A = [q_1, q_2]^{\not\equiv}_A \cap A$. Denote with T_2 the transversal in A that contains both q_2 and $[q_1, q_2]_A$. The set $p'^{\not\equiv} \cap A$ equals $\bar{\pi}^A_{T_1} \cup \bar{\pi}^A_{T_2}$. The point q_1 on the other hand, is noncollinear to $\bar{\pi}^A_{T_1}$ and collinear to some points of $\bar{\pi}^A_{T_2}$. Again using Corollary 6.2, we find that p is collinear to some point of $\bar{\pi}^A_{T_2}$ and noncollinear to $\bar{\pi}^A_{T_1}$. Moreover, the point p is linelike to $q_1 \in T_1$. Considering the possibilities in Lemma 7.46, we can hence indeed conclude that also in this case, $p^{\not\equiv} \cap A = x_p^{\not\equiv} \cap A$ for some point x_p of $T_1 \setminus \{[q_1, q_2]_A\}$. □

7.3. **Special points p and q determine behaviour of the point $[p, q]_A$.** The goal of this section is to prove the following proposition.

Proposition 7.11. *Let p and q be special points, let A be an A_2-plane that contains p and q, and let l be a line through q. The following claims hold:*

(1) The point $[p, q]_A$ is collinear to $l \setminus \{q\}$ if and only if $|p^{\not\equiv} \cap l| = 2$.
(2) The point $[p, q]_A$ is noncollinear to the line l if and only if $|p^{\not\equiv} \cap l| \neq 2$. This is the case if and only if $p^{\not\equiv} \cap l$ is either l or $\{q\}$.

We divide the proof into three parts, namely Lemma 7.50, Lemma 7.51 and Lemma 7.52.

Lemma 7.50. *Let p and q be special points, let A be an A_2-plane that contains p and q, and let l be a line through q. If the point $p \not\equiv l$, then $[p, q]_A \not\equiv l$.*

Proof. It is clear that $[p, q]_A$ is noncollinear to $q \in l$. Suppose for a contradiction that $[p, q]_A$ is collinear to some point of $l \setminus \{q\}$. By Lemma 6.40, the point $[p, q]_A$ is collinear to all points of $l \setminus \{q\}$. Every such point $r \in l \setminus \{q\}$ is then noncollinear to p but collinear to $[p, q]_A$, so Corollary 6.2 implies that $r^{\not\equiv} \cap T = \{p\}$, with T the transversal in A through p and $[p, q]_A$. The set $r^{\not\equiv} \cap A$ is however a conical hyperplane, so there is a line m in π^A_T through q that contains one or two points noncollinear to r. Let T_p be the transversal in A through p different from T, and set $x := T_p \cap m$. By Lemma 6.25 or Lemma 6.26, applied to l and m, we can re-choose $r \in l$

so that r is noncollinear to x. This point r is then noncollinear to both x and p of T_p, and hence by Corollary 6.2, noncollinear to the whole of T_p. Checking the possibilities in Lemma 6.30, and keeping in mind that $r^{\not\equiv} \cap A$ is a conical hyperplane, we find that $r^{\not\equiv} \cap \pi_T^A$ is the union of a line with the transversal T_p. Applying Lemma 6.36 applied to r, l and π_T^A, we see that every point of $l \setminus \{q\}$ is noncollinear to the union of a line and a transversal of π_T^A, which must intersect T in p by assumption, and hence equals T_p. This however implies that T_p is noncollinear to all points of $l \setminus \{q\}$, and by Lemma 6.27, also to q, a contradiction. $\square$

Lemma 7.51. *Let p and q be special points, let A be an A_2-plane that contains p and q and let l be a line through q. If $|p^{\not\equiv} \cap l| = 2$, then $[p, q]_A^{\not\equiv} \cap l = \{q\}$.*

Proof. Suppose for a contradiction that there exists some point of $l \setminus \{q\}$ noncollinear to $[p, q]_A$. The point $[p, q]_A$ is linelike to q, so by Lemma 6.40, the point $[p, q]_A$ is noncollinear to all points of l. Let r be the point of $l \setminus \{q\}$ that is noncollinear to p. Then r is noncollinear to both p and $[p, q]_A$. Denote with T the transversal in A that contains both p and $[p, q]_A$. Using Corollary 6.2, we find that $r \not\equiv T$. Lemma 6.39, with $\pi = \pi_T^A$, implies that $l \not\equiv T$, and in particular that the point p is noncollinear to all points of l, a contradiction. $\square$

Lemma 7.52. *Let p and q be special points, let A be an A_2-plane that contains p and q, and let l be a line through q. If $p^{\not\equiv} \cap l = \{q\}$, then $[p, q]_A \not\equiv l$.*

Proof. Suppose for a contradiction that there exists some point r of $l \setminus \{q\}$ collinear to $[p, q]_A$. Let T denote the transversal in A that contains p and $[p, q]_A$. By Proposition 6.10, the point r is noncollinear to some point p' of T, which, by assumption, has to be different from $[p, q]_A$. This point p' is noncollinear to both q and r of l, and moreover, it is clear that $[p, q]_A = [p', q]_A$.

First suppose that $|p'^{\not\equiv} \cap l| > 2$. Lemma 6.27 implies that $p' \not\equiv l$. We can apply Lemma 7.50 to p', q, A and l and obtain that $[p', q]_A \not\equiv l$. This is a contradiction to the fact that r is collinear to $[p, q]_A = [p', q]_A$.

Next, suppose that $|p'^{\not\equiv} \cap l| = 2$, that is, $p'^{\not\equiv} \cap l = \{q, r\}$. By Axiom $(\mathrm{Im}_1)(ii)$, the point p' and the line l generate an A_2-plane, which we denote here with A'. The point $[p, q]_A$ is linelike to both p' and q, which are both contained in A', and the point p lies on the transversal T through $[p, q]_A$ and p'. We can hence apply Lemma 7.49 to obtain that $p^{\not\equiv} \cap A' = x_p^{\not\equiv} \cap A'$ for some point x_p on $T' \setminus \{[p', q]_{A'}\}$ with T' the transversal in A' that contains p' and $[p', q]_{A'}$. However, by applying Lemma 3.5 to A', we find that such a point x_p is noncollinear to exactly two points of $l \in A'$, a contradiction to the assumption that $p^{\not\equiv} \cap \{l\} = \{q\}$. $\square$

Taking together the results of Lemma 7.50, Lemma 7.51 and Lemma 7.52, we indeed obtain Proposition 7.11.

7.4. **When a point is linelike to some points of an A_2-plane.** In this subsection, we use Proposition 7.11 to obtain more information on what happens when a point is linelike so some point of an A_2-plane.

Lemma 7.53. *Let p be a point and A be an A_2-plane. If p is linelike to some point of A, the set $p^{\not\equiv} \cap A$ is not of the form $\bar{\pi}_1^A \cup \bar{\pi}_2^A$ with π_1 and π_2 dual affine planes in A that intersect in a line.*

Proof. Suppose that p is linelike to some point $q \in A$, and suppose for a contradiction that $p^{\not\equiv} \cap A = \bar{\pi}_1^A \cup \bar{\pi}_2^A$ with π_1 and π_2 dual affine planes in A that intersect in a line l. Using Lemma 7.46, we see that $q \in l$. For $i = 1, 2$, denote with Q_i the transversal of π_i in A through q. The points p and q are linelike, so there exists an A_2-plane A' that contains both p and q in which p and q are linelike. In this A_2-plane A', there is a unique transversal through p that does not contain q. Fix some point $r \neq p$ on that transversal. Observe that $[r, q]_{A'} = p$.

We first discuss the possibilities for $r^{\not\equiv} \cap \pi_1$. The point p is, by assumption, noncollinear to each line m in π_1 through q. We can apply Proposition 7.11, with $A', r, m, p = [r, q]_{A'}$ in the role of $A, p, l, [p, q]_A$, respectively, and obtain that $r^{\not\equiv} \cap m$ is either m or $\{q\}$. Considering the possibilities in Lemma 6.30, and taking into account Proposition 6.10 that says that $r^{\not\equiv} \cap A$ is a conical hyperplane and hence that $r^{\not\equiv} \cap \pi_1$ is not a line, we find that $r^{\not\equiv} \cap \pi_1$ is either π_1, Q_1 or $Q_1 \cup m_1$ with m_1 some line in π_1 through q. In all three cases, Q_1 is contained in $r^{\not\equiv}$. We can apply this same reasoning to π_2 instead of π_1, and obtain that $Q_2 \subseteq r^{\not\equiv}$.

Now let $m \neq l$ again be a line in π_1 through q, and let π_m be the unique dual affine plane in A through m different from π_1. Then Q_2 is a transversal of π_m. The point r is noncollinear to Q_2 and noncollinear to one or all points of m. Using the possibilities in Lemma 6.30, we find that $r^{\not\equiv} \cap \pi_m$ is either π_m, Q_2, or the union $Q_2 \cup m'$ with m' some line in π_m through q. Let $n \neq m$ be a line through q in π_m, then the previous argument shows that $r^{\not\equiv} \cap n = n$ or $\{q\}$. We can again apply Proposition 7.11, and find that $p = [q, r]_{A'}$ is noncollinear to n. This line n is however not contained in $p^{\not\equiv} \cap A = \bar{\pi}_1^A \cup \bar{\pi}_2^A$, a contradiction. This concludes the proof. □

Lemma 7.54. *Let p be a point and let A be an A_2 plane. If p is linelike to some point q of A and noncollinear to a dual affine plane π of A that contains q, it is also noncollinear to $\bar{\pi}_T^A$ with T the transversal of π in A through q.*

Proof. We have that $p \not\equiv \pi$, so it follows form Lemma 5.18 that $p^{\not\equiv} \cap A$ intersects every dual affine plane of A in a degenerate conic. The point p is moreover linelike to q. We can apply Lemma 7.46 and obtain that the set $p^{\not\equiv} \cap A$ is one of the possibilities described in Lemma 7.46. Using Lemma 7.53 and the fact that p is noncollinear to the dual affine plane π of A, one finds that $p^{\not\equiv} \cap A$ is either A or is of the form $z^{\not\equiv} \cap A$ with z some point in A linelike to q. There is only one such point z for which $z^{\not\equiv}$ contains π, namely $T \cap T_\pi^A$ with T the transversal of π in A through q. We hence find that $\bar{\pi}_T^A$ is contained in $z^{\not\equiv} \cap A \subseteq p^{\not\equiv}$. □

Lemma 7.55. *Let p be a point, let A be an A_2-plane containing special points q_1 and q_2. If p is linelike to q_1 and noncollinear to q_2, then p is noncollinear to $\bar{\pi}_{T_1}^A$ with T_1 the transversal of A that contains q_1 and $[q_1, q_2]_A$.*

Proof. Let p, A, q_1, q_2 and T_1 be as stated. Let T_2 be the transversal in A that contains q_2 and $[q_1, q_2]_A$. By assumption, the points p and q_1 are linelike, so there exists some transversal T that contains both p and q_1.

First, we aim to find a point $p_1 \in T$ which is noncollinear to $\bar{\pi}_{T_2}^A$. To that end, let r be a point of $\pi_{T_2}^A$ that is not contained in T_1. Then r is collinear to q_1, so by Corollary 6.4, there is a point p_1 on $T \setminus \{q_1\}$ which is noncollinear to r. This point p_1 is of course also linelike to q_1, so by Lemma 7.44 and Lemma 6.40, we have that $q_1 r$ and T_1 are contained in $p_1^{\not\equiv}$. We can hence conclude that $p_1^{\not\equiv} \cap \bar{\pi}_{T_2}^A$ is equal to either $q_1 r \cup T_1$, or $\bar{\pi}_{T_2}^A$. The point $q_2 \in T_2$, however, is noncollinear to both p and q_1, and is by Corollary 6.2 hence also noncollinear to p_1. This implies that p_1 is noncollinear to $\bar{\pi}_{T_2}^A$.

The point p_1 is linelike to q_1 and is noncollinear to the plane $\pi_{T_2}^A$ which contains q_1. Consequently, we can apply Lemma 7.54 to the point p_1 and obtain that p_1 is noncollinear to the set $\bar{\pi}_{T_1}^A$. Every point in this set is of course also noncollinear to q_1, and is hence noncollinear to two points of T. Using Corollary 6.2, we can conclude that every point of $\bar{\pi}_{T_1}^A$ is noncollinear to $p \in T$. This concludes the proof. □

7.5. **Special points p and q uniquely determine a point $[p, q]$.** For special points p and q, we can always construct a point that is linelike to both of them: take any A_2-plane A that contains p and q, and consider the point $[p, q]_A$. In this subsection, we will prove that this construction is independent of the chosen A_2-plane. Up to this point, we have not used the Axiom (Im_3). This axiom will, however, be crucial in all arguments that follow. We recall:

(Im_3) For points p and q, if $p^{\not\equiv} = q^{\not\equiv}$, then $p = q$.

The main result of this subsection is Proposition 7.12. We first gather two preliminary results.

Lemma 7.56. *Let T be a transversal containing a point p and let x, y be two points for which*

$$x^{\not\equiv} \cap T \setminus \{p\} = y^{\not\equiv} \cap T \setminus \{p\}.$$

Then $x^{\not\equiv} \cap T = y^{\not\equiv} \cap T$.

Proof. Both points x and y are, by Corollary 6.2 and Corollary 6.4, noncollinear to either a unique point of T or to every point of T. From this, we immediately obtain that

$$x \not\equiv p \iff |x^{\not\equiv} \cap T \setminus \{p\}| \neq 1 \iff |y^{\not\equiv} \cap T \setminus \{p\}| \neq 1 \iff y \not\equiv p,$$

which proves the assertion. □

Lemma 7.57. *Let A be an A_2-plane containing a point q, and let x, y be two points for which*

$$x^{\not\equiv} \cap A \cap q^{\equiv} = y^{\not\equiv} \cap A \cap q^{\equiv}.$$

Then $x^{\not\equiv} \cap A = y^{\not\equiv} \cap A$.

Proof. Let p be a point of A, we prove that p is collinear to x if, and only if, it is collinear to y. If p is collinear to q, this follows immediately from the assumption. Suppose that p is special to q. Let T be the transversal in A that contains p but not $[p, q]_A$. The point q is collinear to all points of $T \setminus \{p\}$, so $x^{\not\equiv} \cap T \setminus \{p\} = y^{\not\equiv} \cap T \setminus \{p\}$. Now the assertion follows from Lemma 7.56, in combination with the previous case. Next, suppose that p is linelike to q, let T be the transversal in A that contains p but not q. All points of $T \setminus \{p\}$ are special to q, so the assertion again follows from Lemma 7.56. Finally, suppose that p equals q. Let T be any transversal through p in A. Then all points of $T \setminus \{p\}$ are linelike to q, so with the exact same argument, the assertion holds. □

Proposition 7.12. *Let p and q be special points, and let A and A' be A_2-planes containing p and q. Then $[p, q]_A = [p, q]_{A'}$.*

Proof. First, we claim that $[p, q]_A^{\not\equiv} \cap q^{\equiv} = [p, q]_{A'}^{\not\equiv} \cap q^{\equiv}$. Let x be a point collinear to q, and let l_x be the line that contains x and q. We can apply Proposition 7.11 first to A and then to A' and find that

$$[p, q]_A \equiv x \iff |p^{\not\equiv} \cap l_x| = 2 \iff [p, q]_{A'} \equiv x.$$

This indeed proves the claim.

We proceed by proving that $[p, q]_A^{\not\equiv} = [p, q]_{A'}^{\not\equiv}$. To that end, let x be a point collinear to $[p, q]_A^{\not\equiv}$. We prove that it is also collinear to $[p, q]_{A'}^{\not\equiv}$. If x is collinear to q, the claim immediately follows from the argument above, so we may assume that x is noncollinear to q. Denote with T_p, T_q the transversals in A that contain $[p, q]_A$ and p, q, respectively. Set $\pi := \pi_{T_p}^A$ and note that $q \in \pi$. We claim that there is a line l in π through q for which $|x^{\not\equiv} \cap l| = 2$. Suppose this were not the case, then, using the possibilities in Lemma 6.30, and taking into account that $x^{\not\equiv} \cap A$ is a conical hyperplane, we would obtain that x is noncollinear to T_q, which contains $[p, q]_A$, a contradiction. So let l be a line through q in A for which $|x^{\not\equiv} \cap l| = 2$. By Axiom $(\mathrm{Im}_1)(ii)$ the plane $\langle x, l \rangle$ is an A_2-plane, which we denote with A_l. By the previous claim, we find that $[p, q]_A^{\not\equiv} \cap q^{\equiv} = [p, q]_{A'}^{\not\equiv}$, which immediately implies that

$$[p, q]_A^{\not\equiv} \cap A_l \cap q^{\equiv} = [p, q]_{A'}^{\not\equiv} \cap A_l \cap q^{\equiv}.$$

Lemma 7.57, applied to $[p,q]_A$, $[p,q]_{A'}$ and A_l, then implies that $[p,q]_A \cap A_l = [p,q]_{A'} \cap A_l$. The point x, being collinear to $[p,q]_A$, is hence also collinear to $[p,q]_{A'}$. We can apply this very same argument with A and A' interchanged, and we conclude that $[p,q]_A^{\not\equiv} = [p,q]_{A'}^{\not\equiv}$.

Together with Axiom (Im_3), this last claim immediately implies that $[p,q]_A = [p,q]_{A'}$. $\square$

It is an immediate consequence of Proposition 7.12 that the following is well defined.

Definition 7.26. *For special points p and q, define $[p,q] := [p,q]_A$ for A any A_2-plane that contains p and q.*

An immediate, but useful, corollary of this is the following.

Corollary 7.6. *If a point p is linelike to some point of a line l and collinear to another point of l, then there exists an A_2-plane that contains both p and l.*

Proof. Let q be the point on l that is linelike with p, let A be any A_2-plane containing p and q, and let r be a point in A linelike with p but not with q. Proposition 7.11, with r, q, p in the role of $p, q, [p,q]$, respectively, implies that $|r^{\not\equiv} \cap l| = 2$. We can hence use Axiom (Im_1) (ii) and find an A_2-plane A' that contains r and l. This A_2-plane A' then of course contains $q \in l$ and hence also $[r,q] = p$. $\square$

7.6. **Special points p and q have a unique point linelike to both.** Definition 7.26 associates a point $[p,q]$ to every pair of special points p and q. In this subsection, we prove that this point $[p,q]$ can be characterized as the unique point that is linelike to both p and q. As in Section 7.5, the crucial ingredient will again be Axiom (Im_3).

Lemma 7.58. *Let p and q be special points. Let x be a point linelike or symplectic to p and noncollinear to q. If x is special to q, assume moreover that for every A_2-plane A_q through x and q, the point p is noncollinear to $\pi_{T_q}^{A_q}$, with T_q the transversal in A_q through x and $[x,q]$. Then x is linelike or symplectic to $[p,q]$.*

Proof. Let p, q, x be as stated. Assume for a contradiction that x is neither linelike nor symplectic with $r := [p,q]$. Let A be an A_2-plane that contains p and q. The point x is linelike or symplectic to $p \in A$, so, by Lemma 7.44, x is noncollinear to both transversals in A through p, and hence to r. Together with the assumption that x is not linelike nor symplectic to r, this implies that x is special to r.

We first claim that, if x is special to q, the point $[r,x]$ is noncollinear to any line l_q through x that is contained in an A_2-plane with q. Indeed, assume that x is special to q, and let A_q be an A_2-plane that contains both x and q. Denote with T_q the transversal in A_q that contains x and $[x,q]$. By assumption, we know that p is noncollinear to $\pi_q := \pi_{T_q}^{A_q}$, which contains q. The point p is hence noncollinear to all points of any line through q in π_q. Proposition 7.11 on its turn, then implies that $[p,q] = r$ is also noncollinear to all points of any line through q in π_q, and is hence noncollinear to π_q. This point r is at the same time linelike to q, so with Lemma 7.54, we find that r is noncollinear to $[q,x]^{\not\equiv} \cap A_q$, which implies that $r^{\not\equiv} \cap A_q$ is either $[q,x]^{\not\equiv} \cap A_q$ or A_q itself. Let l_q be any line in A_q through x. Then in any of the two cases, the point r is noncollinear to exactly one or all points of that line. We can apply Proposition 7.11 with $r, x, [r,x], l_q$ taking over the role of $p, q, [p,q], l$, respectively, and obtain that $[r,x]$ is indeed noncollinear to l_q. This proves the claim.

Next, let A_r be an A_2-plane that contains x and r. We claim that $[r,x]^{\not\equiv} \cap A_r$ is contained in $q^{\not\equiv}$. First suppose that q is linelike or symplectic to x. Since q is also linelike to r, we can use Lemma 7.48 to obtain that $q^{\not\equiv} \cap A_r = A_r$ or $[x,r]^{\not\equiv} \cap A_r$, and hence to conclude that $[x,r]^{\not\equiv} \cap A_r$ is contained in $q^{\not\equiv}$. Next, suppose that q is special to x. Let T_r be the transversal in A_r that contains r and $[r,x]$. The point q is linelike to r and noncollinear to x, so Lemma 7.55 implies that q is noncollinear to the set $\bar{\pi}^{A_r}_{T_r}$. Together with Lemma 7.46, this implies that $q^{\not\equiv} \cap A_r$ equals either A_r or $z^{\not\equiv} \cap A_r$ with z some point of T_r. In the former case, the set $[r,x]^{\not\equiv} \cap A_r$ is indeed contained in $q^{\not\equiv}$. Suppose that we are in the latter case, and suppose for a contradiction that $z \neq [r,x]$. Let l_q be a line through x in A_r, not in $\bar{\pi}^{A_r}_{T_r}$. We have that $[r,x] \equiv l_q \setminus \{x\}$, and that $|q^{\not\equiv} \cap l_q| = |z^{\not\equiv} \cap l_q| = 2$. By Axiom (Im_1) (ii), the plane $\langle q, l_q \rangle$ is an A_2-plane, which contains both q and l_q, but the claim above then implies that $[r,x] \not\equiv l_q$, a contradiction. We hence indeed obtain that $[r,x]^{\not\equiv} \cap A_r$ is contained in $y^{\not\equiv}$.

We are now ready to finalize the proof. Let T be the transversal in A that contains r and q, and take $q' \in T \setminus \{r,q\}$. The point x is noncollinear to r and q of T, so, by Corollary 6.2, it is also noncollinear to q'. We can hence repeat the reasoning in the previous paragraph with q' instead of q, and obtain that $[r,x]^{\not\equiv} \cap A$ is contained in $q'^{\not\equiv}$. Every point of $[r,x]^{\not\equiv} \cap A$ is hence noncollinear to both q and q' of T, and is hence contained in $q^{\not\equiv} \cap q'^{\not\equiv} \cap A = \pi^{A}_{T}$, a contradiction to the fact that $[r,x]^{\not\equiv} \cap A$ is a conical hyperplane of A. □

Corollary 7.7. *Let p and q be special points, and let x be a point linelike to p and noncollinear to q. Then x is linelike or symplectic to $[p,q]$.*

Proof. It suffices to show that the conditions of Lemma 7.58 hold. To that end, suppose that x is special to q, and let A_q be an A_2-plane that contains x and q. Denote by T_q the transversal in A_q that contains x and $[x,q]$. The point p is linelike to x and special to q, so, by Lemma 7.55, we find that p is noncollinear to $\bar{\pi}^{A_q}_{T_q}$. The conditions of Lemma 7.58 hence indeed hold, and we obtain that x is linelike or symplectic to $[p,q]$. □

Lemma 7.59. *Let p and q be special points, and let x be a point that is linelike to both p and q. For any A_2-plane A that contains both p and q, the following holds:*

$$x^{\not\equiv} \cap A = [p,q]^{\not\equiv} \cap A.$$

Proof. Let A be an A_2-plane that contains p and q. By assumption, the point x is linelike to both p and q of A, so by Lemma 7.48, the set $x^{\not\equiv} \cap A$ either equals A or $[p,q]^{\not\equiv} \cap A$. It hence suffices to prove that $x^{\not\equiv} \cap A \neq A$. Suppose for a contradiction that this is the case. Let r be a point of A that is linelike to p but not to $[p,q]$, and let s be a point of A that is linelike to r but not to p. By construction, the point r equals $[p,s]$ and is collinear to q. The point x is linelike to p and noncollinear to $s \in A$. By Corollary 7.7, the point x is linelike or symplectic to r. Now consider the line $l := rq$. The point x is linelike to $q \in l$ and linelike or symplectic to $r \in l$, which contradicts Lemma 7.45. □

Lemma 7.60. *Let p and q be special points, and let x be a point linelike to both p and q, then*

$$x^{\not\equiv} \cap q^{\equiv} = [p,q]^{\not\equiv} \cap q^{\equiv}.$$

Proof. It clearly suffices to prove that for every line l through q, we have that $x^{\not\equiv} \cap l = [p,q]^{\not\equiv} \cap l$. So let l be a line through q. First suppose that $[p,q]^{\not\equiv} \cap l = \{q\}$. By Proposition 7.11, the point p is noncollinear to exactly two points of l. Axiom $(\mathrm{Im}_1)(ii)$ implies that $A_l := \langle p, l \rangle$ is an A_2-plane. This plane A_l of course contains p and $q \in l$, so Lemma 7.59 yields $x^{\not\equiv} \cap A_l = [p,q]^{\not\equiv} \cap A_l$. This indeed proves that $x^{\not\equiv} \cap l = [p,q]^{\not\equiv} \cap l$.

Next suppose that $[p,q]^{\not\equiv} \cap l \neq \{q\}$. Proposition 7.11 implies that $[p,q] \not\equiv l$ and that $p^{\not\equiv} \cap l$ is either $\{q\}$ or l. We have to prove that $x \not\equiv l$. Suppose for a contradiction the opposite. Then Corollary 7.6 yields an A_2-plane A_l that contains both x and l. Let π be the dual affine plane in A_l that contains both x and l.

We claim that p is noncollinear to the transversal $T_\pi^{A_l}$. Indeed, first assume that $p^{\not\equiv} \cap l = l$. Let z be any point of π collinear with x. The line xz contains at least two points noncollinear to p, namely x and $xz \cap l$. The point p is linelike to x, so Lemma 6.40 implies that x is noncollinear to the line xz, and hence also to z. From this, we can conclude that $p^{\not\equiv} \cap \pi = \pi$, and, by Corollary 6.2, that p is also noncollinear to $T_\pi^{A_l}$. On the other hand, assume that $p^{\not\equiv} \cap l = \{q\}$. Then we can again take any point z in π collinear to x. The line xz then contains a point $xz \cap l$ collinear to p, so Lemma 6.40 implies that $p^{\not\equiv} \cap xz = \{x\}$. This implies that $p^{\not\equiv} \cap \pi = T_q$ with T_q the transversal in A through x and q. The set $p^{\not\equiv} \cap A$ is however a conical hyperplane, so also in this case, $p \not\equiv T_\pi^{A_l}$. This proves the claim.

The point p is linelike to x and noncollinear to the transversal $T_\pi^{A_l}$. By Lemma 7.55, we find that $p \not\equiv \pi_{T_q}^{A_l}$ with T_q the transversal of A_l that contains q and x. In particular, the point p is noncollinear to T'_q, with T'_q the transversal in A_l through q different from T_l. Let r be a point on $T'_q \setminus \{q\}$, then r is special to x, and $[r, x] = q$. The point p is linelike to x and noncollinear to r, so, by Corollary 7.7, the point p must be linelike or symplectic to $[r, x] = q$, a contradiction to the fact that p is special to q. □

Corollary 7.8. *Let p and q be special points, and let x and y be points that are linelike to both p and q. Then*

$$x^{\not\equiv} \cap q^{\equiv} = y^{\not\equiv} \cap q^{\equiv}.$$

Proof. Lemma 7.60 stipulates that both sets are equal to $[p, q]^{\not\equiv} \cap q^{\equiv}$. □

Lemma 7.61. *Let A be an A_2-plane containing linelike points q_1 and q_2, let T_A be the unique transversal in A that contains q_1 and q_2, and let q be any point on any transversal through q_1 and q_2, but different from q_1 and q_2. Then there exists some point q_A on $T_A \setminus \{q_1, q_2\}$ for which $q^{\not\equiv} \cap A = q_A^{\not\equiv} \cap A$.*

Proof. Let T be a transversal that contains q_1 and q_2, and take $q \in T \setminus \{q_1, q_2\}$. The points q_1 and q_2 on T are noncollinear to the set $\bar\pi_{T_A}^A$, so, by Corollary 6.2, the point q is also noncollinear to $\bar\pi_{T_A}^A$. This point q is moreover linelike to both q_1 and q_2, so, by Lemma 7.46, the set $q^{\not\equiv} \cap A$ is either A or is of the form $q_A^{\not\equiv} \cap A$ with q_A some point of T_A. Assume that q is noncollinear to $q_1^{\not\equiv} \cap A$. Then every point of $q_1^{\not\equiv} \cap A$ is noncollinear to two points of T, namely q and q_1, and is, by Corollary 6.2, hence noncollinear to all points of T, and in particular to q_2, a contradiction. With the same argument, but with q_1 and q_2 interchanged, we also find that q_A is collinear to some point of $q_2^{\not\equiv} \cap A$. Hence $q_A \notin \{q_1, q_2\}$, which concludes the proof. □

Lemma 7.62. *Let p and q be special points, and let x and y be points that are linelike to both p and q, then we have that $x^{\not\equiv} = y^{\not\equiv}$.*

Proof. The points x and y play exactly the same role, it hence suffices to prove that $y^{\not\equiv} \subseteq x^{\not\equiv}$, or equivalently, that $x^{\equiv} \subseteq y^{\equiv}$. Let r be collinear to x. We prove that r is also collinear to y. If r is collinear to q, this follows from Corollary 7.8. So we suppose that this is not the case.

Let T be a transversal that contains both x and q, and take $q' \in T \setminus \{x, q\}$. The point r is collinear to x and noncollinear to q, so, by Corollary 6.2, it is collinear to q'. We can now use Corollary 7.6 to find an A_2-plane A' that contains both q' and the line xr. Let T' be the transversal in A' that contains both x and q'. Then Lemma 7.61, with A', T', x, q', q in the roles of A, T_A, q_1, q_2, q, respectively, implies that $q^{\not\equiv} \cap A = q_A^{\not\equiv} \cap A$ for some point q_A of $T' \subset A'$.

By Corollary 7.8, we have that $x^{\not\equiv} \cap q^{\equiv} = y^{\not\equiv} \cap q^{\equiv}$. In particular, this is true in A', where, $q^{\equiv} \cap A' = q_A^{\equiv} \cap A'$. We hence obtain that

$$x^{\not\equiv} \cap A' \cap q_A^{\equiv} = y^{\not\equiv} \cap A' \cap q_A^{\equiv}.$$

Applying Lemma 7.57, we can conclude that $x^{\not\equiv} \cap A' = y^{\not\equiv} \cap A'$. The point r is hence also collinear to y. □

Proposition 7.13. *For special points p and q, there is exactly one point, namely $[p, q]$, that is linelike to both p and q.*

Proof. It is clear than $[p, q]$ is linelike to both p and q. So let x be any other point linelike to both p and q. By Lemma 7.62, we have that $x^{\not\equiv} = [p, q]^{\not\equiv}$. Axiom (Im_3) then immediately implies that $x = [p, q]$. □

8. Turning Y into a root filtration space

In this section, the partial linear space $Y = (\mathscr{E}, \mathscr{I})$ is a partial linear space satisfying Axioms (Im_1), (Im_2) and (Im_3). We moreover assume that no A_2-plane of Y is defined over the field $\mathbb{F}_3$ or over a field of characteristic two.

Denote with $\mathscr{L}$ the set of transversals of Y. We will prove that $X = (\mathscr{E}, \mathscr{L})$ forms a nondegenerate root filtration space. To that end, we will first gather some extra results in Section 8.1 that will help distinguish linelike, symplectic and special points. Next, in Section 8.2, we translate these results to the language of root filtration spaces, and in particular prove that X satisfies axioms (Rf_1) to (Rf_8) of Definition 2.10. In Section 8.3, we proceed by proving that X also forms a partial linear space, which then implies that it forms a nondegenerate root filtration space.

8.1. Distinguishing linelike, symplectic and special points.

In this subsection, we will gather several results that will help distinguish linelike, symplectic and special points.

In a first step, we consider a point x that is linelike to at least two points of some A_2-plane A, and see if we can determine the set of points in A that are linelike or symplectic to x.

Lemma 8.63. *If a point x is linelike or symplectic to two points of a transversal T, then x is linelike or symplectic to all points of T.*

Proof. Let x be linelike or symplectic to two distinct points q_1 and q_2 of T. Suppose for a contradiction that there exists some point q of T such that x is neither linelike nor symplectic to q. By Lemma 7.44, the point x is also noncollinear to q, implying that x is special to q. Let A be an A_2-plane that contains the special points x and q. The point q_1 is linelike or symplectic to both x and q of A, so we can apply Lemma 7.48 and obtain that $q_1^{\not\equiv} \cap A$ is either A or $[x, q]^{\not\equiv} \cap A$, and in particular, that $[x, q]^{\not\equiv} \cap A \subseteq q_1^{\not\equiv}$. Similarly, we find that $[x, q]^{\not\equiv} \cap A \subseteq q_2^{\not\equiv}$. Every point of A noncollinear to $[x, q]$ is hence noncollinear to both q_1 and q_2 of T, and, by Corollary 6.2, noncollinear to $q \in T$, a contradiction. □

Lemma 8.64. *Let x be a point, let A be an A_2-plane containing a transversal T, and assume that x is linelike to at least two points of T. For every point p of T for which $p^{\not\equiv} \cap A \subseteq x^{\not\equiv}$, the point x is linelike or symplectic to all points of T_p, with T_p the unique transversal in A through p different from T.*

Proof. Let x, A and T be as stated. Let p be any point of T, and denote with T_p the transversal in A through p different from T. The point x is linelike with at least two points of T; denote those with q_1 and q_2. By Lemma 8.63, the point x is linelike or symplectic to all points of T, and in particular also to p.

Assume that $p^{\not\equiv} \cap A \subseteq x^{\not\equiv}$. Denote $\pi_p := \pi^A_{T_p}$, and let q be any point of $\pi_p \setminus T$. Note that q is special to p and noncollinear to x. First suppose that x is linelike to p. Then we can use Corollary 7.7 to obtain that x is indeed linelike or symplectic to all points of T_p. Next, suppose that x is symplectic to p. In particular, we find that p is different from q_1 and q_2, and hence that

$q_1, q_2 \in \pi_p$. We want to apply Lemma 7.58 to x, p and q. To do so, assume that x is special to q, let A_q be any A_2-plane that contains x and q, and denote with T_q the transversal in A_q through x and $[x, q]$. We have to prove that p is noncollinear to $\pi_q := \pi_{T_q}^{A_q}$.

Denote with T_x the transversal in A_q through x different from T_q, and let $y \neq x$ be any point of T_x. We claim that y is noncollinear to T_p. Since every point of T is linelike or symplectic to x, Lemma 7.44 implies that every point of T is noncollinear to T_x, and in particular, that y is noncollinear to T. Moreover, for $i = 1, 2$, the point q_i is linelike to x. If q_i was noncollinear to any point of $\pi_q \setminus T_x$, Lemma 7.55 implies that q_i would be noncollinear to the whole of π_q, which contains q, a contradiction. We hence obtain that $q_i^{\not\equiv} \cap \pi_q = T_x$, and in particular, that q_i is noncollinear to a unique point of the line qy, namely y. Then Axiom $(\mathrm{Im}_1)(i)$ implies that y is noncollinear to a unique point of qq_i, namely q_i. We use this to determine $y^{\not\equiv} \cap \pi_p$. The point y is hence collinear to q, noncollinear to T in π_p, and there are two lines through q in π_p for which y is noncollinear to exactly one point of that line. Lemma 6.30 then implies that $y^{\not\equiv} \cap \pi_p = T$. The set $y^{\not\equiv} \cap A$ is of course a conical hyperplane of A, so this very same lemma implies that y is noncollinear to $T_{\pi_p}^A = T_p$, which proves the claim.

The point $[p, q]$ is linelike to the point q and noncollinear to y, so Lemma 6.40 implies that $[p, q]$ is noncollinear to the line qy. The point p is noncollinear to y, so Proposition 7.11 on its turn then implies that p is noncollinear to the whole of qy. The point y was an arbitrary point on $T_x \setminus \{x\}$, so we indeed obtain that p is noncollinear to the dual affine plane π_q.

As desired, we can now apply Lemma 7.54 to x, p and q, and obtain that x is linelike or symplectic to $[p, q] \in T_p \setminus \{p\}$. The point x is hence linelike or symplectic to at least two points of T_p, namely p and $[p, q]$. Lemma 8.63 concludes the proof. $\square$

Lemma 8.65. *If a point x is linelike to at least two points of a transversal T contained in an A_2-plane A, then the set $x^{\not\equiv} \cap A$ is one of the following:*

1. *The set $p^{\not\equiv} \cap A$ for some point p of T. In this case, the points in A that are linelike or symplectic to x are exactly those points in A that are linelike to p.*
2. *The whole of A. In this case, the points in A linelike or symplectic to x are exactly the points of $\bar{\pi}_T^A$.*

Proof. We can apply Lemma 7.47 to x and A, and find that $x^{\not\equiv} \cap A$ is either A or $p^{\not\equiv} \cap A$ with p some point of T. We have to determine which points of A are linelike or symplectic to x. From Lemma 8.63 it is already clear that x is linelike or symplectic to all points of T.

First assume that $x^{\not\equiv} \cap A = p^{\not\equiv} \cap A$ for some point p of T. We can apply Lemma 8.64 and obtain that x is linelike or symplectic to all points of T_p. The set of points in A that are linelike to p, is exactly $T_p \cup T$, so x is indeed linelike to all points of A that are linelike to p. It follows from Lemma 7.46 that x is not linelike to any other point of A.

Next, assume that $x^{\not\equiv} \cap A = A$. Let p be any point of T, and denote with T_p the transversal in A through p different from T_p. The set $p^{\not\equiv} \cap A$ is contained in A, and hence also in $x^{\not\equiv}$. We can then again apply Lemma 8.64 and obtain that x is linelike or symplectic to all points of T_p. This point p was an arbitrary point of T, so we conclude that x is linelike or symplectic to all points of $\bar{\pi}_T^A$. Let y be any point of A not in $\bar{\pi}_T^A$, then y is collinear to at least one of the points q_1 and q_2. Without loss of generality, we may assume that it is collinear to q_1. Let l be the line through q_1 and y. The point x is linelike to a point of l and by Lemma 7.45 hence not linelike or symplectic to $y \in l$. This concludes the proof. $\square$

The next goal is to prove Corollary 8.9, for which we first gather two smaller results.

Lemma 8.66. *Let T be a transversal containing points q, q_1 and q_2, and let p be a point that is symplectic to q and special to q_1 and q_2. Then we have*

$$[p, q_2]^{\not\equiv} \cap p^{\equiv} = [p, q_1]^{\not\equiv} \cap p^{\equiv}.$$

Proof. The points $[p, q_1]$ and $[p, q_2]$ clearly play the same role (with q_1 and q_2 interchanged). It hence suffices to prove that every point collinear to $[p, q_1]$ and p, is also collinear to $[p, q_2]$. Let l be a line through p containing a point collinear to $[p, q_1]$. Then, by Proposition 7.11, the point q_1 is noncollinear to exactly two points of l. Axiom (Im_1) (ii) implies that $A := \langle q_1, l\rangle$ is an A_2-plane. This plane of course contains q_1, $p \in l$ and $[p, q_1]$. Let T_1 be the transversal in A that contains q_1 and $[p, q_1]$. We can apply Lemma 7.49 with q, q_2, q_1, p in the role of p', p, q_1, q_2, respectively, and find that $q_2^{\not\equiv} \cap A = x_{q_2}^{\not\equiv} \cap A$, with x_{q_2} some point of T_1 different from $[p, q_1]$. This set intersects l in exactly two points, that is, $|q_2^{\not\equiv} \cap l| = 2$. Proposition 7.11 then implies that $[p, q_2]$ is collinear to $l \setminus \{p\}$, which concludes the proof. $\square$

Lemma 8.67. *Let T be a transversal containing points q, q_1 and q_2, and let p be a point that is symplectic to q and special to q_1 and q_2. Then $[p, q_1] = [p, q_2]$.*

Proof. By Axiom (Im_3), it suffices to prove that $[p, q_1]^{\not\equiv} = [p, q_2]^{\not\equiv}$. The points $[p, q_1]$ and $[p, q_2]$ however, play the same role, so it suffices to prove that $[p, q_2]^{\not\equiv} \subseteq [p, q_1]^{\not\equiv}$, or equivalently, that $[p, q_1]^{\equiv} \subseteq [p, q_2]^{\equiv}$. To that end, let r be a point collinear to $[p, q_1]$. If r is collinear to p, it follows from Lemma 8.66 that r is also collinear to $[p, q_2]$. So suppose that r is noncollinear to p.

Let T_1 be a transversal through p and $[p, q_1]$, and let p' be a point of $T_1 \setminus \{p, [p, q_1]\}$. The point r is collinear to $[p, q_1]$ and noncollinear to p, so, by Corollary 6.2, it is collinear to $p' \in T_1$. We then apply Corollary 7.6 to the point p' and the line $r[p, q_1]$, and obtain an A_2-plane A that contains p', r and $[p, q_1]$. By Lemma 8.66, we have that $[p, q_1]^{\not\equiv} \cap p^{\equiv} = [p, q_2]^{\not\equiv} \cap p^{\equiv}$. In particular, we find that

$$[p, q_1]^{\not\equiv} \cap A \cap p^{\equiv} = [p, q_2]^{\not\equiv} \cap A \cap p^{\equiv}.$$

Let T_A be the transversal in A through p' and $[p, q_1]$. We apply Lemma 7.61 with $p, p', [p, q_1]$ in the role of q, q_1, q_2, respectively, and obtain that $p^{\not\equiv} \cap A = p_A^{\not\equiv} \cap A$ for some point p_A of T_A. Hence,

$$[p, q_1]^{\not\equiv} \cap A \cap p_A^{\equiv} = [p, q_2] \cap A \cap p_A^{\equiv},$$

for some point p_A of A. Using Lemma 7.57, we obtain that $[p, q_1]^{\not\equiv} \cap A = [p, q_2]^{\not\equiv} \cap A$. The point r is contained in A, and is hence collinear to $[p, q_2]$. This concludes the proof. $\square$

Corollary 8.9. *Let T be a transversal containing points q and x. If a point p is linelike or symplectic to q and special to x, then $[p, x]$ is linelike to all points of $T \setminus \{q\}$.*

Proof. Let q' be any point of $T \setminus \{x, q\}$. By Lemma 8.63, the point p is special to q'. We now use Lemma 8.67 with p, q, q', x in the role of p, q_1, q_2, q, respectively, and obtain $[p, q] = [p, q']$. The point $[p, q']$ is linelike to q', so $[p, q]$ is, too. $\square$

Remark 8.19. *If in Corollary 8.9, the point p is linelike to q, then q is linelike to both p and x, implying that $q = [p, x]$.*

Next, we consider an A_2-plane A containing two special points q_1 and q_2. We suppose that a point p is linelike to q_1 and symplectic to q_2, and try to produce extra points in A to which p is linelike. To that end, we distinguish between the case where p is collinear to some points of A (Lemma 8.68) and the case where p is noncollinear to all points of A (Lemma 8.69). We summarize the results in Corollary 8.10.

Lemma 8.68. *Let q_1 and q_2 be two special points, let A be an A_2-plane containing q_1 and q_2 and let T_1 be the transversal in A that contains q_1 and $[q_1, q_2]$. If some point p is linelike to q_1, symplectic to q_2 and if $p^{\not\equiv} \cap A = [q_1, q_2]^{\not\equiv} \cap A$, then p is linelike to all points of $T_1 \setminus \{[q_1, q_2]\}$.*

Proof. Let T be a transversal that contains both p and q_1, and let x_1 be any point of $T_1 \setminus \{q_1, [q_1, q_2]\}$. We claim that there exists some point x of T such that x is noncollinear to $x_1^{\not\equiv} \cap A$, while being linelike to two points of T. Indeed, let T_{x_1} be the transversal in A through x_1 different from T_1, and let y be any point of $\pi^A_{T_{x_1}} \setminus T_1$. Note that y is collinear to $q_1, [q_1, q_2]$ and p. By Corollary 6.4, the point y is noncollinear to at least one point of T, say x, which of course is different from p and q_1. The point q_2 is symplectic to $p \in T$ and special to $q_1 \in T$, so, by Corollary 8.9, the point $[q_1, q_2]$ is linelike to all points of $T \setminus \{p\}$, and in particular to x. The point x is hence linelike to the two points q_1 and $[q_1, q_2]$ of T_1. We can use this to determine $x^{\not\equiv} \cap A$; by Lemma 7.47, it is either equal to A or of the form $q^{\not\equiv} \cap A$ with q some point of T_1. In the latter case, we can use the fact that x is noncollinear to y to conclude that $q = x_1$. In any case, the point x is indeed noncollinear to $x_1^{\not\equiv} \cap A$, which proves the claim.

We can now apply Lemma 8.64 to x, and obtain that x is linelike or symplectic to all points of T_{x_1}. Let z be a point of T_{x_1}, then z is linelike or symplectic to x and special to q_1. Corollary 8.9 then implies that $[z, q_1] = x_1$ is linelike to p. This concludes the proof. □

Lemma 8.69. *Let q_1 and q_2 be two special points, let A be an A_2-plane containing q_1 and q_2 and let T_1 be the transversal in A that contains q_1 and $[q_1, q_2]$. If a point p is linelike to q_1, symplectic to q_2 and noncollinear to A, then p is linelike to each, but at most one, point of T_1.*

Proof. Let T be a transversal that contains q_1 and p, and let x be any point on $T \setminus \{q_1, p\}$. The point q_2 is symplectic to p and special to $q_1 \in T$, we can hence use Corollary 8.9 to obtain that $[q_1, q_2]$ is linelike to all points of $T \setminus \{p\}$, and in particular to x. We claim that x is linelike to all points of T_1. To that end, we first determine $x^{\not\equiv} \cap A$. By Corollary 6.2, every point noncollinear to p and q_1 is noncollinear to all points of T, and hence to x. This implies that $q_1^{\not\equiv} \cap A$ is contained in $x^{\not\equiv}$. Moreover, any point in $x^{\not\equiv} \cap A$ is noncollinear to x and p, and again by Corollary 6.2, also to q_1. We conclude that $x^{\not\equiv} \cap A = q_1^{\not\equiv} \cap A$. Take z in A linelike to q_1 but not on T_1. By Lemma 8.65, applied to x, we find that x is linelike or symplectic to z. We can now apply Lemma 8.68 to x, and find that x is linelike to all points of $T_1 \setminus \{q_1\}$. The point x is obviously also linelike to q_1, which proves the claim. The point x was an arbitrary point of $T \setminus \{p, q_1\}$, so we conclude that each point of $T \setminus \{p\}$ is linelike to each point of T_1.

Next, let x_1 be any point of $T_1 \setminus \{q_1\}$. By the previous paragraph, the point x_1 is linelike to all points of $T \setminus \{p\}$, so Lemma 8.63 implies that x_1 is linelike or symplectic to p. Let A' be an A_2-plane that contains T. Assume that x_1 is symplectic to p, we aim to prove that $x_1^{\not\equiv} \cap A'$ contains $p^{\not\equiv} \cap A'$. The point x_1 is linelike to at least two points of T, so, using Lemma 8.65, we find that $x_1^{\not\equiv} \cap A'$ is either A' or equals $x^{\not\equiv} \cap A'$, for some point x on T. In the latter case, x_1 is moreover linelike or symplectic to all points of T'_x, with T'_x the transversal in A' through x different from T. In the former case, we immediately obtain that $p^{\not\equiv} \cap A'$ is contained in $x_1^{\not\equiv}$. Therefore, suppose that we are in the latter case. Lemma 8.68 implies that x_1 is linelike to all points of $T \setminus \{x\}$. The point x_1 is assumed to be symplectic to $p \in T$, so this implies that $x = p$. We again obtain that $p^{\not\equiv} \cap A'$ is contained in $x_1^{\not\equiv}$.

Suppose for a contradiction that p is symplectic to another point x'_1 of $T_1 \setminus \{q_1\}$. Then, by the previous paragraph, both x_1 and x'_1 are noncollinear to the set $p^{\not\equiv} \cap A'$. Using Corollary 6.2, we find that every point of T is noncollinear to the set $p^{\not\equiv} \cap A'$, including q_1, a contradiction. We conclude that p is indeed linelike to each, but at most one, point of T_1. □

Corollary 8.10. *Let x be a point and T a transversal. If x is linelike to at least two points of T, it is linelike to all but at most one point of T. If x is moreover not linelike to $p \in T$, then $p^{\not\equiv} \cap A \subseteq x^{\not\equiv}$ for any A_2-plane A that contains T.*

Proof. Let A be an A_2-plane that contains T. Using Lemma 7.47, we see that $x^{\not\equiv} \cap A$ is either A or is of the form $p^{\not\equiv} \cap A$ with p some point of T. In any case, we can pick a point p of T such that $p^{\not\equiv} \cap A \subseteq x^{\not\equiv}$. Let q be a point in A linelike to p but not on T. By Lemma 7.58, the point x is linelike or symplectic to q. If x is linelike to q, we use Proposition 7.13 to see that x coincides with p, which proves the assertion. Suppose that x is symplectic to q. If $x^{\not\equiv} \cap A = p^{\not\equiv} \cap A$, then, by Lemma 8.68, the point x is linelike to $T \setminus \{p\}$. If on the other hand, $x^{\not\equiv} \cap A = A$, then we use Lemma 8.69 to conclude that x is linelike to all but at most one point of T. □

We of course want to obtain a stronger version of Corollary 8.10, namely that a point is linelike to zero, one or all points of a transversal. We will prove this in Proposition 8.14. Once again Axiom (Im_3) will play a crucial role. We first prove an in-between lemma.

Lemma 8.70. *Let p be a point, and let T_1 and T_2 be two transversals through p. If some point q_1 of T_1 is symplectic to some point q_2 of T_2 but linelike to all other points of T_2, then q_2 is linelike to all points of T_1 different from q_1.*

Proof. Let x_2 be any point of $T_2 \setminus \{q_2\}$. By assumption, this point is linelike to both p and q_1 of T_1, so Corollary 8.10 implies that there is at most one point on T_1 that is not linelike to x_2. In particular, we find some point $x_1 \in T_1 \setminus \{p, q_1\}$ that is linelike to x_2.

We claim that x_1 is linelike to q_2. Suppose not. Let A be an A_2-plane that contains T_2. Both points q_1 and x_1 are then linelike to p and x_2 of T_2, but not to q_2. Corollary 8.10 then implies that both q_1 and x_1 are noncollinear to $q_2^{\not\equiv} \cap A$. The points q_1 and x_1 are both contained in the transversal T_1, so Corollary 6.2 then implies that $q_2^{\not\equiv} \cap A$ is noncollinear to all points of T_1, in particular to $p \in A$, a contradiction. The claim follows. The point q_2 is linelike to both p and x_1 on T_1, and is symplectic to $q_1 \in T_1$. We can again use Corollary 8.10 and obtain that q_2 is linelike to $T_1 \setminus \{q_1\}$. □

Proposition 8.14. *If a point is linelike to at least two points of a transversal, it is linelike to all points of that transversal.*

Proof. Let q_1 be a point and T_2 a transversal, and suppose that q_1 is linelike to at least two points of T_2. By Corollary 6.2, we have that q_1 is linelike or symplectic to all points of T_2. Suppose for a contradiction that there is some point q_2 on T_2 that is symplectic to q_1.

By Corollary 8.10, the point q_1 is linelike to $T_2 \setminus \{q_2\}$. Take any point p on $T_2 \setminus \{q_2\}$, and let T_1 be a transversal that contains q_1 and p. By Lemma 8.70, the point q_2 is linelike to $T_1 \setminus \{q_1\}$. We will prove that $q_1^{\not\equiv} = q_2^{\not\equiv}$. The points q_1 and q_2 clearly play the same role, so it suffices to show $q_1^{\not\equiv} \subseteq q_2^{\not\equiv}$. Let x be any point noncollinear to q_1. We prove that x is also noncollinear to q_2.

First assume that x is linelike or symplectic to q_1. By Lemma 7.44, the point x is noncollinear to every point that is linelike to q_1. The point q_1 is, however, linelike to $T_2 \setminus \{q_2\}$. So, x is noncollinear to $T_2 \setminus \{q_2\}$. By Corollary 6.2, the point x is then noncollinear to all points of T_2, including q_2.

Now assume that x is special to q_1 and denote $x_1 := [x, q_1]$. The point x_1 is linelike to q_1, and is, by the reasoning above, hence noncollinear to all points of T_2. We claim that x_1 is linelike or symplectic to q_2. If x_1 is linelike or symplectic to at least two points of T_2, Lemma 8.63 implies that it is linelike or symplectic to all points of T_2, including q_2. If not, then there is a point $x_2 \in T_2 \setminus \{q_2\}$ that is special to x_1. In this case, both x_1 and x_2 are linelike to q_1, so Proposition 7.13 yields $q_1 = [x_1, x_2]$. Let A_x be any A_2-plane containing x_1 and x_2, and let T_x be the transversal in A_x through q_1 and x_2. The point q_1 is linelike to $T_2 \setminus \{q_2\}$, so by Lemma 8.70, the point q_2 is linelike to $T_x \setminus \{q_1\}$. Then Corollary 8.10 implies that q_2 is noncollinear to $q_1^{\not\equiv} \cap A_x$ and Lemma 8.65 then implies that q_2 is linelike or symplectic to all points of A_x that are linelike to q_1. The point x_1 being such a point, we indeed obtain that q_2 is linelike or symplectic to x_1. Using Lemma 7.44, we find that q_2 is noncollinear to all points that are linelike to x_1. The point x is of course linelike to x_1, so we conclude that x is indeed noncollinear to q_2.

We have obtained that $q_1^{\not\equiv} = q_2^{\not\equiv}$. Axiom (Im_3) implies $q_1 = q_2$, a contradiction. This concludes the proof of the proposition. □

We finish this subsection by gathering two corollaries which will be useful later on.

Corollary 8.11. *Let q_1 and q_2 be two special points. If a point p is linelike to q_1 and symplectic to q_2, it is linelike to $[q_1, q_2]$.*

Proof. Let T be a transversal through p and q_1. The point q_2 is symplectic to p and special to q_1, so, by Corollary 8.9, the point $[q_1, q_2]$ is linelike to $T \setminus \{p\}$. Using Proposition 8.14, we immediately obtain that $[q_1, q_2]$ is linelike to all points of T, and hence also to p. □

Corollary 8.12. *Let q_1 and q_2 be two special points. If a point p is symplectic to both q_1 and q_2, it is linelike or symplectic to $[q_1, q_2]$.*

Proof. Assume for a contradiction that p is special to $[q_1, q_2]$. For $i = 1, 2$, the point q_i is linelike to $[q_1, q_2]$ and symplectic to p. Using Corollary 8.11, we find that q_i is linelike to $[p, [q_1, q_2]]$. This point $[p, [q_1, q_2]]$ is hence linelike to both points q_1 and q_2, and, by Proposition 7.13, equals $[q_1, q_2]$, a contradiction. □

8.2. **Translation to the language of root filtration spaces.** We define a new line set $\mathscr{L}$ on $\mathscr{E}$, and define relations on $\mathscr{E}$ that will turn out to define the filtration on $(\mathscr{E}, \mathscr{L})$. One should note that these relations are actually just a rebranding of those considered in Definition 6.25.

Definition 8.27. *We define the following relations on $\mathscr{E}$:*

$$\begin{aligned}
\mathscr{E}_{-2} &:= \{(x, y) \mid x = y\},\\
\mathscr{E}_{-1} &:= \{(x, y) \mid x \text{ and } y \text{ are linelike}\},\\
\mathscr{E}_{0} &:= \{(x, y) \mid x \text{ and } y \text{ are symplectic}\},\\
\mathscr{E}_{1} &:= \{(x, y) \mid x \text{ and } y \text{ are special}\},\\
\mathscr{E}_{2} &:= \{(x, y) \mid x \text{ and } y \text{ are collinear}\}.
\end{aligned}$$

Let $\mathscr{L}$ denote the set of transversals of Y. We will denote with X the point-line geometry $(\mathscr{E}, \mathscr{L})$, equipped this filtration $\{\mathscr{E}_i\}_{-2 \le i \le 2}$.

Lemma 8.71. *The sets $\mathscr{E}_i$, with $-2 \le i \le 2$ provide a partition of $\mathscr{E} \times \mathscr{E}$ into five symmetric relations. Every element of $\mathscr{L}$ contains at least six points.*

Proof. It is clear from Definition 6.25 that the relations are symmetric. The fact that the relations form a partition follows from Lemma 6.37, Proposition 6.9 and Corollary 6.5. An element T of $\mathscr{L}$ is a transversal of Y, and is hence a transversal in some A_2-plane of Y, which is defined over a field with at least five elements, implying that T contains at least six points. □

Axioms (Rf_1) and (Rf_2) hold by definition in X, we hence start with proving Axiom (Rf_3).

Lemma 8.72. *Axiom (Rf_3) holds in X.*

Proof. Let x and y be special points, then, by Proposition 7.13, there is a unique point $[x, y]$ that is linelike to both x and y. We check that $[x, y]$ indeed satisfies the Axiom (Rf_3). Let z be any point in $\mathscr{E}_i(x) \cap \mathscr{E}_j(y)$, we aim to prove that z is contained in $\mathscr{E}_{\leq i+j}([x, y])$. It suffices to check this for $i \leq j$ and for $i + j \leq 1$.

- Suppose that $i = -2$. Then z equals x, which automatically means that y is special to z (i.e. $j = 1$), and that $[x, y]$ is indeed linelike to z.
- Suppose that $i = -1$. If $j = -1$, then, by Proposition 7.13, the point z equals $[x, y]$. If $j = 0$, then it follows from Corollary 8.11 that z is linelike to $[x, y]$. If $j = 1$, then it follows from Corollary 7.7 that z is linelike or symplectic to $[x, y]$. If $j = 2$, then it suffices to prove that z is noncollinear to $[x, y]$. But z is linelike to x and is by Lemma 7.44 noncollinear to all points linelike to x, in particular indeed to $[x, y]$.
- Suppose that $i = 0$. If $j = 0$, then it follows from Corollary 8.12 that z is linelike or symplectic to $[x, y]$. If $j = 1$, we have to prove that z is noncollinear to $[x, y]$. The point z is symplectic to x and is by Lemma 7.44 noncollinear to all points linelike to x, in particular to $[x, y]$.

□

Lemma 8.73. *Axioms* $(\mathrm{Rf}_4) - (\mathrm{Rf}_8)$ *hold in* X*.*

Proof. Axiom (Rf_4) holds by Lemma 7.45. Axiom (Rf_5) holds by Corollary 6.2 for $i = 1$, Lemma 8.63 for $i = 0$ and Proposition 8.14 for $i = -1$. Axiom (Rf_6) holds by Corollary 6.4. Any two collinear points of Y give rise to an element of $\mathscr{E}_2$, which implies that Axiom (Rf_7) holds. The space Y is connected, so in order to prove that X is connected, it suffices to find a path in X that connects any pair (x, y) of $\mathscr{E}_2$. Such a pair however lies on a line of Y, and is, by Lemma 6.42 contained in an A_2-plane. Inside such an A_2-plane, we of course find a path in X connecting x and y. This proves that Axiom (Rf_8) holds. □

8.3. **A tedious yet unavoidable detail: X forms a partial linear space.** In order to be able to conclude that X is a root filtration space, we still have to verify that X is a partial linear space, that is, two linelike points of Y are contained in a unique common transversal of Y. This is what we will do in this section. In order to do so, we will use several results from [4] that hold for (nondegenerate) root filtration spaces. Whenever we do so, the results do not depend on this root filtration space being a partial linear space.

Lemma 8.74. *For each* $(p, q) \in \mathscr{E}_0$ *and each* $(x, y) \in \mathscr{E}_{-1}$ *with* $x, y \in \mathscr{E}_{-1}(p) \cap \mathscr{E}_{-1}(q)$*, there is exactly one element of* $\mathscr{L}$ *that contains* x *and* y*.*

Proof. [4, Proposition 11], it is proved that the subspace $\mathscr{E}_{-1}(p) \cap \mathscr{E}_{-1}(q)$ satisfies the following properties:

(1) No point is linelike with all other points.
(2) Every point is linelike with one or all points of every transversal (contained in the subspace).

A space with these properties is however always a partial linear space (see for example [24, Theorem 7.3.6]). There is hence at most one element of $\mathscr{L}$ containing x and y that is itself contained in $\mathscr{E}_{-1}(p) \cap \mathscr{E}_{-1}(q)$. Since $\mathscr{E}_{-1}(p)$ and $\mathscr{E}_{-1}(q)$ are subspaces in X, any element of $\mathscr{L}$ that contains x and y is contained in $\mathscr{E}_{-1}(p) \cap \mathscr{E}_{-1}(q)$. □

Lemma 8.75. *If there is some* $(x, y) \in \mathscr{E}_{-1}$*, for which there exists some* $p, q \in \mathscr{E}_{-1}(x) \cap \mathscr{E}_{-1}(y)$ *for which* $(p, q) \in \mathscr{E}_0$*, then* X *is a partial linear space.*

Proof. [4, Theorem 13] implies that, if there is some point pair $(x,y) \in \mathscr{E}_{-1}$ for which this holds, this holds for all point pairs in $\mathscr{E}_{-1}$. The result then follows from Lemma 8.74. □

Lemma 8.76. *Let $(x,y) \in \mathscr{E}_{-1}$ be a pointpair such that $M_{x,y} := \mathscr{E}_{\leq -1}(x) \cap \mathscr{E}_{\leq -1}(y)$ consists of mutually linelike points, then for each v with $\emptyset \neq \mathscr{E}_{\leq -1}(v) \cap M_{x,y} \neq M_{x,y}$, the set $M_{x,y} \cap \mathscr{E}_{\leq}(v)$ is a proper hyperplane of $M_{x,y}$. In particular, $M_{x,y} \cap \mathscr{E}_1(v) \neq \emptyset$.*

Proof. It follows from [4, Lemma 16] that $\mathscr{E}_{\leq 0}(v) \cap M_{x,y}$ forms a proper subspace of $M_{x,y}$. In particular, there exists some element w of $M_{x,y} \setminus \mathscr{E}_{\leq 0}(v)$. There is some point of $M_{x,y}$ linelike to both v and w, implying that $w \in \mathscr{E}_1(v)$. □

We are now ready to prove that X is a partial linear space. The proof is based on the idea in [4, Lemma 17].

Lemma 8.77. *The point-line geometry X is a partial linear space.*

Proof. Assume that X is not a partial linear space, then there exists linelike points x and y, with two different transversals T_1 and T_2 through x and y. Without loss of generality, we find a point z_2 on $T_2 \setminus T_1$. Let A_1 be an A_2-plane through T_1. Lemma 7.61 yields $z_2^{\not\equiv} \cap A_1 = z_1^{\not\equiv} \cap A_1$ for some point z_1 of $T_1 \setminus \{x,y\}$. Select z in A_1 linelike to z_1 but not on T_1. Then, by Lemma 8.64, the point z_2 is linelike or symplectic to z. If it was linelike to z, then, by Proposition 7.13, $z_2 = [z,x] = z_1$, a contradiction to $z_2 \notin T_2$. We hence obtain that z_2 is symplectic to z.

The set X is not a partial linear space, so, by Lemma 8.75, the sets

$$M_{x,y} := \mathscr{E}_{\leq -1}(x) \cap \mathscr{E}_{\leq -1}(y) \text{ and } M_{z_1,z} := \mathscr{E}_{\leq -1}(z) \cap \mathscr{E}_{\leq -1}(z_1)$$

both consist of mutually linelike points. Note that $z_1, z_2 \in M_{x,y}$. The point z_2 is linelike to z_1 but not to z, so by Lemma 8.76, there is some point $w \in M_{z_1,z}$ that is special to z_2. Let T be a transversal through z and w. The point x is linelike to z_1 and special to z, so, by Lemma 8.76, it is linelike or symplectic to some point w' of T. Note that $w' \neq z$. This point w' is contained in T, which is contained in $M_{z_1,z}$ and is hence linelike to z_1. Since w' is linelike or symplectic to x and z_1 of T_1, it follows from Lemma 8.63 that w' is also linelike or symplectic to y. This point w' is hence also linelike or symplectic to $z_2 \in T_2$. The point z_2 is linelike or symplectic to both z and w' of T, and hence also to w, a contradiction. □

In particular, we obtain:.

Proposition 8.15. *The space X is a nondegenerate root filtration space.*

8.4. **Last step in the proof of the Main Theorem.** In this subsection, we finish the proof of the Main Theorem. In particular, we prove that $X = (\mathscr{E}, \mathscr{L})$ is a long root geometry, which is defined over a field (not $\mathbb{F}_3$ and of characteristic not two), and that Y is the imaginary geometry of X. We first apply the classification theorem of root filtration spaces.

Proposition 8.16 ([4] and [13])**.** *The point-line geometry X is a hexagonal root shadow space (possibly of infinite rank) which is defined over a field of characteristic not two, different from $\mathbb{F}_3$.*

We will use the correspondence of X and Y to prove that X is not just a root shadow space, but also a long root geometry. To that end, we first describe how we can reconstruct the partial linear space Y from X, cf. Construction 2.

Construction 3. *Two points p and q are opposite in X if and only if they are collinear in Y. In this case, we can reconstruct the line pq in Y as follows. Take any two paths (p,p_1,p_2,q) and (p,q_1,q_2,q) in X such that p_1 is special to q_1 and p_2 is special to q_2.*

$$pq = \{[x,y] \,|\, x \in p_1p_2, y \in q_1q_2 \text{ with } x \text{ special to } y\}.$$

Proof. Follows immediately from the fact that the points p, p_1, p_2, q, q_1 and q_2 generate an A_2-plane in Y. Note that this construction is independent of the chosen points p_i, q_i precisely because Y is a partial linear space. $\square$

Proposition 8.17. *X is a hexagonal long root geometry and Y is the imaginary geometry of X.*

Proof. It follows from Proposition 8.16 that X is a hexagonal root shadow space. If X would be a long root geometry, then it follows immediately from Construction 2 and Construction 3 that Y is the imaginary geometry of X. It hence suffices to show that X is a long root geometry.

First suppose that X does not have infinite rank. Then X is related to a thick, spherical building Δ of rank $n \geq 2$. If X is related to a spherical Moufang building Δ, one easily checks that the fact that Y is a partial linear space which can be obtained from X using Construction 3, implies that X is indeed a long root geometry. We prove that Δ is Moufang. If $n > 2$, this follows immediately. Suppose that $n = 2$, then X is either of type $\mathsf{A}_{2,\{1,2\}}$ or $\mathsf{G}_{2,1}$. In the former case, the points of X coincide with the points of an A_2-plane of Y, which is assumed to be defined over a field, implying that Δ is Moufang. In the latter case, X is a thick generalized hexagon, as noted in Remark 2.1. In the language of generalized polygons however, Construction 3 translates to the fact that the lines of X are *distance-3-regular* (see [20], also [28, Section 1.9.16]). Also, the existence of an A_2-plane through every pair of opposite lines implies readily that in the dual generalized hexagon, with the terminology of [21], all intersection sets have size 1. The main result of [21] (see also [28, Theorem 6.3.4]) now implies that Δ is Moufang, and hence that X is a long root geometry.

Next, suppose that X has infinite rank. If X is of type $\mathscr{E}(\mathbb{P}, \mathbb{H})$, it is automatically a long root geometry. If X is a line Grassmannian of a polar space Γ of infinite rank, one again checks that the fact that Y is a partial linear space which can be obtained from X using Construction 3, implies that Γ is an orthogonal polar space, and hence that X is a long root geometry. $\square$

This concludes the proof of the Main Theorem.

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