

Research Article

# Korovkin approximation of non-convex set-valued continuous functions

*Dedicated to Professor Ioan Raşa, on the occasion of his 75th birthday*

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**ABSTRACT.** In this paper, we continue the study of Korovkin approximation in cones of Hausdorff continuous set-valued functions. Besides some general results in the context of set-valued functions with compact and convex values, we consider for the first time the possibility of lacking the convex assumption using different methods, namely parametrizations and starshaped conditions. The interest of the new results consist also in considering local conditions which depend on directions of  $\mathbb{R}^d$ .

**Keywords:** Korovkin approximation, approximation of set-valued functions, non convex set-valued functions.

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## 1. INTRODUCTION AND NOTATION

Korovkin-type approximation theory has been successfully extended to sequences of monotone operators between cones of Hausdorff continuous set-valued functions since the pioneering work by Keimel and Roth [10, 11].

Thereafter, many papers have been subsequently devoted to this study and to different problems arising in connection with this kind of approximation (see, e.g. [7, 8]). The most recent results in this direction are collected in [5, 6].

Recently, one of the most interesting topics in Korovkin approximation has been the study of set-valued functions with compact but not necessarily convex values. However, in this setting a Korovkin-type approximation theory involving these kind of functions has not been yet developed.

In [9], some cases of particular interest have been indicated. In particular, the authors suggest to consider the possibility of replacing the convex condition with suitable subsets which admit a parametrization of their boundary or, alternatively, starshaped convex sets.

Here, our aim is to state some Korovkin type results in both these cases. We begin with some notations. Let  $\mathbf{B}^d$  denote the closed unit ball in  $\mathbb{R}^d$  and let

- $\mathcal{F}(\mathbb{R}^d)$  the cone of all non empty subsets of  $\mathbb{R}^d$ ,
- $\mathcal{K}(\mathbb{R}^d)$  the cone of all non empty compact subsets of  $\mathbb{R}^d$ ,
- $\mathcal{K}_c(\mathbb{R}^d)$  the cone of all non empty compact convex subsets of  $\mathbb{R}^d$ .

All the above cones are endowed with the natural addition and multiplication by positive scalars.

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In  $\mathcal{F}(\mathbb{R}^d)$  it is also defined the Hausdorff distance

$$d_H(A, B) := \max \left\{ \sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\| \right\}.$$

If  $X$  is a Hausdorff topological space and  $\mathcal{A}(\mathbb{R}^d)$  denotes one of the above cones, we consider the cone  $C(X, \mathcal{A}(\mathbb{R}^d))$  of all continuous set-valued functions defined in  $X$  and with values in  $\mathcal{A}(\mathbb{R}^d)$ .  $C(X, \mathcal{A}(\mathbb{R}^d))$  is endowed with the Hausdorff uniform distance and is naturally ordered by inclusion.

If  $F \in C(X, \mathcal{F}(\mathbb{R}^d))$ , we shall denote by  $\text{Sel}(F)$  the set of all continuous selections of  $F$ . If  $F \in C(X, \mathcal{K}_c(\mathbb{R}^d))$  then  $\text{Sel}(F)$  is a convex subset of  $C(X, \mathcal{K}_c(\mathbb{R}^d))$ . Moreover, an operator  $L : C(X, \mathcal{A}(\mathbb{R}^d)) \rightarrow C(X, \mathcal{A}(\mathbb{R}^d))$  is called linear if  $L(F+G) = L(F)+L(G)$  and  $L(\lambda F) = \lambda L(F)$  for every  $F, G \in C(X, \mathcal{A}(\mathbb{R}^d))$  and  $\lambda \geq 0$  and monotone if the condition  $F \leq G$  (i.e.,  $F(x) \subset G(x)$  for every  $x \in X$ ) implies  $L(F) \leq L(G)$ . Finally, we recall the definition of Korovkin subset of  $C(X, \mathcal{A}(\mathbb{R}^d))$ . For the sake of simplicity, we shall refer only to the identity operator.

A subset  $\mathcal{H}$  of  $C(X, \mathcal{A}(\mathbb{R}^d))$  is a Korovkin subset in  $C(X, \mathcal{A}(\mathbb{R}^d))$  with respect to equicontinuous nets of monotone operators (respectively, monotone linear operators) if, for every equicontinuous net  $(L_i)_{i \in I}^{\leq}$  of monotone (respectively, monotone linear) operators from  $C(X, \mathcal{A}(\mathbb{R}^d))$  into itself satisfying  $\lim_{i \in I} L_i(H) = H$  for every  $H \in \mathcal{H}$ , we also have  $\lim_{i \in I} L_i(F) = F$  for every  $F \in C(X, \mathcal{A}(\mathbb{R}^d))$ .

One of the main results concerning the compact convex case is the following.

**Theorem 1.1.** *Let  $X$  be a compact Hausdorff topological space and let  $\mathcal{H}$  be a subset of  $C(X, \mathcal{K}_c(\mathbb{R}^d))$  such that, for every  $F \in C(X, \mathcal{K}_c(\mathbb{R}^d))$ ,  $x_0 \in X$  and  $\varepsilon > 0$ , there exists  $H \in \mathcal{H}$  satisfying the following conditions*

$$(1.1) \quad F \leq H, \quad H(x_0) \subset F(x_0) + \varepsilon \cdot \mathbf{B}.$$

Then,  $\mathcal{H}$  is a Korovkin system in  $C(X, \mathcal{K}_c(\mathbb{R}^d))$ .

From the above result it is possible to obtain different criteria and examples of Korovkin subsets. Hence, we consider the possibility of extending this result to set-valued functions which do not take their values in  $\mathcal{K}_c(\mathbb{R}^d)$ . In the next sections, we consider different methods.

## 2. PARAMETRIZATION OF COMPACT SUBSETS

The first method under consideration is based on the possibility of considering a parametrization of convex compact which can be extended to more general non convex subsets. In this way we can try to obtain Korovkin approximation results for the parametrized sets.

A natural way to parametrize a compact convex subset is to consider its support function (see, e.g., [8, 13]). Namely, denote by  $S^{d-1}$  be the unit sphere surface in  $\mathbb{R}^d$  ( $S^{d-1} := \mathbf{B}^d \setminus \overset{\circ}{\mathbf{B}}^d$ ).

The support function  $\delta^* : \mathcal{K}_c(\mathbb{R}^d) \times S^{d-1} \rightarrow \mathbb{R}$  is defined by setting, for every  $A \in \mathcal{K}_c(\mathbb{R}^d)$  and every direction  $d \in S^{d-1}$ ,

$$\delta^*(A, d) := \max_{a \in A} \langle d, a \rangle.$$

Many properties of the map  $\delta^*$  are well-known and are listed, e.g., in [8, 9, 13]. In particular,  $\delta^*(A, \cdot)$  is continuous for every  $A \in \mathcal{K}_c(\mathbb{R}^d)$ . Moreover, a subset  $A \in \mathcal{K}_c(\mathbb{R}^d)$  may be recovered from its support function and is given by

$$A = \bigcap_{d \in S^{d-1}} \{x \in \mathbb{R}^d \mid \langle d, x \rangle \leq \delta^*(A, d)\}.$$

Now, if  $X$  be a compact Hausdorff topological space, we can associate to every set-valued function  $F \in C(X, \mathcal{K}_c(\mathbb{R}^d))$  its parametrization  $p_F : X \times S^{d-1} \rightarrow \mathbb{R}$  defined by setting, for every  $(x, d) \in X \times S^{d-1}$ ,

$$p_F(x, d) := \delta^*(F(x), d) \quad (= \max_{y \in F(x)} \langle d, y \rangle).$$

It is well-known that  $f$  is Hausdorff continuous if and only if  $p_F$  is (uniformly) continuous on  $X \times S^{d-1}$ . Thus, we obtain a continuous operator  $P : C(X, \mathcal{K}_c(\mathbb{R}^d)) \rightarrow C(X \times S^{d-1}, [0, +\infty[)$  defined by

$$P(F) := p_F, \quad F \in C(X, \mathcal{K}_c(\mathbb{R}^d)).$$

Since convex compact subsets may be recovered from their support function, the map  $P$  is invertible. The above operator allows us to study Korovkin approximation in  $C(X, \mathcal{K}_c(\mathbb{R}^d))$  using Korovkin approximation results in  $C(X \times S^{d-1}, \mathbb{R})$ .

As regards to the space  $C(X \times S^{d-1}, [0, +\infty[)$ , we know many examples of Korovkin systems with respect to positive operators both in  $C(X, \mathbb{R})$  than in  $C(S^{d-1}, \mathbb{R})$  (see, e.g. [1, Chapter 4] and [2]) and hence we can easily obtain Korovkin systems in  $C(X \times S^{d-1}, \mathbb{R})$  (see [1] for a rather complete treatment of Korovkin systems on a product space). We also observe that many classical examples are just made up by positive functions. At this point, we can fix a Korovkin system  $D$  in  $C(X \times S^{d-1}, \mathbb{R})$  consisting of positive functions and consider the following subset of  $C(X, \mathcal{K}_c(\mathbb{R}^d))$

$$\mathcal{D} := \{F \in C(X, \mathcal{K}_c(\mathbb{R}^d)) \mid p_F \in D\}.$$

We have the following result.

**Theorem 2.2.** *The set  $\mathcal{D}$  is a Korovkin system in  $C(X, \mathcal{K}_c(\mathbb{R}^d))$  with respect to equicontinuous nets of monotone operators.*

*Proof.* Let  $(L_i)_{i \in I}^{\leq}$  be an equicontinuous net of monotone operators from  $C(X, \mathcal{K}_c(\mathbb{R}^d))$  into itself satisfying  $\lim_{i \in I} L_i(H) = H$  for every  $H \in \mathcal{D}$ . For every  $i \in I$ , consider the operator  $M_i : C(X \times S^{d-1}, [0, +\infty[) \rightarrow C(X \times S^{d-1}, [0, +\infty[)$  defined by  $\xi \in C(X \times S^{d-1}, [0, +\infty[)$  and  $(x, d) \in X \times S^{d-1}$ ,

$$M_i(\xi)(x, d) := \delta^*(L_i(P^{-1}(\xi))(x), d).$$

In this way, we obtain a net  $(M_i)_{i \in I}^{\leq}$  of operators from  $C(X \times S^{d-1}, [0, +\infty[)$  into itself which are obviously monotone and satisfy  $\lim_{i \in I} M_i(\eta) = \eta$  for every  $\eta \in D$ . Since  $D$  is a Korovkin system in  $C(X \times S^{d-1}, [0, +\infty[)$ , we have also  $\lim_{i \in I} M_i(\xi) = \xi$  for every  $\xi \in C(X \times S^{d-1}, [0, +\infty[)$ . This yields, for every  $F \in C(X, \mathcal{K}_c(\mathbb{R}^d))$ ,  $\lim_{i \in I} P(L_i(F)) = P(F)$  and finally  $\lim_{i \in I} L_i(F) = F$ , which completes the proof.  $\square$

The above representation has been very useful in order to use well-known results for single-valued functions for the Korovkin approximation of set-valued Hausdorff continuous functions with compact and convex values.

However, we cannot use the above representation to extend our results to the Korovkin approximation of set-valued functions with non convex values. Indeed, if we consider a compact non convex subset  $K$  of  $\mathbb{R}^d$ , we can still consider the support function of  $K$ , but if we try to recover  $K$  from the support function we obtain the convex hull of  $K$  instead of  $K$ . Hence, this representation is useful only in the context of convex subsets of  $\mathbb{R}^d$ .

In order to overcome this problem, we consider a different representation which allows to recover uniquely every compact subset of  $\mathbb{R}$ . Namely, we consider the squared distance function  $\rho^* : \mathcal{K}(\mathbb{R}^d) \times \mathbb{R}^d \rightarrow [0, +\infty[$  defined by setting, for every  $K \in \mathcal{K}(\mathbb{R}^d)$  and  $y \in \mathbb{R}^d$ ,

$$\rho^*(K, y) := \inf_{x \in K} \|x - y\|^2.$$

The squared distance function has many regularity properties since it is connected to the Moreau-Yosida regularization of the indicator function. Moreover,  $\rho^*(K, \cdot)$  is a convex function whenever  $K \in \mathcal{K}_c(\mathbb{R}^d)$ .

Now, consider a set-valued Hausdorff continuous function  $F : X \rightarrow \mathcal{K}(\mathbb{R})$ ; we can associate the function  $\rho_F : X \times \mathbb{R}^d \rightarrow \mathbb{R}$  defined by setting, for every  $x \in X$  and  $y \in \mathbb{R}^d$ ,

$$\rho_F(x, y) := \rho^*(F(x), y) \quad (= \inf_{z \in F(x)} \|z - y\|^2).$$

The function  $\rho_F$  is continuous; indeed, if we fix  $(x_0, y_0) \in X \times \mathbb{R}^d$ , for every  $(x, y) \in X \times \mathbb{R}^d$  we have

$$\begin{aligned} |\rho_F(x, y) - \rho_F(x_0, y_0)| &\leq |\rho_F(x, y) - \rho_F(x_0, y)| + |\rho_F(x_0, y) - \rho_F(x_0, y_0)| \\ &= \left| \inf_{z \in F(x)} \|z - y\|^2 - \inf_{z \in F(x_0)} \|z - y\|^2 \right| + |\rho^*(F(x_0), y) - \rho^*(F(x_0), y_0)| \end{aligned}$$

and the first addend in the last sum tends to 0 as  $x \rightarrow x_0$  due to the Hausdorff continuity of  $F$ , while the second addend tends to 0 as  $y \rightarrow y_0$  due to the continuity of squared distance function. With the same argument it can be shown that  $\rho_F$  is uniformly continuous on every compact subset  $X \times B$ , where  $B$  is a closed ball containing the image  $\bigcup_{x \in X} F(x)$  of  $F$ . Now, consider the set

$$\Gamma := \{\rho_F \mid F \in C(X, \mathcal{K}(\mathbb{R}^d))\}.$$

We have  $\Gamma \subset C(X \times \mathbb{R}^d, \mathbb{R})$  and we can define on  $\Gamma$  a cone structure by defining

$$\rho_F \oplus \rho_G := \rho_{F+G}, \quad \lambda \odot \rho_F := \rho_{\lambda F}, \quad F, G \in C(X, \mathcal{K}(\mathbb{R}^d)), \quad \lambda \geq 0,$$

and assuming  $\rho_F \preceq \rho_G$  if and only if  $G \leq F$  (the reversed inclusion is justified by the fact that the distance is smaller from larger sets).

Finally, it can be readily seen that a sequence  $(F_n)_{n \in \mathbb{N}}$  in  $C(X, \mathcal{K}(\mathbb{R}^d))$  converges to a set-valued function  $F \in C(X, \mathcal{K}(\mathbb{R}^d))$  with respect to the uniform Hausdorff distance is and only if the sequence  $(\rho_{F_n})_{n \in \mathbb{N}}$  converges to  $\rho_F$  uniformly on compact subsets  $X \times B \subset X \times \mathbb{R}^d$ . These properties allows us to state the following main result of this section.

**Theorem 2.3.** *Let  $X$  be a compact Hausdorff topological space and let  $\mathcal{H}$  be a subset of  $C(X, \mathcal{K}(\mathbb{R}^d))$  such that, for every  $F \in C(X, \mathcal{K}(\mathbb{R}^d))$ ,  $x_0 \in X$ ,  $y_0 \in \mathbb{R}^d$  and  $\varepsilon > 0$ , there exist  $G, H \in \mathcal{H}$  satisfying the following conditions*

$$(2.2) \quad G \leq F \leq H, \quad d(G(x_0), y_0) < d(H(x_0), y_0) + \varepsilon.$$

*Then,  $\mathcal{H}$  is a Korovkin system in  $C(X, \mathcal{K}(\mathbb{R}^d))$  with respect to equicontinuous nets of monotone operators.*

*Proof.* Let  $(L_i)_{i \in I}^{\preceq}$  be an equicontinuous net of monotone operators from  $C(X, \mathcal{K}(\mathbb{R}^d))$  into itself satisfying  $\lim_{i \in I} L_i(H) = H$  for every  $H \in \mathcal{H}$ . We observe that, for every  $\rho \in \Gamma$ , there exists a unique  $F \in C(X, \mathcal{K}(\mathbb{R}^d))$  such that  $\rho = \rho_F$ . This allows us to define, for every  $i \in I$ , the operator  $\tau_i : \Gamma \rightarrow \Gamma$  by setting, for every  $F \in C(X, \mathcal{K}(\mathbb{R}^d))$ ,

$$\tau_i(\rho_F) := \rho_{L_i(F)}.$$

The operator  $\tau_i$  is monotone since, for every  $F, G \in C(X, \mathcal{K}(\mathbb{R}^d))$ ,

$$\rho_F \preceq \rho_G \Rightarrow G \leq F \Rightarrow L_i(G) \leq L_i(F) \Rightarrow \tau_i(\rho_F) = \rho_{L_i(F)} \preceq \rho_{L_i(G)} = \tau_i(\rho_G)$$

(in the same manner, it can be shown that  $\tau_i$  is linear if  $L_i$  is linear). Hence, we obtain a net  $(\tau_i)_{i \in I}^{\preceq}$  of monotone operators on  $\Gamma$  which satisfies  $\lim_{i \in I} \tau_i(\rho_H) = \rho_H$  uniformly on compact subsets  $X \times B \subset X \times \mathbb{R}^d$  for every  $H \in \mathcal{H}$ .

Let  $F \in C(X, \mathcal{K}(\mathbb{R}^d))$  and  $\varepsilon > 0$ . From our assumptions, for every  $(x_0, y_0) \in X \times \mathbb{R}^d$ , there exist  $G, H \in \mathcal{H}$  satisfying the following conditions

$$G \leq F \leq H, \quad d(G(x_0), y_0) < d(H(x_0), y_0) + \min \left\{ \varepsilon, \frac{\varepsilon}{2d(F(x_0), y_0) + \varepsilon} \right\}.$$

Thus, we have  $\rho_H \preceq \rho_F \preceq \rho_G$ ; moreover, since

$$d(G(x_0), y_0) - d(F(x_0), y_0) \leq d(G(x_0), y_0) - d(H(x_0), y_0) \leq \varepsilon,$$

we have  $d(G(x_0), y_0) \leq d(F(x_0), y_0) + \varepsilon$  and consequently

$$\begin{aligned} 0 &\leq \rho_G(x_0, y_0) - \rho_H(x_0, y_0) = d^2(G(x_0), y_0) - d^2(H(x_0), y_0) \\ &= (d(G(x_0), y_0) - d(H(x_0), y_0))(d(G(x_0), y_0) + d(H(x_0), y_0)) \\ &\leq \frac{\varepsilon}{2d(F(x_0), y_0) + \varepsilon} (d(F(x_0), y_0) + \varepsilon + d(F(x_0), y_0)) = \varepsilon. \end{aligned}$$

Since  $\rho_H(x_0, y_0) \leq \rho_F(x_0, y_0) \leq \rho_G(x_0, y_0)$ , we have also

$$\rho_G(x_0, y_0) \leq \rho_F(x_0, y_0) + \varepsilon, \quad \rho_F(x_0, y_0) \leq \rho_H(x_0, y_0) + \varepsilon,$$

and therefore

$$\rho_G(x, y) - 2\varepsilon \leq \rho_F(x, y) \leq \rho_H(x, y) + 2\varepsilon,$$

for  $(x, y)$  in a suitable neighborhood of  $(x_0, y_0)$ . Since, for every  $i \in I$ , the operator  $\tau_i$  is monotone, we have also

$$\tau_i(\rho_H) \preceq \tau_i(\rho_F) \preceq \tau_i(\rho_G).$$

Finally, if we fix a compact subset  $B$  of  $\mathbb{R}^d$  such that  $(x_0, y_0)$  is in the interior of  $X \times B$ , we have  $\lim_{i \in I} \tau_i(\rho_H) = \rho_H$  and  $\lim_{i \in I} \tau_i(\rho_G) = \rho_G$  uniformly in  $X \times B$  and therefore we can consider  $\alpha \in I$  such that, for every  $i \geq \alpha$  and  $(x, y) \in X \times B$ ,

$$\rho_H(x, y) - \varepsilon \leq \tau_i(\rho_H)(x, y) \leq \tau_i(\rho_F)(x, y) \leq \tau_i(\rho_G)(x, y) \leq \rho_G(x, y) + \varepsilon.$$

Collecting the last inequalities, we obtain

$$\tau_i(\rho_F)(x, y) - 3\varepsilon \leq \rho_F(x, y) \leq \tau_i(\rho_F)(x, y) + 3\varepsilon,$$

in a suitable neighborhood of  $(x_0, y_0)$ . At this point, a straightforward argument applied to compact subsets  $X \times B \subset X \times \mathbb{R}^d$  yields the convergence of  $(\tau_i(\rho_F))_{i \in I}^{\preceq}$  to  $\rho_F$  uniformly on compact subsets. This is equivalent to the convergence of  $(L_i(F))_{i \in I}^{\preceq}$  to  $F$  and this completes the proof.  $\square$

Theorem 2.3 extends Korovkin-type approximation theory to set-valued Hausdorff continuous functions with non convex values. This is not the only noteworthy point. Indeed, in Theorem 1.1 and all previous results, only global assumption independent of direction have been considered. On the contrary, in Theorem 2.3, the second condition in (2.2) is a local assumption that depends on each direction. We also observe that in (2.2) there is an additional internal condition which is due to the possible failure of the convexity condition. However, in some cases of particular interest this condition can be weakened. If  $f \in C(X, \mathbb{R})$ , we shall denote by  $\{f\}$  the set-valued function defined by setting, for every  $x \in X$ ,  $\{f\}(x) := \{f(x)\}$ . Moreover, we shall need to consider this corollary.

**Corollary 2.1.** *Let  $X$  be a compact Hausdorff topological space and let  $\mathcal{C}$  a subcone of  $C(X, \mathcal{K}(\mathbb{R}^d))$  such that  $\{f\} \in \mathcal{C}$  for every  $f \in C(X, \mathbb{R}^d)$  and such that, for every  $F \in \mathcal{C}$  and  $x \in X$ ,*

$$F(x) = \bigcup_{f \in \text{Sel}(F)} \{f(x)\}.$$

Moreover, let  $\mathcal{H}$  be a subset of  $\mathcal{C}$  such that

- i) There exists a Korovkin system  $S \subset C(X, \mathbb{R}^d)$  such that, for every  $s \in S$ ,  $\{s\} \in \mathcal{H}$ .
- ii) For every  $F \in \mathcal{C}$ ,  $x_0 \in X$ ,  $y_0 \in \mathbb{R}^d$  and  $\varepsilon > 0$ , there exists  $H \in \mathcal{H}$  satisfying the following conditions

$$(2.3) \quad F \leq H, \quad d(F(x_0), y_0) < d(H(x_0), y_0) + \varepsilon.$$

Then,  $\mathcal{H}$  is a Korovkin system in  $\mathcal{C}$  with respect to equicontinuous nets of monotone operators.

*Proof.* Since  $S$  is a Korovkin system, assumption i) ensures the convergence on every function  $\{f\}$  with  $f \in C(X, \mathbb{R}^d)$ .

Now, let  $F \in \mathcal{C}$ ,  $x_0 \in X$ ,  $y_0 \in \mathbb{R}^d$  and  $\varepsilon > 0$  and consider the set-valued function  $H \in \mathcal{H}$  such that  $F \leq H$  and  $d(F(x_0), y_0) < d(H(x_0), y_0) + \varepsilon/2$ . From the definition of  $d(F(x_0), y_0)$ , there exists  $z_0 \in F(x_0)$  such that  $d(z_0, y_0) < d(F(x_0), y_0) + \varepsilon/2$ . Moreover, we can consider a continuous selection  $g$  of  $F$  such that  $g(x_0) = z_0$ . If we set  $G := \{g\}$ , we have  $G \in \mathcal{C}$ ,  $G \leq F$  and

$$d(G(x_0), y_0) = d(z_0, y_0) < d(F(x_0), y_0) + \frac{\varepsilon}{2} < d(H(x_0), y_0) + \varepsilon.$$

Hence, condition (2.2) is satisfied and this completes the proof.  $\square$

Different Korovkin systems in cones of set-valued functions are obtained starting with Korovkin systems in  $C(X, \mathbb{R}^d)$  and therefore condition i) in Theorem 2.3 is not too restrictive. The usage of Korovkin systems for single-valued continuous functions is deepened in the next section for particular set-valued functions with non convex values.

### 3. KOROVKIN RESULTS FOR STARSHAPED SET-VALUED FUNCTIONS

Set-valued functions with compact starshaped values have a particular interest for the possibility of extending classical operators to more general cones. We recall that a subset  $S$  of  $\mathbb{R}^d$  is called starshaped set if there exists  $x_0 \in S$  such that the segment joining  $x_0$  and  $x$  is contained in  $S$  whenever  $x \in S$ .

We shall denote by  $\mathcal{S}(\mathbb{R}^d)$  the set of all starshaped compact subsets of  $\mathbb{R}^d$ .  $\mathcal{S}(\mathbb{R}^d)$  is obviously a cone endowed with the natural addition and multiplication by positive scalars. If  $A \in \mathcal{S}(\mathbb{R}^d)$ , we observe that the kernel  $\ker A$  of  $A$ , defined by

$$\ker(A) := \{x_0 \in A \mid \forall x \in A, \forall \lambda \in [0, 1] : x_0 + \lambda(x - x_0) \in A\},$$

is compact and convex. Indeed,  $\ker(A)$  is obviously bounded and if we set, for every  $x_0 \in A$ ,

$$A(x_0) := \{x \in A \mid \forall \lambda \in [0, 1] : x_0 + \lambda(x - x_0) \in A\},$$

we can easily show that  $A(x_0)$  is closed and  $\ker(A) = \bigcap_{x_0 \in A} A(x_0)$ . The convexity of  $\ker(A)$  is obvious. Hence, for every  $F \in C(X, \mathcal{S}(\mathbb{R}^d))$ , we can consider the *kernel set-valued function*  $\ker(F) : X \rightarrow \mathcal{K}_c(\mathbb{R}^d)$  defined by

$$(\ker F)(x) = \ker(F(x)).$$

Unfortunately, we cannot ensure in general that  $\ker(F)$  is Hausdorff continuous and this does not allow us to consider a kernel selection, that is, a continuous selection of  $\ker(F)$ .

Now, assume that  $\ker F$  admits a continuous selection  $\varphi_F$ , which acts as a center selection. We can associate to  $\varphi_F$  the radial function  $\psi_F : X \times S^{d-1} \rightarrow [0, +\infty[$  defined by

$$\psi_F(x, y) := \max\{t \in [0, +\infty[ \mid \varphi_F(x) + ty \in A\}.$$

Even under our restrictive assumptions, we cannot ensure the continuity of  $\psi_F$ ; for example, consider the set-valued function  $F : [-\pi/2, \pi/2] \rightarrow \mathcal{S}_R(\mathbb{R}^2)$  defined by setting, for every  $\theta_0 \in [-\pi/2, \pi/2]$ ,

$$F(\theta_0) := \{(\rho \cos \theta, \rho \sin \theta) \in \mathbb{R}^2 \mid 0 \leq \rho \leq 1, |\theta_0| \leq |\theta| \leq \pi\}.$$

If we take the constant function  $\varphi_F(x) := (-1/2, 0)$  as central selection, then the radial function associated to  $\varphi_F$  is not continuous at  $\theta_0 = 0$ . Therefore, we shall restrict our attention to the subcone  $\mathcal{C}$  of  $C(X, \mathcal{S}_R(\mathbb{R}^d))$  consisting of all  $F \in C(X, \mathcal{S}_R(\mathbb{R}^d))$  which admit a continuous central selection  $\varphi_F$  of  $\ker(F)$  and a continuous radial function  $\psi_F$  with positive values.

It is clear that we can associate a unique set-valued function  $F_{\varphi, \psi} : X \rightarrow \mathcal{S}_R(\mathbb{R}^d)$  to every pair  $(\varphi, \psi) \in C(X, \mathbb{R}^d) \times C(X \times S^{d-1}, [0, +\infty])$  by setting, for every  $x \in X$ ,

$$(3.4) \quad F_{\varphi, \psi}(x) := \{\varphi(x) + ty \in \mathbb{R}^d \mid y \in S^{d-1}, t \in [0, \psi(x, y)]\}.$$

In the next Proposition, we show that  $F_{\varphi, \psi}$  is Hausdorff continuous.

**Proposition 3.1.** *Let  $\varphi : X \rightarrow \mathbb{R}^d$  and  $\psi : X \times S^{d-1} \rightarrow \mathbb{R}$  be continuous and assume that  $\psi$  takes strictly positive values. Then, the set-valued function  $F_{\varphi, \psi} : X \rightarrow \mathcal{S}_R(\mathbb{R}^d)$  defined by (3.4) is Hausdorff continuous.*

*Proof.* Since the Hausdorff distance is invariant under translations, we can assume  $\varphi = 0$ . Moreover, for every  $x_0, x \in X$ ,

$$d_H(F(x_0), F(x)) \leq \sup_{y \in S^{d-1}} |\psi(x, y) - \psi(x_0, y)|$$

and since  $\psi$  is uniformly continuous

$$\lim_{x \rightarrow x_0} \left( \sup_{y \in S^{d-1}} |\psi(x, y) - \psi(x_0, y)| \right) = 0$$

and this completes the proof.  $\square$

Hence, we can associate to every pair  $(\varphi, \psi) \in C(X, \mathbb{R}^d) \times C(X \times S^{d-1}, [0, +\infty])$  a set-valued function in the cone  $\mathcal{C}$ . The converse property is not valid since we may have more than one central selection and radial function which are associated with the same set-valued function. Hence, we have only a partial Korovkin-type result in the cone  $\mathcal{C}$ .

First, we recall that a linear operator  $T : C(X, \mathbb{R}^d) \rightarrow C(X, \mathbb{R}^d)$  is called a convexity monotone operator if for every  $\varphi_1, \varphi_2 \in C(X, \mathbb{R}^d)$  having disjoint graphs

$$(3.5) \quad \varphi \in \text{co}(\varphi_1, \varphi_2) \Rightarrow T(\varphi) \in \text{co}(T(\varphi_1), T(\varphi_2)).$$

Many examples of Korovkin systems in spaces of vector-valued continuous functions with respect to equicontinuous nets of convexity monotone operators have been obtained in [3] and, more recently, also in [4, 5, 6]. We fix a Korovkin  $S$  in  $C(X, \mathbb{R}^d)$  with respect to equicontinuous nets of convexity monotone operators.

Similarly, we have already observed in the preceding section that many examples of Korovkin systems with respect to positive linear operators are well-known both in  $C(X, \mathbb{R})$  than in  $C(S^{d-1}, \mathbb{R})$  (see, e.g. [1, Chapter 4] and [2]) and hence we can easily obtain Korovkin systems in  $C(X \times S^{d-1}, \mathbb{R})$ . Hence, we can also fix a Korovkin system  $D$  in  $C(X \times S^{d-1}, \mathbb{R})$  consisting of positive functions. At this point, we can state the following convergence result. The proof is based on a simple property, which can be easily shown:

If  $(\varphi_i)_{i \in I}^{\leq}$  is a net of functions in  $C(X, \mathbb{R}^d)$  converging uniformly to  $\varphi \in C(X, \mathbb{R}^d)$  and if  $(\psi_i)_{i \in I}^{\leq}$  is a net of positive functions in  $C(X \times S^{d-1}, \mathbb{R})$  converging uniformly to a positive

function  $\psi \in C(X \times S^{d-1}, \mathbb{R})$ , we have that the net  $(F_{\varphi_i, \psi_i})_{i \in I}^{\leq}$  converges to  $F_{\varphi, \psi}$  for the Hausdorff distance.

**Theorem 3.4.** *Let  $X$  be a compact Hausdorff topological space and let  $S$  be a Korovkin system in  $C(X, \mathbb{R}^d)$  with respect to equicontinuous nets of linear convexity monotone operators and  $D$  be a Korovkin system in  $C(X \times S^{d-1}, \mathbb{R})$  consisting of positive functions with respect to equicontinuous nets of positive linear operators. Consider a net  $(S_i)_{i \in I}^{\leq}$  of linear convexity monotone operators from  $C(X, \mathbb{R}^d)$  into itself and assume that, for every  $\varphi \in S$ , we have  $\lim_{i \in I} S_i(\varphi) = \varphi$ .*

*Moreover, let  $(T_i)_{i \in I}^{\leq}$  a net of linear positive operators from  $C(X \times S^{d-1}, \mathbb{R})$  into itself and assume that, for every  $\psi \in D$ , we have  $\lim_{i \in I} T_i(\psi) = \psi$ .*

*Then, the net  $(F_{S_i(\varphi), T_i(\psi)})_{i \in I}^{\leq}$  converges to  $F_{\varphi, \psi}$  for every  $\varphi \in C(X, \mathbb{R}^d)$  and  $\psi \in C(X \times S^{d-1}, [0, +\infty])$ .*

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