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Research Article

More on remoteness functions of exact slow NIM

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ABSTRACT. Given integer n and k such that $0 < k < n$ and n piles of stones, two players make moves alternately, reducing by every move exactly k piles from n, by one stone each. The player who must move but cannot is the loser. We call this game exact slow NIM and denote it $\text{NIM}^1_=(n,k)$. In recent articles [25, 26, 27, 28], a very simple explicit rule was suggested choosing a move in each position of this game. Furthermore, in case $n = k + 1$, it was proven that each move chosen by this rule reduces the Smith's remoteness function of the game exactly by one. This result allows us to compute the remoteness function efficiently, thus, solving the game. In the present article, we apply the same rule for two new cases: $k = 2, n = 4, 5$ and show that it still works efficiently: The remoteness function is reduced by one with almost every move. Exceptions are sparse and have a regular pattern. We suggest simple formulas explaining all known exceptions and, thus, solve the two considered new families of games too.

Keywords: Remoteness function, Sprague-Grundy function, slow NIM.

2020 Mathematics Subject Classification: 91A05, 91A46, 91A68.

1. INTRODUCTION

We assume that the reader is familiar with basic concepts of impartial game theory (see e.g., [1, 2] for an introduction) and also with the recent paper [26], where the Smith remoteness function of the game $\text{NIM}^1_=(n,k)$, so-called exact slow NIM, was computed for the case $n = k + 1$. Here, we extend this result and cover two more cases: $\text{NIM}^1_=(4,2)$ and $\text{NIM}^1_=(5,2)$.

1.1. **Exact Slow NIM.** Game exact slow NIM was introduced in [22] as follows: Given two integers n and k such that $0 < k \le n$ and n piles containing $x_1, \dots, x_n$ stones. By one move it is allowed to reduce any k piles by exactly one stone each. Two players alternate turns. One who has to move but cannot is the loser in the normal version of the game and s/he is the winner of its misère version. In [22], this game was denoted $\text{NIM}^1_=(n,k)$. Here, we simplify this notation to $\text{NIM}(n,k)$, for short. Notice that this game is naturally partitioned into k independent subgames, since $x_1 + \dots + x_n \mod k$ is an invariant: It cannot be changed by a move.

Game $\text{NIM}(n,k)$ is trivial if $k = 1$ or $k = n$. In the first case it ends after $x_1 + \dots + x_n$ moves and in the second one–after $\min(x_1, \dots, x_n)$ moves. In each case everything depends only on the parity of the corresponding number and nothing on players' skills. All other cases are more complicated.

The game was solved for $k = 2$ and $n \le 6$. In [24], an explicit formula for the Sprague-Grundy (SG) function was found for $n \le 4$, for both the normal and misère versions. This formula allows us to compute the SG function in time linear in n, k and $\omega(x) = \sum_{i=1}^{n} \log(|x_i| + 1)$. Then, in [8] the P-positions of the normal version were found for $n \le 6$. For the subgame

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with even $x_1 + \ldots + x_n$ a very simple formula for the P-positions was found, which allows to verify in linear time if x is a P-position and, if not, to find a move from it to a P-position. The subgame with odd $x_1 + \ldots + x_n$ is more difficult. Still a (more sophisticated) formula for the P-positions was found, providing a recognition algorithm linear in time n, k and $\omega(x)$.

1.2. **Case $n = k + 1$, the Normal Version.** In [26], the normal version was solved in case $n = k + 1$ by the following simple rule:

- (o) if all piles are odd, keep a largest one and reduce all other,
- (e) if there exist even piles, keep a smallest one of them and reduce all other.

This rule is well-defined: It defines a unique move $x \to x'$ in each position x[1]. This rule and the corresponding moves is called the M-rule and M-moves; the sequence of successive M-moves is called the M-sequence. Of course, $n > 1$ is required. If $n = 1$, then x_1 will reach an even value in at most one M-move and stop. Since this case is trivial, we can assume that $n > 1$ without any loss of generality.

It is also easily seen that no M-move can result in a position whose all entries are odd. Hence, for an M-sequence, part (o) of the M-rule can be applied at most once, at the beginning; after this only part (e) works.

Given a position $x = (x_1, \ldots, x_n)$, assume that both players follow the M-rule and denote by $\mathcal{M}(x)$ the number of moves from x to a terminal position. In [26], it was proven that $\mathcal{M} = \mathcal{R}$, where $\mathcal{R}$ is the classical remoteness function introduced by Smith [39]. Thus, M-rule solves the game moreover, it allows to win as fast as possible in an N-position and to resist as long as possible in a P-position.

An algorithm computing $\mathcal{M} = \mathcal{R}$ (and in particular, the P-positions) in time polynomial in n, k and $\omega(x)$ was given in [26]. Let us also note that an explicit formula for the P-positions is known only for $n \leq 4$ and already for $n = 3$ it is pretty complicated; see [24] and also [26, Appendix].

1.3. **Related Works on NIM.** By definition, the present game NIM(n, k) is the exact slow version of the famous Moore's NIM$_k$ [34]. In the latter game a player, by one move, reduces arbitrarily (not necessarily by one stone) at most k piles from n. The case $k = 1$ corresponds to the classical NIM whose P-position was found by Bouton [7] for both the normal and misère versions.

Remark 1.1. *Actually, the Sprague-Grundy (SG) values of NIM were also computed in Bouton's paper, although were not defined explicitly in general. This was done later by Sprague* [41] *and Grundy* [18] *for arbitrary disjunctive compounds of impartial games; see also* [11, 39].

In fact, the concept of a P-position was also introduced by Bouton in [7]*, but only for the (acyclic) digraph of NIM, not for all impartial games. In its turn, this is a special case of the concept of a kernel, which was introduced for arbitrary digraphs by von Neumann and Morgenstern* [35]*. Also the misère version was introduced by Bouton in* [7]*, but only for NIM, not for all impartial games. The latter was done by Grundy and Smith* [17]*; see also* [11, 19, 20, 23, 39].

Moore [34] obtained an elegant explicit formula for the P-positions of NIM$_k$ generalizing the Bouton's case $k = 1$. Even more generally, the positions of the SG-values 0 and 1 were efficiently characterized by Jenkins and Mayberry [30]; see also [4, Section 4]. Also in [30], the SG function of NIM$_k$ was computed explicitly for the case $n = k + 1$ (in addition to the case $k = 1$). In general, no explicit formula, nor even a polynomial algorithm computing the SG-values (larger than 1) is known. The smallest open case: 2-values for $n = 4$ and $k = 2$.

[1]Obviously, permuting the piles with the same number of stones we keep the game unchanged.

The remoteness function of NIM_k was recently studied in [5] along with some other classical games: Euclid [9, 10, 16, 19, 20, 31, 32, 33, 37] and Wythoff [43] with some of its generalizations [6, 13, 14, 15, 16, 21, 33, 36, 38, 40, 42, 44]. Further generalizations of the exact slow NIM were considered in [25, 26, 27, 28].

Let us also mention the exact (but not slow) game $\text{NIM}_=(n,k)$ [3, 4] in which, by one move, exactly k from n piles are reduced, but by an arbitrary number of stones each, rather than by 1. The SG-function was efficiently computed in [3, 4] for $n \le 2k$. Otherwise, even a polynomial algorithm looking for the P-positions is not known (unless $k = 1$, of course). The smallest open case is $n = 5$ and $k = 2$.

2. New Results

In this paper, we consider the remoteness function of the exact slow $\text{NIM}(n,k)$ for $(n,k) = (4,2)$ and $(n,k) = (5,2)$. We extend the M-rule, introduced in [26] for the case $n = k+1$, to any k such that $1 < k < n$. The obtained GM-rule works efficiently for $\text{NIM}(4,2)$ and $\text{NIM}(5,2)$, that is, it frequently reduces the remoteness function $\mathcal{R}(x)$ by 1. The exceptions are sparse. Moreover, all exceptions found by computer form a regular pattern described by simple formulas. However, these formulas are not proven.

Let us recall that the remoteness function $\mathcal{R}(x)$ has the following properties: The player who makes a move from x can guarantee a victory in at most $\mathcal{R}(x)$ moves whenever $\mathcal{R}(x)$ is odd, that is, x is an N-position, and s/he can guarantee that the play will last at least $\mathcal{R}(x)$ moves if $\mathcal{R}(x)$ is even, in this case x is a P-position. In both cases the corresponding optimal move reduces $\mathcal{R}(x)$ by 1.

2.1. **GM-Rule for NIM(n,k) and Its Basic Properties.** Fix integers n,k such that $0 < k < n$, and consider a nonnegative integer vector $x = (x_1, \dots, x_n)$. Assume that x is defined up to a permutation of its entries. Hence, without loss of generality, we will require monotonicity: $x_1 \le \dots \le x_n$.

Denote by $m = m(x)$ the number of even entries of x. If $m \ge n-k$, choose the smallest $n-k$ of even entries. If $m < n-k$, choose all m even entries and add arbitrary $n-k-m$ others, for example the largest ones. Note that in both cases the choice may be not unique, since entries of x may be equal. In case of such ambiguity we use the following:

Tie-breaking rule: Always choose the rightmost entries, that is, x_i with the largest i.

The chosen $n-k$ entries of x will be called pivotal and their indices-pivots.

Given n,k and x, define a discrete dynamical system by the next GM-(n,k)-rule. By one step transform x to x' as follows: the $n-k$ pivotal entries of x are not changed, while the remaining k, non-pivotal, entries are reduced by 1. This transformation $x \to x'$ and the corresponding sequence $x = x^0 \to x^1 \to \dots \to x^j \to \dots$ will be called GM-(n,k)-move and GM-(n,k)-sequence, respectively. We will omit the arguments (n,k) whenever their values are unambiguous from the context and simply write GM-rule, GM-move, and GM-sequence, for short.

By definition, a GM-move $x \to x'$ is unique for any given x. We begin with several simple properties of a GM-move $x \to x'$.

Observation 2.1. *GM-moves respect monotonicity, that is, $x'_1 \le \dots \le x'_n$ whenever $x_1 \le \dots \le x_n$.*

Proof. It follows immediately from the tie-breaking rule. □

Recall that $m = m(x)$ denotes the number of even entries in x.

Observation 2.2. *GM-moves respect inequality $n-k \le m(x)$, that is, $n-k \le m(x')$ whenever $n-k \le m(x)$.*

Proof. Let a move $x \to x'$ de-pivot $i \in [n] = \{1, \dots, n\}$, that is, i is a pivot of x but not of x'. Then, x_i is even and $x'_i = x_i$, since i is a pivot of x. In contrast, i is not a pivot of x'. Hence, by the tie-breaking rule, there exist $n-k$ pivots of x' each of which is either smaller than i, or larger than i, but its pivotal value equals $x_i = x'_i$. Thus, i may stop to be a pivot, after a GM-move $x \to x'$, only if a replacing pivot appears in x'. □

Observation 2.3. *For an arbitrary x^0, in the corresponding (unique) GM-sequence $x^0 \to x^1 \to \dots \to x^j \to \dots$ inequality $n-k \le m(x^j)$ holds for any $j \ge 1$.*

Proof. Suppose that $n-k > m = m(x^0)$. By definition, m even pivotal entries of x^j are not changed, while $n-k-m$ odd pivotal entries are reduced by 1, by the (unique) GM-move. Clearly, in one GM-move each of these $n-k-m$ entries becomes even. Then, by the GM-rule, it becomes a pivot. Moreover, it remains in this status in x^j until $n-k \ge m(x^j)$. Thus, the set of even pivots is expanding until their number reaches $n-k$, which happens just in one GM-move. After this even pivots may be replaced, but still, by Observation 2.2, their number remains $n-k$. In other words, the difference $m(x^j) - (n-k)$ in one GM-move becomes and then remains non-negative. □

Yet, it may decrease after the GM-move $x^j \to x^{j+1}$ for some j, because some pivots of x^j may disappear after this move, if $m(x^j) \ge n-k$. Also, we will assume that $m(x) \ge n-k$ holds, that is, all pivotal entries are even. By Observation 2.2 this property always holds after a GM-move. However, we do not know when this GM-move is optimal.

2.2. **Exact Slow NIM**$(4,2)$**.** Our computations show that for this game a GM-move reduces the remoteness value $\mathcal{R}(x)$ by 1 (provided $m(x) \ge n-k = 2$) except the following cases:

$$x = (x_1, x_2, x_3, x_4) = (2a_1+1, 2a_2, 2a_3, 2(a_3+a_2-a_1)+),$$

where $0 \le a_1 < a_2 \le a_3$. Here, $y+$ means any integer number not smaller than y. Note that $m(x) \ge n-k = 2$. Furthermore, $\mathcal{R}(x) = 2(a_3+a_2)-1$ is always odd. In other words, only winning positions may be exceptional. In these exceptional positions the entries x_1 and x_4 are not changed by a (unique) move reducing $\mathcal{R}(x)$ by 1, while the GM-move $x \to x'$ has pivots 2 and 3.

For the GM-move we have $\mathcal{R}(x') = \mathcal{R}(x)+1$, whenever $x_4 > 2(a_3+a_2-a_1)$. This means that the GM-move is still optimal but it delays the fastest winning by 2 moves. If $x_4 = 2(a_3+a_2-a_1)$ then $\mathcal{R}(x') = \mathcal{R}(x)$, which means that the GM-move is losing in a winning position.

2.3. **Exact Slow NIM**$(5,2)$**.** Our computations show that for this game a GM-move reduces the remoteness value $\mathcal{R}(x)$ by 1 (provided $m(x) \ge n-k = 3$) except for the following cases:

$$x = (x_1, x_2, x_3, x_4, x_5) = (2a_1, 2a_2+1, 2a_3, 2a_4, 2(a_4+a_3-a_2+a_1)+),$$

where $0 \le a_1 \le a_2 < a_3 \le a_4$. Note that $m(x) \ge n-k = 3$. Furthermore, $\mathcal{R}(x) = 2(a_4+a_3+a_1)-1$ is always odd. In other words, only winning positions may be exceptional. In these exceptional positions the entries x_1, x_2, and x_5 are not changed by a (unique) move reducing $\mathcal{R}(x)$ by 1, while the GM-move $x \to x'$ has pivots $1, 3$ and 4.

For the GM-move we have $\mathcal{R}(x') = \mathcal{R}(x)+1$, whenever $x_5 > 2(a_4+a_3-a_2+a_1)$. This means that the GM-move is still optimal but it delays the fastest winning by 2 moves. If $x_5 = 2(a_4+a_3-a_2+a_1)$ then $\mathcal{R}(x') = \mathcal{R}(x)$, which means that the GM-move is losing in a winning position.

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