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Research Article

Inequalities for the normalized determinant of positive operators in Hilbert spaces via Tominaga and Furuichi results

SILVESTRU SEVER DRAGOMIR*

ABSTRACT. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp \langle \ln Ax, x \rangle$. In this paper, we prove among others that, if $0 < mI \leq A \leq MI$, then

$$1 \leq \exp \left\langle \ln S \left(\left(\frac{M}{m} \right)^{\frac{1}{2}I - \frac{1}{M-m}\left|A - \frac{1}{2}(m+M)I\right|} \right) x, x \right\rangle \leq \frac{\Delta_x(A)}{m^{\frac{M-\langle Ax,x \rangle}{M-m}} M^{\frac{\langle Ax,x \rangle - m}{M-m}}} \leq S\left(\frac{M}{m}\right)$$

for $x \in H$, $\|x\| = 1$, where $S(\cdot)$ is Specht's ratio.

Keywords: Positive operators, normalized determinants, inequalities.

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1. INTRODUCTION

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol, $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of self-adjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [3, 4], introduced the normalized determinant $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x \rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. Some of the fundamental properties of normalized determinant are as follows, [3].

For each unit vector $x \in H$, see also [6] or [7], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x \rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x \rangle$;
(iii) *continuous mean*: $\langle A^p x, x \rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x \rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality*: $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

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*Corresponding author: Silvestru Sever Dragomir; sever.dragomir@vu.edu.au

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a}, & b \neq a \\ \\ a, & b = a \end{cases}.$$

In [3] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m and M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$. The famous Young inequality for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)\,a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called ν-weighted arithmetic-geometric mean inequality. We recall that Specht's ratio is defined by [8]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)}, & h \in (0,1)\cup(1,\infty) \\ \\ 1, & h = 1 \end{cases}. \tag{1.3}$$

It is well known that $\lim_{h\to1} S(h) = 1$ and $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$. In [4], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$. Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (1.4) for A^{-1} we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right) \tag{1.5}$$

for $x \in H$, $\|x\| = 1$. The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$\left(a^{1-\nu}b^{\nu} \leq\right) S\left(\left(\frac{a}{b}\right)^{r}\right) a^{1-\nu}b^{\nu} \leq (1-\nu)\,a + \nu b \leq S\left(\frac{a}{b}\right) a^{1-\nu}b^{\nu}, \tag{1.6}$$

where a, $b > 0$, $\nu \in [0,1]$ and $r = \min\{1-\nu, \nu\}$. The second inequality in (1.6) is due to Tominaga [9] while the first one is due to Furuichi [5].

2. Main Results

Our first main result is as follows:

Theorem 2.1. *If* $0 < mI \leq A \leq MI$ *for positive numbers* m *and* M*, then*

$$1 \leq \exp \left\langle \ln S \left(\left(\frac{M}{m} \right)^{\frac{1}{2}I - \frac{1}{M-m}\left|A - \frac{1}{2}(m+M)I\right|} \right) x, x \right\rangle \leq \frac{\Delta_x(A)}{m^{\frac{M - \langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle - m}{M-m}}} \leq S\left(\frac{M}{m}\right) \tag{2.7}$$

for $x \in H$, $\|x\| = 1$.

Proof. Assume that $t \in [m, M]$ and consider $\nu = \frac{t-m}{M-m} \in [0, 1]$. Then

$$\min\{1-\nu, \nu\} = \frac{1}{2} - \left|\nu - \frac{1}{2}\right| = \frac{1}{2} - \left|\frac{t-m}{M-m} - \frac{1}{2}\right| = \frac{1}{2} - \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|,$$

$$(1-\nu)m + \nu M = \frac{M-t}{M-m}m + \frac{t-m}{M-m}M = t$$

and

$$m^{1-\nu}M^{\nu} = m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}}.$$

By using the inequality (1.6), we deduce

$$m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} \leq S\left(\left(\frac{M}{m}\right)^{\frac{1}{2} - \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|} \right) m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} \leq t \leq S\left(\frac{M}{m}\right) m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} \tag{2.8}$$

for $t \in [m, M]$. By taking the log in (2.8), we get

$$\begin{aligned} &\frac{M-t}{M-m}\ln m + \frac{t-m}{M-m}\ln M \\ &\leq \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2} - \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|} \right) + \frac{M-t}{M-m}\ln m + \frac{t-m}{M-m}\ln M \\ &\leq \ln t \\ &\leq \ln S\left(\frac{M}{m}\right) + \frac{M-t}{M-m}\ln m + \frac{t-m}{M-m}\ln M \end{aligned} \tag{2.9}$$

for $t\in[m,M]$. If $0<mI\leq A\leq MI$, then by using the continuous functional calculus for self-adjoint operators, we get from (2.9) that

$$\begin{aligned}
&\ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m}\\
&\leq\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|}\right)+\ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m}\\
&\leq\ln A\\
&\leq\ln S\left(\frac{M}{m}\right)I+\ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\ln m\frac{M-\langle Ax,x\rangle}{M-m}+\ln M\frac{\langle Ax,x\rangle-m}{M-m}\\
&\leq\left\langle\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|}\right)x,x\right\rangle\\
&+\ln m\frac{M-\langle Ax,x\rangle}{M-m}+\ln M\frac{\langle Ax,x\rangle-m}{M-m}\\
&\leq\langle\ln Ax,x\rangle\leq\ln S\left(\frac{M}{m}\right)+\ln m\frac{M-\langle Ax,x\rangle}{M-m}+\frac{\langle Ax,x\rangle-m}{M-m}\ln M
\end{aligned}$$

for $x\in H$, $\|x\|=1$. This inequality can also be written as

$$\begin{aligned}
&\ln\left(m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}\right)\\
&\leq\ln\left(\exp\left\langle\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|}\right)x,x\right\rangle\right)\\
&+\ln\left(m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}\right)\\
&\leq\langle\ln Ax,x\rangle\leq\ln S\left(\frac{M}{m}\right)+\ln\left(m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}\right)
\end{aligned}\tag{2.10}$$

for $x\in H$, $\|x\|=1$. If we take the exponential in (2.10), then we get

$$\begin{aligned}
&m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}\\
&\leq\left(\exp\left\langle\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|}\right)x,x\right\rangle\right)m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}\\
&\leq\exp\langle\ln Ax,x\rangle\leq S\left(\frac{M}{m}\right)m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}
\end{aligned}$$

and the inequality (2.7) is proved. □

Remark 2.1. *From (1.4) and (2.7), we derive the following inequalities in terms of Specht's ratio*

$$\frac{m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}}{S\left(\frac{M}{m}\right)}\leq\frac{\langle Ax,x\rangle}{S\left(\frac{M}{m}\right)}\leq\Delta_x(A)\leq S\left(\frac{M}{m}\right)m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}\tag{2.11}$$

for $x\in H$, $\|x\|=1$.

Corollary 2.1. *With the assumption of Theorem 2.1, we get*

(2.12)
$$1 \le \exp \left\langle \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{m^{-1}-M^{-1}}\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|}\right)x,x\right\rangle \le \frac{M^{\frac{m^{-1}-\left\langle A^{-1}x,x\right\rangle}{m^{-1}-M^{-1}}}m^{\frac{\left\langle A^{-1}x,x\right\rangle-M^{-1}}{m^{-1}-M^{-1}}}}{\Delta_x(A)} \le S\left(\frac{M}{m}\right)$$

for $x \in H$, $\|x\| = 1$.

Proof. If we write the inequality for A^{-1} that satisfies the condition $0 < M^{-1}I \le A^{-1} \le m^{-1}I$, then

$$1 \le \exp \left\langle \ln S\left(\left(\frac{m^{-1}}{M^{-1}}\right)^{\frac{1}{2}I-\frac{1}{m^{-1}-M^{-1}}\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|}\right)x,x\right\rangle \le \frac{\Delta_x(A^{-1})}{M^{-\frac{m^{-1}-\langle A^{-1}x,x\rangle}{m^{-1}-M^{-1}}}m^{-\frac{\langle A^{-1}x,x\rangle-M^{-1}}{m^{-1}-M^{-1}}}} \le S\left(\frac{m^{-1}}{M^{-1}}\right),$$

namely

$$1 \le \exp \left\langle \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{m^{-1}-M^{-1}}\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|}\right)x,x\right\rangle \le \frac{\Delta_x(A^{-1})}{M^{-\frac{m^{-1}-\langle A^{-1}x,x\rangle}{m^{-1}-M^{-1}}}m^{-\frac{\langle A^{-1}x,x\rangle-M^{-1}}{m^{-1}-M^{-1}}}} \le S\left(\frac{M}{m}\right)$$

or

$$1 \le \exp \left\langle \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{m^{-1}-M^{-1}}\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|}\right)x,x\right\rangle \le \frac{\left[\Delta_x(A)\right]^{-1}}{\left(M^{\frac{m^{-1}-\langle A^{-1}x,x\rangle}{m^{-1}-M^{-1}}}m^{\frac{\langle A^{-1}x,x\rangle-M^{-1}}{m^{-1}-M^{-1}}}\right)^{-1}} \le S\left(\frac{M}{m}\right),$$

which is equivalent to the desired result (2.12). □

Corollary 2.2. *If* $0 < mI \le A, B \le MI$ *for positive numbers* m *and* M*, then*

$$\frac{\frac{m^{\frac{M-1}{M-m}}M^{\frac{1-m}{M-m}}}{\ln\left(\frac{M}{m}\right)}\Theta\left(A,B,m,M,x\right)}{S\left(\frac{M}{m}\right)} \le \frac{\left\langle \frac{A+B}{2}x,x\right\rangle}{S\left(\frac{M}{m}\right)} \le \int_0^1 \Delta_x((1-t)A+tB)dt \le S\left(\frac{M}{m}\right)\frac{m^{\frac{M-1}{M-m}}M^{\frac{1-m}{M-m}}}{\ln\left(\frac{M}{m}\right)}\Theta\left(A,B,m,M,x\right), \tag{2.13}$$

where

$$\Theta\left(A,B,m,M,x\right) := \begin{cases} \frac{\left(\frac{M}{m}\right)^{\frac{\langle (B-A)x,x\rangle}{M-m}}-1}{\frac{\langle (B-A)x,x\rangle}{M-m}}, & \langle (B-A)x,x\rangle \ne 0 \\ 1, & \langle (B-A)x,x\rangle = 0 \end{cases}$$

for $x \in H$, $\|x\| = 1$.

Proof. From (2.11), we get

$$\frac{m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}}{S\left(\frac{M}{m}\right)} \le \frac{\langle[(1-t)A+tB]x,x\rangle}{S\left(\frac{M}{m}\right)} \le \Delta_x((1-t)A+tB) \le S\left(\frac{M}{m}\right)m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}$$

for $t \in [0,1]$. If we take the integral over $t \in [0,1]$, then we get

$$\frac{\int_0^1 m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}dt}{S\left(\frac{M}{m}\right)} \le \frac{\left\langle \frac{A+B}{2}x,x\right\rangle}{S\left(\frac{M}{m}\right)} \le \int_0^1 \Delta_x((1-t)A+tB)dt \le S\left(\frac{M}{m}\right)\int_0^1 m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}dt. \tag{2.14}$$

Observe that

$$\int_0^1 m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}dt = m^{\frac{M}{M-m}}M^{\frac{-m}{M-m}}\int_0^1\left(\frac{M}{m}\right)^{\frac{\langle[(1-t)A+tB]x,x\rangle}{M-m}}dt = m^{\frac{M}{M-m}}M^{\frac{-m}{M-m}}\left(\frac{M}{m}\right)^{\frac{1}{M-m}}\int_0^1\left(\frac{M}{m}\right)^{t\frac{\langle(B-A)x,x\rangle}{M-m}}dt = m^{\frac{M-1}{M-m}}M^{\frac{1-m}{M-m}}\int_0^1\left(\frac{M}{m}\right)^{t\frac{\langle(B-A)x,x\rangle}{M-m}}dt.$$

Since for $a > 0$, $a \neq 1$ and $b \in \mathbb{R}$, we have

$$\int_0^1 a^{bx} dx = \frac{a^b - 1}{b \ln a}$$

then, for $\langle (B - A) x, x \rangle \neq 0$

$$\int_0^1 \left(\frac{M}{m}\right)^{t \frac{\langle (B-A)x,x\rangle}{M-m}} dt = \frac{\left(\frac{M}{m}\right)^{\frac{\langle (B-A)x,x\rangle}{M-m}} - 1}{\frac{\langle (B-A)x,x\rangle}{M-m} \ln\left(\frac{M}{m}\right)}$$

and by (2.14), we derive (2.13). □

3. Related Results

We also have this theorem.

Theorem 3.2. *With the assumption of Theorem 2.1, we have that*

$$1 \leq S\left(\left(\frac{M}{m}\right)^{\frac{1}{2} - \frac{1}{\ln M - \ln m}\left|\langle \ln Ax, x\rangle - \frac{\ln M + \ln m}{2}\right|}\right) \tag{3.15}$$

$$\leq \frac{\frac{\ln M - \langle \ln Ax, x\rangle}{\ln M - \ln m} m + \frac{\langle \ln Ax, x\rangle - \ln m}{\ln M - \ln m} M}{\Delta_x(A)}$$

$$\leq S\left(\frac{M}{m}\right)$$

for $x \in H$, $\|x\| = 1$.

Proof. Assume that $m^{1-\nu}M^{\nu} = \exp s$, then $s = (1-\nu)\ln m + \nu \ln M \in [\ln m, \ln M]$, which gives that

$$\nu = \frac{s - \ln m}{\ln M - \ln m}.$$

Also,

$$\min\{1-\nu, \nu\} = \frac{1}{2} - \left|\frac{s - \ln m}{\ln M - \ln m} - \frac{1}{2}\right|$$
$$= \frac{1}{2} - \frac{1}{\ln M - \ln m}\left|s - \frac{\ln M + \ln m}{2}\right|.$$

From (2.7), we get

$$\exp s \leq S\left(\left(\frac{M}{m}\right)^{\frac{1}{2} - \frac{1}{\ln M - \ln m}\left|s - \frac{\ln M + \ln m}{2}\right|}\right) \exp s$$
$$\leq \frac{\ln M - s}{\ln M - \ln m} m + \frac{s - \ln m}{\ln M - \ln m} M$$
$$\leq S\left(\frac{M}{m}\right) \exp s,$$

namely

$$
\begin{aligned}
1 &\leq S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|s-\frac{\ln M+\ln m}{2}\right|}\right) \\
&\leq \frac{\frac{\ln M-s}{\ln M-\ln m}m+\frac{s-\ln m}{\ln M-\ln m}M}{\exp s} \\
&\leq S\left(\frac{M}{m}\right)
\end{aligned}
$$

for $s \in [\ln m, \ln M]$. If $0 < m \leq A \leq M$ and $x \in H$, $\|x\| = 1$, then $\ln m \leq \langle \ln Ax, x\rangle \leq \ln M$ and for $s = \langle \ln Ax, x\rangle$, we deduce

$$
\begin{aligned}
1 &\leq S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle-\frac{\ln M+\ln m}{2}\right|}\right) \\
&\leq \frac{\frac{\ln M-\langle \ln Ax,x\rangle}{\ln M-\ln m}m+\frac{\langle \ln Ax,x\rangle-\ln m}{\ln M-\ln m}M}{\exp \langle \ln Ax, x\rangle} \\
&\leq S\left(\frac{M}{m}\right),
\end{aligned}
$$

which is equivalent to (3.15). □

Corollary 3.3. *With the assumption of Theorem 2.1, we get*

$$
\begin{aligned}
1 &\leq S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle-\frac{\ln m+\ln M}{2}\right|}\right) \\
&\leq \frac{\Delta_x(A)}{\left(\frac{\langle \ln Ax,x\rangle-\ln m}{\ln M-\ln m}M^{-1}+\frac{\ln M-\langle \ln Ax,x\rangle}{\ln M-\ln m}m^{-1}\right)^{-1}} \\
&\leq S\left(\frac{M}{m}\right)
\end{aligned}
\tag{3.16}
$$

for $x \in H$, $\|x\| = 1$.

Proof. If we write the inequality (3.15) for A^{-1} that satisfies the condition $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then we obtain

$$
\begin{aligned}
1 &\leq S\left(\left(\frac{m^{-1}}{M^{-1}}\right)^{\frac{1}{2}-\frac{1}{\ln m^{-1}-\ln M^{-1}}\left|\left\langle \ln A^{-1}x,x\right\rangle-\frac{\ln m^{-1}+\ln M^{-1}}{2}\right|}\right) \\
&\leq \frac{\frac{\ln m^{-1}-\left\langle \ln A^{-1}x,x\right\rangle}{\ln m^{-1}-\ln M^{-1}}M^{-1}+\frac{\left\langle \ln A^{-1}x,x\right\rangle-\ln M^{-1}}{\ln m^{-1}-\ln M^{-1}}m^{-1}}{\Delta_x(A^{-1})} \\
&\leq S\left(\frac{m^{-1}}{M^{-1}}\right),
\end{aligned}
$$

namely

$$\begin{aligned}1 &\leq S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle-\frac{\ln m+\ln M}{2}\right|}\right)\\ &\leq \frac{\frac{\langle \ln Ax,x\rangle-\ln m}{\ln M-\ln m}M^{-1}+\frac{\ln M-\langle \ln Ax,x\rangle}{\ln M-\ln m}m^{-1}}{\Delta_x(A^{-1})}\\ &\leq S\left(\frac{M}{m}\right)\end{aligned}$$

for $x \in H$, $\|x\| = 1$. This proves (3.16). □

As further research, the author plan to investigate the applications of other inequalities like the ones from [1] and [2].

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Silvestru Sever Dragomir
Victoria University
College of Engineering & Science
Melbourne City, MC 8001, Australia
University of the Witwatersrand
School of Computer Science & Applied Mathematics
Private Bag 3, Johannesburg 2050, South Africa
Email address: sever.dragomir@vu.edu.au